On the Mean Curvatures Sharp Estimates of Hypersurfaces

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Abstract

Sharp estimates for the mean curvatures of hypersurfaces in Riemannian manifolds are known from the works of Jorge-Xavier [3], Markvorsen [6] and Vlachos [11]. We first give a simplified proof of these estimates. This proof shows that a similar original result holds for hypersurfaces in Einstein manifolds which are warped product of $\mathbb{R}$ by Ricci-flat manifolds.

1 Introduction and notations

For $n \geq 1$ and $c \in \mathbb{R}$, let $(M = \mathbb{M}_{n+1}(c), g = \langle \cdot, \cdot \rangle)$ be the $(n+1)$-dimensional simply connected space form of constant curvature $c$, $d$ its Riemannian distance, $\nabla$ its Levi-Civita connection and $\nabla^2$ its Hessian operator. If $N$ is a closed (compact without boundary) connected hypersurface of $M$, we endow $N$ with the induced metric, also denoted by $\langle \cdot, \cdot \rangle$. The induced connection and Hessian are denoted by $\nabla$ and $\nabla^2$ respectively. By the generalized Jordan theorem, $N$ is orientable and divides $M$ into two connected components, one of which (the interior) is relatively compact and has $N$ as its oriented boundary. Let $\eta$ the smooth unit inner normal vector field of $N$, $h$ its second fundamental form and $A$ its shape operator. We recall that the mean curvatures of $N$ are the functions $(H_i)_{0 \leq i \leq n}$ defined by $\sum_{i=0}^{n} \sigma_i X^i \equiv \sum_{i=0}^{n} \binom{n}{i} H_i X^i$ where $(k_i)_{1 \leq i \leq n}$ are the principal curvatures of $N$, $\sigma_i$ the $i^{th}$-elementary symmetric polynomial in the $k_i$'s and $\binom{n}{i}$ the binomial coefficient. The notation $\|H_i\|$ will mean the uniform norm of $H_i$ on $N$. We also introduce the radius of $N$ in $M$, defined by $\text{rad } N := \min_{p \in M} \max_{q \in N} d(p, q)$ and which is the radius of the smallest closed ball(s) in $(M, d)$ containing $N$. We recall that a geodesic sphere of radius $r$ (with $r < \pi/\sqrt{c}$ if $c > 0$) in $M$ is a totally umbilical hypersurface: more precisely, if $s_n$ is the solution of the differential equation $\dot{y}(t) = cy(t)$ with the initial conditions $(y(0), \dot{y}(0)) = (0, 1)$ and $\cot_c = \frac{\sin c}{\sin c}$ its logarithmic derivative, its principal curvatures are all equal to $-\cot_c(r)$.
Thanks to the works of Jorge-Xavier, Markvorsen and Vlachos, we have the following estimates for the mean curvatures of $N$ and this leads to a characterization of geodesic spheres:

**Theorem 1** [3, 6, 11]. Let $N$ be a closed connected hypersurface of $\mathbb{R}^n$. We assume that $\text{rad } N < \pi/(2\sqrt{c})$ if $c$ is positive. Then

1. For any integer $k \in \{1, \ldots, n\}$, we have $\|H_k\| \geq \cot_c(\text{rad } N)$.
   In other words, we obtain a sharp lower bound for the radius of $N$:
   $$\text{rad } N \geq \cot_c^{-1}\left(\min_{k\in\{1,...,n\}} \|H_k\|^{1/k}\right).$$

2. If there exists an integer $k \in \{1, \ldots, n\}$ for which $\|H_k\| = \cot_c(\text{rad } N)$, then $N$ is a geodesic sphere.

We will present here a simplified proof of this result. The key point is to choose an appropriate function and to consider the Newton $(1,1)$-tensors introduced by Reilly. The papers quoted above did not show the necessity of the assumption "rad $M < \pi/(2\sqrt{c})" when $c$ is positive: we fill this gap by producing a counter-example at the end of section 2. Moreover, our approach allows the discovering of a new similar result (section 3, theorem 3).

