Trace inequalities for logarithms and powers of $J$-Hermitian matrices

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**Abstract**

Spectral inequalities are stated for the trace of the exponential or the logarithmic of certain $J$-Hermitian matrices, $J = I_r \oplus -I_{n-r}$, $0 < r < n$. The obtained inequalities are established in the context of indefinite inner product spaces, and they are known to be valid for Hilbert space operators or operator algebras.

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**1. Introduction**

In the sequel $M_n$ will denote the algebra of $n \times n$ complex matrices. We will consider $\mathbb{C}^n$ with a Krein space structure induced by the indefinite inner product $[x, y] := y^* J x$, $x, y \in \mathbb{C}^n$, where $J = I_r \oplus -I_{n-r}$.

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A matrix \( A \in M_n \) is said to be \( J \)-Hermitian if \( A = A^\# \) where \( A^\# \) is defined by \([Ax, y] = [x, A^\# y] \), \( x, y \in \mathbb{C}^n \), that is, \( A^\# = JAJ \). These matrices appear in many problems of physics, such as in applications of the theory of small oscillations of a mechanical system, in relativistic quantum mechanics or in the theory of algebraic models in quantum physics \([20,21]\). For \( J \)-Hermitian matrices \( A, B \in M_n \), the \( J \)-order relation \( A \succ B \) is defined by \([Ax, x] \succ [Bx, x], x \in \mathbb{C}^n \), which means that \( JA - JB \) is a positive semi-definite Hermitian matrix. Ando \([2]\) and Sano \([25]\) obtained a Löwner inequality of indefinite type, namely if \( I \succ A \succ B \), then \( I \succ A^\# \succ B^\# \) for any \( \alpha \in [0, 1] \). In this direction, Sano \([25]\) proved that if \( r > 0, p > 0, q \geq 1 \) satisfy \((1 + 2r)q \geq p + 2r \) then \( I \succ A \succ B \) implies \( A^{(p+2r)/q} \succ (A^r B^p A^r)^{1/q} \). This can be viewed as an indefinite version of Furuta inequality \([15,16]\) for a certain class of \( J \)-Hermitian matrices. Motivated by these results, some authors \([7-10]\) derived indefinite versions of inequalities valid in the context of Hilbert spaces.

There exists a rich theory of inequalities for Hermitian matrices. In contrast with these matrices whose spectrum is real, the spectrum of a \( J \)-Hermitian matrix is symmetric relatively to the real axis \([4]\). Henceforth, this property prevents the derivation of inequalities for these matrices, except for some special classes. In this note, spectral inequalities involving traces of logarithms and traces of powers of certain \( J \)-Hermitian matrices are derived.

### 2. Trace inequalities for logarithms

Before discussing trace inequalities, we recall some useful notions and notation. We denote by \( \sigma(A) \) the spectrum of \( A \in M_n \) and by \( \sigma_j^\pm(A) \) the sets of the eigenvalues of \( A \) with associated eigenvectors \( x \) such that \( x^* J x = \pm 1 \). A matrix \( U \in M_n \) is said to be \( J \)-unitary if \( UU^\# = I_n \). The \( J \)-unitary matrices form a locally compact group \( U_{r,n} \), called the \( J \)-unitary group. We recall that a \( J \)-Hermitian matrix \( A \) is diagonalizable under a \( J \)-unitary similarity transformation (or simply is \( J \)-unitarily diagonalizable) if and only if every eigenvalue of \( A \) belongs either to \( \sigma_j^+(A) \) or to \( \sigma_j^-(A) \) \([4]\). We will be concerned with the class \( \mathcal{J} \) of \( J \)-Hermitian matrices \( X \in M_n \) such that:

1. \( \sigma_j^+(X) = \{\lambda_1, \ldots, \lambda_r : \lambda_1 \geq \cdots \geq \lambda_r\} \), \( \sigma_j^-(X) = \{\lambda_{r+1}, \ldots, \lambda_n : \lambda_{r+1} \geq \cdots \geq \lambda_n\} \);
2. The eigenvalues of \( X \) do not interlace, that is, either \( \lambda_r > \lambda_{r+1} \) or \( \lambda_n > \lambda_1 \).

For \( A, B \in \mathcal{J} \) with positive spectra, we define the entropy of \( A \) by \( S(A) := -\text{Tr}(A \log A) \) (by convention, \( x \log x = 0 \) if \( x = 0 \)) and the relative entropy of \( A \) and \( B \) as \( S(A, B) := \text{Tr}(A \log A - \log B) \). These concepts have been extensively studied for positive semi-definite Hermitian matrices (see \([11]\) and references therein).

**Theorem 2.1.** Let \( A, B \in \mathcal{J} \) have positive eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \), respectively. Then statements (i) and (ii) hold:

(i) If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then

\[
\text{Tr} \left( A(\log A - \log B) \right) \geq \sum_{j=1}^{n} \alpha_j \log \frac{\alpha_j}{\beta_j}.
\]

(ii) If \( \alpha_r > \alpha_{r+1} \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_n > \alpha_1 \) and \( \beta_n > \beta_1 \)), then

\[
\text{Tr} \left( A(\log A - \log B) \right) \leq \sum_{j=1}^{r} \alpha_j \log \frac{\alpha_j}{\beta_{r-j+1}} + \sum_{j=r+1}^{n} \alpha_j \log \frac{\alpha_j}{\beta_{n+r-j+1}}.
\]

**Proof.** Since \( A, B \in \mathcal{J} \), the \( J \)-Hermitian matrices \( A, B \) are \( J \)-unitarily diagonalizable. Without loss of generality, we may assume that \( A = \text{diag}(\alpha_1, \ldots, \alpha_n), B = UB_0U^\# \) with \( U \in U_{r,n-r} \) and \( B_0 = \text{diag}(\beta_1, \ldots, \beta_n) \). Let

\[
\psi(U) = \text{Tr} \left( A \left( \log A - \log \left( UB_0U^\# \right) \right) \right) = \text{Tr} \left( A \log A - AU \log B_0U^\# \right).
\]

(1)
Under the restrictions on the eigenvalues of $A$, $B$, we show that $\psi(U)$ ranges over a closed real half-line. In order to determine its endpoint, we begin by investigating the critical points of $\psi$. We say that $U$ is critical if $\frac{d}{dt} \psi(e^{i t S} U) \bigg|_{t=0} = 0$ for an arbitrary $J$-Hermitian matrix $S$. After some computations, we get

$$\frac{d}{dt} \psi \left( e^{i t S} U \right) \bigg|_{t=0} = -i \text{Tr} \left( ASU \log B_0 U^# - AU \log B_0 U^# S \right) = i \text{Tr} \left( SAU \log B_0 U^# - SU \log B_0 U^# A \right).$$

Thus, $U$ is critical if $\text{Tr}(S[A, U \log B_0 U^#]) = 0$ for any $J$-Hermitian matrix $S$, where $[X, Y]$ stands for the Lie bracket $XY - YX$. Since $S$ is arbitrary, we get $[A, U \log B_0 U^#] = 0$. Supposing that the eigenvalues $\alpha_1, \ldots, \alpha_n$ are all distinct, this implies that

$$U \text{diag}(\log \beta_1, \ldots, \log \beta_n) U^# = \text{diag}(\log \beta_{\sigma(1)}, \ldots, \log \beta_{\sigma(n)}).$$

being $\sigma \in S_n$, the symmetric group of degree $n$. If the eigenvalues $\alpha_1, \ldots, \alpha_n$ are not all distinct, suppose $A = \bigoplus_{k=1}^s \alpha_k I_{n_k}$, with $\alpha_1', \ldots, \alpha_s'$ distinct. Then $U \log B_0 U^#$ is a direct sum of block matrices,