2 A short proof of theorem 1

Let $h_c$ be the primitive of $\sin_c$ which vanishes at 0. For a fixed point $p$ of $\mathbb{M}$, we introduce the smooth modified distance function $h_c \circ d_p$ on $\mathbb{M}$ which Hessian is proportional to the metric:

**Proposition [9].** - The function $h_c \circ d_p$ satisfies: $\nabla^2(h_c \circ d_p) = (\sin_c \circ d_p) \cdot g$.

In the sequel, let $p$ be a point of $\mathbb{M}$ for which $N$ is included in the closed ball $B_d(p, \text{rad } N)$. If we set $F = h_c \circ d_p$, let $\mathbb{f} = F\rvert_N$ be the restriction of $F$ to $N$ and $q_0$ a point of $N$ where $\mathbb{f}$ achieves its maximum. By the Hopf principle, we have $\langle \nabla \mathbb{f}(q_0), X \rangle = 0$ and $\nabla^2 \mathbb{f}(q_0)(X, X) \leq 0$ for any $X \in T_{q_0}N$. But

$$\langle \nabla \mathbb{f}(q_0), X \rangle = \langle \nabla F(q_0), X \rangle = \sin_c(d_p(q_0)) \cdot \langle \dot{\gamma}(d_p(q_0)), X \rangle$$

where $\gamma : [0, d_p(q_0)] \to \mathbb{M}$ is the unique unit-speed geodesic in $\mathbb{M}$ joining $p$ to $q_0$. This shows that $\dot{\gamma}(d_p(q_0)) \in (T_{q_0}N)^\perp$. On the other side, by the above proposition

$$\nabla^2 \mathbb{f}(q_0)(X, X) = \nabla^2 F(q_0)(X, X) + \langle \nabla F(q_0), h_{q_0}(X, X) \rangle$$

$$= \sin_c(d_p(q_0)) \cdot |X|^2 + \langle \nabla F(q_0), h_{q_0}(X, X) \rangle$$

In this way,
\[ \nabla^2 f(q_0)(X, X) \geq \sin_c(d_p(q_0)) \cdot |X|^2 - \sin_c(d_p(q_0)) \cdot \langle A_q X, X \rangle \]  
(1)

This shows that the principal curvatures \((k_1(q_0))_{1 \leq i \leq n}\) of \(N\) at \(q_0\) are all greater than or equal to \(\cot_c(d_p(q_0))\). Since \(d_p(q_0) \leq \text{rad } N\), as \(\cot_c\) is a decreasing function and as \(\cot_c(\text{rad } N)\) is positive (we use here the assumption on the radius if \(c\) is positive), we have \(\|H_k\| \geq \cot_c^k(\text{rad } N)\) and this shows the first point of the theorem.

If there is equality, i.e. if \(\|H_k\| = \cot_c^k(\text{rad } N)\), then \(d_p(q_0) = \text{rad } N\) and all the principal curvatures of \(N\) at \(q_0\) are equal to \(\cot_c(\text{rad } N)\) which is positive. Let \(U\) be an open neighborhood of \(q_0\) in \(N\) such that the principal curvatures are all positive on \(U\). On \(U\), we will use the classical inequalities [2]:

In [7], Reilly introduced a family \((T_k)_{k \in \{0, \ldots, n\}}\) of \((1,1)\)-tensors on \(N\) defined by the formulae: \(T_0 = \text{Id}\) (identity map) and \(T_{k+1} = \sigma_{k+1} \text{Id} - AT_k\) for \(0 \leq k \leq n - 1\) which satisfy the following formula:

\[ \text{Div}(T_k \nabla f) = (n - k) \cdot \binom{n}{k} \cdot \left\{ (\sin_c \circ d_p) \cdot H_k + \langle \nabla F, \eta \rangle \cdot H_{k+1} \right\} \]  
(2)

where \(\text{Div}\) is the divergence operator on \(N\). As \(\cot_c\) is a decreasing function, then for any point \(q\) of \(U\), we have by equation (2):

\[ \text{Div} \left( T_{k-1} \nabla f(q) \right) = \binom{n}{k} \cdot \left\{ (\sin_c \circ d_p(q)) \cdot H_{k-1}(q) + \langle \nabla F(q), \eta(q) \rangle \cdot H_k(q) \right\} \]

Now \(H_k(q) = \|H_k\|^{1/k} \cdot H_k^{(k-1)/k}(q) \leq \|H_k\|^{1/k} \cdot H_{k-1}(q)\). This shows that \(\text{Div} \left( T_{k-1} \nabla f \right)\) is nonnegative on \(U\). As \((\cdot \mapsto \text{Div} \left( T_{k-1} \nabla \cdot \right))\) is an elliptic operator on smooth functions on \(U\) ([4]), the function \(f\) is therefore constant on \(U\) by the maximum principle for these operators. Hence, the non-empty closed subset \(\{q \in N/f(q) = f(q_0)\}\) of \(N\) is also open. The connectedness of \(N\) implies that \(N\) is included in the geodesic sphere \(F_{\text{rad }}(F_p(q_0))\). As this geodesic sphere is also connected and \(n\)-dimensional, \(N\) coincides with this geodesic sphere.