$$U \log B_0 U^# = \bigoplus_{k=1}^s B_k,$$

with $B_k \in M_{n_k}, k = 1, \ldots, s$ and $n_1 + \ldots + n_s = n$. Since $n_1 + \ldots + n_k = r$, for some $k$, each $B_k$ is Hermitian and so it may be diagonalized by a unitary matrix $V_k$. Then the matrix $V = \bigoplus_{k=1}^s V_k$ is unitary as well as $J$-unitary and satisfies $[A, V] = 0$ as well as $[A, V^#] = 0$. So $U \log B_0 U^#$ may be taken in diagonal form

$$P_\sigma^T \text{diag}(\log \beta_1, \ldots, \log \beta_n) P_\sigma,$$

where $P_\sigma$ is the permutation matrix associated with

$$\sigma = \sigma_1 \circ \sigma_2, \quad \sigma_1(j) = j, \quad j = r + 1, \ldots, n, \quad \sigma_2(j) = j, \quad j = 1, \ldots, r. \quad (2)$$

Thus, a critical $U$ is essentially a permutation matrix of the form $U_\sigma = \delta_{\sigma(j)}$ and

$$\psi(U_\sigma) = \sum_{j=1}^n \alpha_j \left( \log \alpha_j - \log \beta_{\sigma(j)} \right). \quad (3)$$

Let $J = I_r \oplus -I_{n-r} = \text{diag} (\epsilon_1, \ldots, \epsilon_n)$. Under the assumptions on the eigenvalues of $A$ and $B$ in (i), we have

$$\epsilon_k \epsilon_j (\alpha_k - \alpha) (\log \beta_j - \log \beta) < 0, \quad k, j = 1, 2, \ldots, n,$$

where

$$\alpha = \frac{\alpha_1 + \alpha_n}{2}, \quad \beta = \frac{\beta_1 + \beta_{r+1}}{2} \quad \text{if} \quad \alpha_n > \alpha_1 \quad \text{and} \quad \beta_r > \beta_{r+1},$$

$$\alpha = \frac{\alpha_1 + \alpha_n}{2}, \quad \beta = \frac{\beta_1 + \beta_n}{2} \quad \text{if} \quad \alpha_r > \alpha_{r+1} \quad \text{and} \quad \beta_n > \beta_1.$$
\[
\psi (V_{kl}U_{\sigma}) = \sum_{j=1}^{n} \alpha_j (\log \alpha_j - \log \beta_{\sigma(j)}) + \sinh^2 t (\alpha_k - \alpha_l) (\log \beta_{\sigma(l)} - \log \beta_{\sigma(k)}) \geq \psi (U_{\sigma})
\]
and so \( \psi (U) \) ranges over a closed right half-line. For \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), and any \( U \in U_{r,n-r} \), we show that
\[
\psi (U) > \sum_{j=1}^{n} \alpha_j (\log \alpha_j - \log \beta_j).
\]
Indeed, suppose that \( \sigma_1 (j) = j, j = 1, \ldots, l - 1, \sigma_1 (l) \neq l \) and consider \( 1 < k \leq r \) such that \( \sigma_1 (k) = l \). Having in mind (3), we get
\[
\sum_{j=1}^{r} \alpha_j (\log \alpha_j - \log \beta_{\sigma_1(j)}) + \alpha_l (\log \alpha_l - \log \beta_l) + \alpha_k (\log \alpha_k - \log \beta_{\sigma_1(l)}) + \sum_{j=r+1}^{n} \alpha_j (\log \alpha_j - \log \beta_{\sigma_1(l)}) - \psi (U_{\sigma}) = - (\alpha_l - \alpha_k) (\log \beta_l - \log \beta_{\sigma_1(l)}) \leq 0,
\]
because \( k > l \) and \( \sigma_1 (l) > l \). For \( \sigma_1, \sigma_2 \) in (2), let \( \xi \in S_n \) be such that \( \xi (l) = l, \xi (k) = \sigma_1 (l), \xi (j) = \sigma_1 (j), j = 1, \ldots, r, j \neq k, l \), and \( \xi (j) = \sigma_2 (j), j = r + 1, \ldots, n \).