**Remark 1.** For \(k = 1\), theorem 1 can be stated if one replaces \(M_{m+1}(c)\) by a manifold \(M\) with sectional curvature bounded from above by \(c\): this has been done by Markvorsen [6], Jorge-Xavier [3] and can be derived easily from the above equations: indeed, for a hypersurface in an arbitrary manifold, (2) is still true for \(k = 0\). On the other hand, we have the following comparison result:
Lemma ([8], p 153). - Let $M$ be a complete Riemannian manifold with sectional curvature bounded from above by a constant $c$ ($c \in \mathbb{R}$), $d$ the distance of $M$ and $p$ a point of $M$. Then if $q_0$ is not a cut point of $p$, the function $d_p$ is smooth at $q_0$ and for any vector $X \in T_{q_0}M$ which is normal at $q_0$ to the unique unit-speed geodesic joining $p$ to $q_0$, we have

$$\nabla^2 d_p(q_0)(X, X) \geq \cot_c(d_p(q_0)) \cdot |X|^2.$$

which makes true inequation (1). The end of proof is similar.

Remark 2: a counter-example without the radius assumption. Let $n$ be an integer $\geq 2$, $c$ a positive number, $j$ and $k$ two integers $\geq 1$ with $j + k = n$ and $s$ a number of $]0, \pi/2[$. We will write $\mathbb{R}^{n+2} = \mathbb{R}^{j+1} \times \mathbb{R}^{k+1}$ and any point $x$ of $\mathbb{R}^{n+2}$ will be decomposed as $x = (y, z)$ where $(y, z) \in \mathbb{R}^{j+1} \times \mathbb{R}^{k+1}$. In [10], the author proves that $N := S^j(\cos(s)/\sqrt{c}) \times S^k(\sin(s)/\sqrt{c})$ is a compact connected hypersurface of $M_{n+1}(c) = S^{n+1}(1/\sqrt{c}) = \{x = (y, z) \in \mathbb{R}^{j+1} \times \mathbb{R}^{k+1}/ |y|^2 + |z|^2 = 1/c\}$ which principal curvatures at any point are $(-\sqrt{c} \tan s)$ et $\sqrt{c} \cot s$ with multiplicities $j$ and $k$ respectively (for $c=j=k=1$ and $s = \pi/4$, one recognizes the Clifford torus in $S^3$).

Moreover, We claim that

$$\text{rad } N = \frac{1}{\sqrt{c}} \cdot \left\{\frac{3\pi}{4} - \left|\frac{s - \pi}{4}\right|\right\},$$

which proof is a straightforward calculation: let $p = (y_p, z_p)$ and $q = (y, z)$ be arbitrary points of $S^{n+1}(1/\sqrt{c})$ and $N$ respectively. In $S^{n+1}(1/\sqrt{c})$, the distance between $p$ and $q$ is $d(p, q) = (1/\sqrt{c}) \cdot \cos^{-1}(c(p, q))$. By Cauchy-Schwarz inequality, one obtains $c(p, q) = c(|y_p|^2 + |z_p|^2) \geq -c(|y_p||y| + |z_p||z|) = -\sqrt{c}(|y_p| \cdot \cos s + |z_p| \cdot \sin s)$. So $\max_{p \in N} d(p, q) \leq (1/\sqrt{c}) \cdot \cos^{-1}(-\sqrt{c}(|y_p| \cdot \cos s + |z_p| \cdot \sin s))$. Moreover, this inequality is sharp (indeed, if $y_p$ and $z_p$ are both non zero, take $(y, z) = (\frac{-y_p}{|y_p|} \cos s, \frac{-z_p}{|z_p|} \sin s)$ and if $y_p = 0$, then $z_p$ is nonzero necessarily and take $z = (z_p/|z_p|) \sin s$ and any point of $S^j(\cos(s)/\sqrt{c})$ for $y$). Using the relation $\cos^{-1} a + \cos^{-1}(-a) = \pi$, we deduce that

$$\text{rad } N = \min_{p \in S^{n+1}(1/\sqrt{c})} \left\{\frac{\pi}{\sqrt{c}} - \cos^{-1}\{\sqrt{c} \cdot (|y_p| \cdot \cos s + |z_p| \cdot \sin s)\}\right\},$$