If the equality does not hold in (4), then \( \psi (U_{\xi}) < \psi (U_{\sigma}) \), a contradiction, since \( \psi (U_{\sigma}) \) is the minimum of \( \psi (U) \). Therefore, the equality in (4) holds and the point \( \psi (U_{\xi}) \) is also the minimum. Hence, we can take \( \xi \) as new \( \sigma_1 \) in (4). Repeating the argument, we conclude that \( \sigma_1 (j) = j, j = 1, \ldots, r \). Thus, \( \sigma_1 \) can be assumed the identity. Similarly, it can be shown that \( \sigma_2 \) is the identity, and so
\[
\sum_{j=1}^{n} \alpha_j (\log \alpha_j - \log \beta_j)
\]
is the minimum of \( \psi (U) \). This proves (i).

Now, we prove (ii). Under the assumptions on the eigenvalues of \( A \) and \( B \) in (ii), we find
\[
e_{k} e_{l} (\alpha_k - \alpha_l) (\log \beta_j - \log \beta) > 0, \quad k, j = 1, 2, \ldots, n,
\]
where
\[
\alpha = \frac{\alpha_1 + \alpha_n}{2}, \quad \beta = \frac{\beta_1 + \beta_n}{2} \quad \text{if} \quad \alpha_n > \alpha_1 \quad \text{and} \quad \beta_n > \beta_1, \n\]
\[
\alpha = \frac{\alpha_r + \alpha_{r+1}}{2}, \quad \beta = \frac{\beta_r + \beta_{r+1}}{2} \quad \text{if} \quad \alpha_r > \alpha_{r+1} \quad \text{and} \quad \beta_r > \beta_{r+1}.
\]
Repeating the above arguments, we get
\[
\psi (U) \leq \sum_{j=1}^{n} \alpha_j \log \alpha_j - (\alpha_1 \log \beta_r + \cdots + \alpha_r \log \beta_1 + \alpha_{r+1} \log \beta_n + \cdots + \alpha_n \log \beta_{r+1})
\]
for any \( U \in U_{r,n-r} \) and so (ii) holds.

The following corollary is an indefinite version of Klein inequality [23].

**Corollary 2.1.** Let \( A, B \in \mathcal{J} \) be under the conditions of Theorem 2.1. If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then
\[
\text{Tr} (A (\log A - \log B)) > \text{Tr} (A - B).
\]
Proof. By Theorem 2.1 (i), we obtain
\[
\text{Tr} \left( A(\log A - \log B) - A + B \right) \geq \sum_{j=1}^{n} \left( \alpha_j \log \frac{\alpha_j}{\beta_j} - \alpha_j + \beta_j \right) = \sum_{j=1}^{n} \beta_j \left( \frac{\alpha_j}{\beta_j} \log \frac{\alpha_j}{\beta_j} - \frac{\alpha_j}{\beta_j} + 1 \right) \geq 0,
\]
the last inequality being justified by the fact that \( x \log x - x + 1 \geq 0 \) for \( x > 0 \). Clearly, the equality occurs if and only if \( \alpha_j/\beta_j = 1, j = 1, \ldots, n \). But this is impossible because we are assuming that \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), so the strict inequality holds. □

The following corollary can be seen as an indefinite version of the Peierls–Bogoliubov inequality.

Corollary 2.2. Let \( A, B \in \mathcal{J} \) have eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \), respectively. If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then
\[
\frac{\text{Tr} e^B}{\text{Tr} e^A} > \exp \left( \frac{\text{Tr} \left( (B - A)e^A \right)}{\text{Tr} e^A} \right).
\]

Proof. We observe that \( e^A/\text{Tr} e^A, e^B/\text{Tr} e^B \in \mathcal{J} \). Moreover, their eigenvalues are positive and satisfy analogous conditions to those of \( A, B \). Replacing in Corollary 2.1, \( A \) and \( B \) by \( e^A/\text{Tr} e^A \) and \( e^B/\text{Tr} e^B \), respectively, we find
\[
\text{Tr} \left( \frac{e^A}{\text{Tr} e^A} (A - B) + \frac{e^A}{\text{Tr} e^A} \log \frac{\text{Tr} e^B}{\text{Tr} e^A} \right) > 0.
\]

By applying the exponential to both sides of the inequality
\[
\log \frac{\text{Tr} e^B}{\text{Tr} e^A} > \frac{\text{Tr} \left( (B - A)e^A \right)}{\text{Tr} e^A},
\]
the result follows. □

The following corollary is an indefinite version of the thermodynamic inequality [5,6,22].

Corollary 2.3. Let \( A, B \in \mathcal{J} \) have eigenvalues \( \alpha_1, \ldots, \alpha_n \geq 0 \) and \( \beta_1, \ldots, \beta_n \), respectively. Assume also that \( \text{Tr} A = 1 \).

If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then
\[
\log \text{Tr} e^B > \text{Tr}(AB) + S(A).
\]

Proof. Under the hypothesis, we may conclude that \( \log \alpha_1 \geq \cdots \geq \log \alpha_r \) belong to \( \sigma^+_J(\log A) \) and \( \log \alpha_{r+1} \geq \cdots \geq \log \alpha_n \) belong to \( \sigma^-_J(\log A) \). Moreover, \( \log \alpha_n > \log \alpha_1 \) (or \( \log \alpha_r > \log \alpha_{r+1} \)). Replacing \( A \) by \( \log A \) in Corollary 2.2 and having in mind that \( \text{Tr} A = 1 \), we get
\[
\text{Tr} e^B > \exp \text{Tr} ((B - \log A)A) = \exp (\text{Tr}(AB) + S(A)).
\]

By the monotonicity of the logarithmic function, the result follows. □

Now, we present a chain of equivalent statements to the indefinite Peierls–Bogoliubov inequality. For simplicity of notation, we denote by \( \mathcal{J}' \) the class of all pairs of matrices \( (A, B) \) such that \( A, B \in \mathcal{J} \) have eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \), respectively, satisfying \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)).
Theorem 2.2. The following statements hold and are mutually equivalent:

(i) $\text{Tr } e^B / \text{Tr } e^A > \exp(\text{Tr}((B - A)e^A) / \text{Tr } e^A)$ for $(A, B) \in \mathcal{J}^+$;

(ii) $\log(\text{Tr } e^B) > \text{Tr}(AB) + S(A)$, whenever $(A, B) \in \mathcal{J}^+$, $\text{Tr } A = 1$ and $\sigma(A) \subset \mathbb{R}^+$;

(iii) $\frac{1}{n} S((\text{Tr } A)I_n, (\text{Tr } B)I_n) < S(A, B)$, whenever $(A, B) \in \mathcal{J}^+$, $\sigma(A), \sigma(B) \subset \mathbb{R}^+$;

(iv) $S(A, B) > 0$, whenever $(A, B) \in \mathcal{J}^+$, $\text{Tr } A = \text{Tr } B$ and $\sigma(A), \sigma(B) \subset \mathbb{R}^+$.

Proof. By Corollary 2.2, (i) holds and the implication (i) $\Rightarrow$ (ii) has been proved in Corollary 2.3.

(ii) $\Rightarrow$ (iii): Assume that the eigenvalues of $A, B$ are positive. Replacing in (ii) $A$ and $B$ by $A/\text{Tr}(A)$ and $\log B$, respectively, we get

$$\log \text{Tr } B > \frac{\text{Tr}(A \log B) + S(A)}{\text{Tr } A} + \log \text{Tr } A.$$ 

Taking into account that $\text{Tr } A > 0$, we find

$$-\frac{1}{n} S((\text{Tr } A)I_n, (\text{Tr } B)I_n) = \text{Tr } A (\log \text{Tr } B - \log \text{Tr } A)$$

$$> \text{Tr}(A \log B) + S(A)$$

$$= -S(A, B).$$

(iii) $\Rightarrow$ (iv): Considering $\text{Tr } A = \text{Tr } B$ in (iii), then (iv) is trivially obtained.