$$= \frac{\pi}{\sqrt{c}} - (1/\sqrt{c}) \cos^{-1}\left\{\min_{p \in S^{n+1}(1/\sqrt{c})} \left\{\sqrt{c} \cdot (|y_p| \cdot \cos s + |z_p| \cdot \sin s)\right\}\right\},$$

$$= \frac{\pi}{\sqrt{c}} - (1/\sqrt{c}) \cos^{-1}\left\{\min_{0 \leq |y_p| \leq (1/\sqrt{c})} \left\{\sqrt{c} \cdot (|y_p| \cdot \cos s + \sqrt{1/c - |y_p|^2} \cdot \sin s)\right\}\right\},$$

$$= \frac{\pi}{\sqrt{c}} - (1/\sqrt{c}) \cos^{-1}\{\cos s, \sin s\},$$

$$= \frac{\pi}{\sqrt{c}} - (1/\sqrt{c}) \max\{s, \pi/2 - s\},$$

$$= (1/\sqrt{c}) \cdot \{3\pi/4 - |s - \pi/4|\}. \quad \square$$

In the particular case where $s = \pi/4$ and $j = k = n/2$, the radius of $N$ is $3\pi/(4\sqrt{c})$ and $\cot_c(\text{rad } N) = -\sqrt{c}$. Since the mean curvatures of $N$ satisfy the relation $\sum_{i=0}^{n} \binom{n}{i} \cdot H_i \cdot X^i =$
(1 - X √c) i ∙ (1 + X √c) j = (1 - cX^2)^j = \sum_{l=0}^{j} (-c)^l \binom{j}{l} X^{2l}, \text{ the mean curvatures of odd order all vanish and}

\[ H_{2j} \| = \left( \frac{j}{2^j} \right) \cdot c^j \begin{cases} \cot \frac{2j}{2\ell} (\text{rad } N) & \text{if } \ell = j \\ < \cot \frac{2j}{2\ell} (\text{rad } N) & \text{if } \ell < j \end{cases} \]

As N is not a geodesic sphere (not even homeomorphic), this shows that the radius assumption cannot be omitted in theorem 1.

3 A New result

The Hessian of D is proportional to the metric g of \( M_{n+1}(c) \). This remark has simplified a lot the calculation of \( \nabla^2 f \) in section 2. We are naturally led to ask for natural questions:

**Question 1.** “Which are the complete Riemannian manifolds \( (M, g, \nabla) \) admitting a smooth function \( F \) which Hessian satisfy \( \nabla^2 F = \lambda g \) for some function \( \lambda \)?

**Question 2.** “Among them, which ones admit totally umbilical hypersurfaces?”

Other manifolds than space forms satisfying this both questions exist:

**Example.** Consider a \( n \)-dimensional complete Ricci-flat manifold \( (N, g_*) \) and consider the Riemannian manifold \( M = \mathbb{R} \times e^{2ct} N_* \) with the warped product metric \( g = dt^2 + e^{2ct} g_* \) where \( c \) is a constant. The function \( F : M \rightarrow \mathbb{R} : (t, x) \mapsto e^{ct} \) satisfies \( \nabla^2 F = c^2 F \cdot g \) and the levels \( \{ t \} \times N_* \) of \( F \) are totally umbilical hypersurfaces of \( M \) with principal curvatures all equal to \(-c\). We also remark that the Ricci formulae for warped products, which may be found in Besse book [1], show that \( M = \mathbb{R} \times e^{2ct} N_* \) is an Einstein manifold with (constant) scalar curvature \(-n(n + 1)c^2\).

Fortunately, question 1 has been studied since 1925 ([5]) and solved: the manifold has to be conformally diffeomorphic to either a space form either the Riemannian product \( I \times N_* \) of an open interval \( I \) of \( \mathbb{R} \) by an arbitrary \( n \)-dimensional complete manifold. If one considers only Einstein manifolds, the second question is also settled:

**Theorem 2** [5]. Let \( (M, g) \) be an \( (n + 1) \)-dimensional complete connected Einstein manifold admitting a smooth function \( F \) which Hessian satisfies \( \nabla^2 F = \lambda g \) where \( \lambda \) is non-identically zero. Then \( M \) is isometric to a space form or the above example.