(iv) $\Rightarrow$ (i): The $J$-Hermitian matrices $e^A/\text{Tr } e^A, e^B/\text{Tr } e^B$ have trace one and positive eigenvalues. Replacing $A, B$ in (iv) by $e^A/\text{Tr } e^A, e^B/\text{Tr } e^B$, respectively, (i) follows. \qed

3. Trace inequalities for powers

For $A, B \in \mathcal{J}$ with positive eigenvalues and $\lambda \in (0, 1]$, we define the Tsallis entropy of $A$ as $S_{\lambda}(A) := \frac{1}{\lambda} \text{Tr}(A^{1-\lambda} - A)$ and the Tsallis relative entropy of $A$ and $B$ as $D_{\lambda}(A, B) := \frac{1}{\lambda} \text{Tr}(A - A^{1-\lambda} B^{\lambda})$. These concepts are familiar in the context of positive semi-definite Hermitian matrices [12–14] and are still meaningful for the class $\mathcal{J}$. For $x \geq 0$ the function $e^x = (1 + \lambda x)^{\frac{1}{\lambda}}$ and its inverse $\text{ln}_\lambda x = \frac{x^{\lambda}-1}{\lambda}$ converge to $e^x$ and $\log x$, respectively, as $\lambda \to 0$. Thus, the Tsallis entropy and the Tsallis relative entropy can be written as $S_{\lambda}(A) = -\text{Tr}(A^{1-\lambda} \text{ln}_\lambda A)$ and $D_{\lambda}(A, B) = \text{Tr}(A^{1-\lambda} (\text{ln}_\lambda A - \text{ln}_\lambda B))$ and as $\lambda \to 0$ they converge to $S(A)$ and $S(A, B)$, respectively. The proof of the next result follows similar steps to the proof of Theorem 2.1, so we only sketch it, the details being left to the reader.

Theorem 3.1. Let $A, B \in \mathcal{J}$ be under the conditions of Theorem 2.1. For $\lambda \in (0, 1]$, the statements (i) and (ii) hold:

(i) If $\alpha_n > \alpha_1$ and $\beta_r > \beta_{r+1}$ (or $\alpha_r > \alpha_{r+1}$ and $\beta_n > \beta_1$), then

$$\text{Tr } \left(A^{1-\lambda} (\text{ln}_\lambda A - \text{ln}_\lambda B)\right) \geq \sum_{j=1}^{n} \frac{\alpha_j - \alpha_j^{1-\lambda} \beta_j^{\lambda}}{\lambda}.$$ 

(ii) If $\alpha_r > \alpha_{r+1}$ and $\beta_r > \beta_{r+1}$ (or $\alpha_n > \alpha_1$ and $\beta_n > \beta_1$), then

$$\text{Tr } \left(A^{1-\lambda} (\text{ln}_\lambda A - \text{ln}_\lambda B)\right) \leq \sum_{j=1}^{r} \frac{\alpha_j - \alpha_j^{1-\lambda} \beta_{r-j+1}^{\lambda}}{\lambda} + \sum_{j=r+1}^{n} \frac{\alpha_j - \alpha_j^{1-\lambda} \beta_{n-r-j+1}^{\lambda}}{\lambda}.$$ 

Proof. For $\lambda = 1$ and with the usual convention $A^0 = I$, we have $\text{Tr } (A - B) = \sum_{j=1}^{n} (\alpha_j - \beta_j)$, So we consider $\lambda \in (0, 1)$. Without loss of generality, we may assume that $A = \text{diag}(\alpha_1, \ldots, \alpha_n), B = UB^0U^\#$ for $U \in U_{r,n-r}$ and $B_0 = \text{diag}(\beta_1, \ldots, \beta_n)$. Let
\[ \psi(U) = \text{Tr} \left( A - A^{1-\lambda} (UB_0U^\#)^\lambda \right). \]

The matrix \( U \) is critical if \( \frac{d}{dt} \psi(e^{itS})\big|_{t=0} = 0 \), for an arbitrary \( J \)-Hermitian matrix \( S \). Now,

\[ \frac{d}{dt} \psi(e^{itS})\big|_{t=0} = -i\text{Tr} \left( A^{1-\lambda} \left[ S, UB_0^\# U^\# \right] \right) = i\text{Tr} \left( S \left[ A^{1-\lambda}, UB_0^\# U^\# \right] \right) = 0. \]

Since \( S \) is arbitrary, this implies that

\[ U\text{diag} \left( \beta_1^\lambda, \ldots, \beta_n^\lambda \right) U^\# = \text{diag} \left( \beta_1^\lambda, \ldots, \beta_n^\lambda \right), \]

being \( \sigma = \sigma_1 \circ \sigma_2 \in S_n \), with \( \sigma_1(j) = j, j = r + 1, \ldots, n \), and \( \sigma_2(j) = j, j = 1, \ldots, r \). Thus, a critical \( U \) is a permutation matrix \( U_\sigma = (\delta_{\sigma(j)k}) \) and

\[ \psi(U_\sigma) = \sum_{j=1}^n \alpha_j - \alpha_j^{1-\lambda} \beta_{\sigma(j)}^\lambda \]

For \( k \leq r \), \( l \geq r + 1 \), let \( V_{kl} \in U_{r,n-r} \) be the \( J \)-unitary matrix obtained from \( I_n \) replacing the entries \((k,k), (l,l)\) and \((k,l), (l,k)\) by \( \cosh t \) and \( \sinh t \), respectively. Then

\[ \psi(V_{kl}U_\sigma) = \sum_{j=1}^n \alpha_j - \alpha_j^{1-\lambda} \beta_{\sigma(j)}^\lambda + \sinh^2 t \frac{1}{\lambda} \left( \alpha_k - \alpha_1 \right) \left( \beta_1^{\lambda} - \beta_{\sigma(k)}^{\lambda} \right). \]

For \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)) and any \( U \in U_{r,n-r} \), after some calculations we may conclude that

\[ \psi(U) \geq \sum_{j=1}^n \frac{\alpha_j - \alpha_j^{1-\lambda} \beta_j^\lambda}{\lambda} \]

and we infer (i).

If \( \alpha_r > \alpha_{r+1} \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_n > \alpha_1 \) and \( \beta_n > \beta_1 \)) and any \( U \in U_{r,n-r} \), we find

\[ \psi(U) \leq \frac{1}{\lambda} \sum_{j=1}^n \alpha_j - \frac{1}{\lambda} \left( \alpha_1^{1-\lambda} \beta_r^{\lambda} + \cdots + \alpha_r^{1-\lambda} \beta_1^{\lambda} + \alpha_{r+1}^{1-\lambda} \beta_n^{\lambda} + \cdots + \alpha_n^{1-\lambda} \beta_{r+1}^{\lambda} \right) \]

and so (ii) holds. \( \square \)

**Corollary 3.1.** Let \( \lambda \in (0,1) \) and let \( A,B \in \mathcal{J} \) be under the conditions of Theorem 2.1. If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then

\[ \text{Tr} \left( A^{1-\lambda} (\ln A - \ln B) \right) > \text{Tr}(A-B). \]

**Proof.** Firstly, we show that for \( \lambda \in (0,1) \), we have \( f(x) = \frac{1}{\lambda} (x - x^{1-\lambda}) - x + 1 \geq 0 \) for \( x > 0 \). Indeed, \( f'(x) = \frac{1}{\lambda} (1-\lambda) (1-x^{1-\lambda}), f''(x) = (1-\lambda) x^{-1-\lambda}. \) Since \( f'(x) = 0 \) if and only if \( x = 1 \) and \( f''(x) > 0 \), we have \( f(x) \geq f(1) = 0 \) as desired. Now, by Theorem 3.1 (i), we obtain

\[ \text{Tr} \left( A - A^{1-\lambda} \frac{B^\lambda}{\lambda} - A + B \right) \geq \sum_{j=1}^n \left( \frac{\alpha_j - \alpha_j^{1-\lambda} \beta_j^\lambda}{\lambda} - \alpha_j + \beta_j \right) \]

\[ = \sum_{j=1}^n \beta_j \left( \frac{\alpha_j}{\beta_j^\lambda} - \frac{\alpha_j^{1-\lambda}}{\beta_j^\lambda} \right) \geq 0. \]