Moreover, if \( c \) is the Einstein constant (i.e. the constant for which the Ricci curvature \( \text{Ric} \) of \( (M, g) \) satisfies \( \text{Ric} = n cg \)), there exists constants \( s \) and \( t \) such that \( \lambda = -cF^2 + s \) and \( |\nabla F|^2 = -cF^2 + 2sF + t \). In particular, \( \lambda \) and \( |\nabla F| \) are constant on the level sets of \( F \). At last, the non empty level sets of \( F \) above regular values are totally umbilical hypersurfaces of \( M \), with principal curvatures all equal to \(-\lambda/|\nabla F|\).
So we are naturally led to expect a similar result to theorem 1 with hypersurfaces of \( \mathbb{R} \times e^{2\tau} N_\tau \). This is done below:

**Theorem 3.** Let \( N \) be a closed connected hypersurface of \( M = \mathbb{R} \times e^{2\tau} N_\tau \) with the warped product metric \( g = dt^2 + e^{2\tau} g_\tau \), \( (N_\tau, g_\tau) \) being an \( n \)-dimensional compact connected Ricci-flat manifold \( (c > 0) \). Then

i) For any integer \( k \in \{1, \ldots, n\} \), we have \( \|H_k\| \geq c^k \).

ii) If \( \|H_1\| = c \) or \( \|H_2\| = c^2 \), then \( N = \{t\} \times N_\tau \) for some real \( t \) and is a totally umbilical hypersurface of \( M \) with principal curvatures all equal to \( -c \).

We refer again the reader to [1] for numerous examples of compact Ricci-flat manifolds. The proof of the above result is quite similar and only sketched: we apply the Hopf principle to the function \( f = F_N \) and obtain

\[
\nabla^2 f(q_0)(X, X) \geq c^2 F(q_0) \cdot |X|^2 - cF(q_0) \cdot \langle A_{q_0} X, X \rangle
\]

which shows the first part of theorem 3. To study the equality case, we claim that formula (1) is still true for \( k = 0 \) and \( k = 1 \): indeed, it is a straightforward calculation for \( k = 0 \). For \( k = 1 \), an examination of Reilly’s proof shows that

\[
\text{Div}(T_1 \nabla f) = n(n - 1) \left\{ (\text{sn}_c \circ d_p) \cdot H_1 + \langle \nabla F, \eta \rangle \cdot H_2 \right\} + \sum_{i=1}^{n} \langle \nabla f, (\nabla_{e_i} T_1) e_i \rangle
\]

where \( \{e_i\}_{i=1}^n \) is a local orthonormal basis of \( N \). In space forms, the Codazzi equation implies that \( T_1 \) is divergence-free that is \( \sum_{i=1}^{n} (\nabla_{e_i} T_1) e_i = 0 \). It is still zero in the present case: fix a point \( x \) in \( N \) and let us choose an orthonormal basis \( \{e_i\}_{i=1}^n \) with \( \nabla_{e_i} e_j(x) = 0 \) for all \( i \) and \( j \). Denoting by \( \hat{R} \) the Riemann tensor of \( M \), by \( \hat{\text{Ric}} \) its Ricci curvature, using Codazzi equation and Bianchi identities, we have at the point \( x \)

\[
\sum_{i=1}^{n} (\nabla_{e_i} T_1) e_i = \sum_{i=1}^{n} (\nabla_{e_i} (\sigma_1 Id) - (\nabla_{e_i} A) e_i)
= \nabla\sigma_1 - \sum_{i=1}^{n} (\nabla_{e_i} A) e_i
= \nabla\sigma_1 - \sum_{i,j=1}^{n} (\nabla_{e_i} A) e_i e_j
= \nabla\sigma_1 - \sum_{i,j=1}^{n} (\nabla_{e_i} A) e_i (\nabla_{e_j} A) e_j
= \sum_{i,j=1}^{n} (e_i, (\nabla_{e_j} A) e_j) e_j
= \sum_{i,j=1}^{n} (e_i, (\nabla_{e_j} A) e_j) e_j + \hat{R}(e_i, e_j) \eta e_j
= \sum_{i=1}^{n} (e_i, \hat{R}(e_i, e_j) \eta) e_j
= \sum_{j=1}^{n} \hat{\text{Ric}}(e_j, \eta) e_j
= 0.
\]

In the case of equality, the same argument implies that \( \Delta f(q) \geq ncF(q) \{ \|H_1\| - H_1(q) \} \geq 0 \) or \( \text{Div}(T_1 \nabla f) \geq n(n - 1)cF(q) \{ \|H_2\|^{1/2} H_1(q) - H_2(q) \} \geq 0 \) on \( \mathcal{U} \) and we conclude as above.

\[ \Box \]

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References


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