The equality occurs if and only if \( \alpha_j/\beta_j = 1, j = 1, \ldots, n \), which is impossible because we are assuming that \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)). \( \square \)
Corollary 3.2. Let \( \lambda \in (0, 1) \) and let \( A, B \in \mathcal{J} \) be under the conditions of Theorem 2.1. If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then \( \text{Tr}(A^{1-\lambda}B^\lambda) < (\text{Tr} A)^{1-\lambda}(\text{Tr} B)^\lambda \).

Proof. Replacing in Corollary 3.1, \( A \) and \( B \) by the \( J \)-Hermitian matrices \( A/\text{Tr} A \) and \( B/\text{Tr} B \) respectively, we find

\[
\frac{1}{\lambda} \text{Tr} \left( \frac{A}{\text{Tr} A} - \frac{A^{1-\lambda}B^\lambda}{(\text{Tr} A)^{1-\lambda}(\text{Tr} B)^\lambda} \right) = \frac{1}{\lambda} \left( 1 - \frac{\text{Tr}(A^{1-\lambda}B^\lambda)}{(\text{Tr} A)^{1-\lambda}(\text{Tr} B)^\lambda} \right) > 0
\]

and the result follows since \( \lambda > 0 \) and \( A, B \) have positive trace. \( \square \)

Corollary 3.3. Let \( \lambda \in (0, 1) \) and let \( A, B \in \mathcal{J} \) have eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \), respectively. If \( \alpha_n > \alpha_1 \) and \( \beta_r > \beta_{r+1} \) (or \( \alpha_r > \alpha_{r+1} \) and \( \beta_n > \beta_1 \)), then

\[
\frac{\text{Tr} e^{B}_\lambda}{\text{Tr} e^{A}_\lambda} > \exp_{\lambda} \left( \frac{\text{Tr}((B - A)(e^{A}_\lambda)^{1-\lambda})}{\text{Tr} e^{A}_\lambda} \right).
\]

Proof. Under the hypothesis, \( e^{A}_\lambda \) is a \( J \)-Hermitian and \( J \)-unitarily diagonalizable matrix with positive eigenvalues \( (1 + \lambda \alpha_1)^{1/2}, \ldots, (1 + \lambda \alpha_n)^{1/2} \). Then \( (1 + \lambda \alpha_1)^{1/2} \geq (1 + \lambda \alpha_2)^{1/2} \geq \cdots \geq (1 + \lambda \alpha_n)^{1/2} \) belong to \( \sigma_j^+(e^{A}_\lambda) \), \( (1 + \lambda \alpha_r)^{1/2} \geq (1 + \lambda \alpha_{r+1})^{1/2} \) belong to \( \sigma_j^-(e^{A}_\lambda) \) and satisfy \( (1 + \lambda \alpha_n)^{1/2} > (1 + \lambda \alpha_1)^{1/2} \). Analogous conditions hold for the eigenvalues of \( e^{B}_\lambda \).

If \( A, B \) have positive eigenvalues, from Corollary 3.2 we infer

\[
\frac{\text{Tr}((A^{1-\lambda}B^\lambda) - \text{Tr} A)}{\lambda} < \frac{(\text{Tr} A)^{1-\lambda}(\text{Tr} B)^\lambda - \text{Tr} A}{\lambda}
\]

and we easily get

\[
\ln_{\lambda} \frac{\text{Tr} B}{\text{Tr} A} > -\frac{D_{\lambda}(A, B)}{\text{Tr} A}.
\]

Replacing in (5) \( A \) and \( B \) by \( e^{A}_\lambda \) and \( e^{B}_\lambda \), respectively, we obtain

\[
\ln_{\lambda} \frac{\text{Tr} e^{B}_\lambda}{\text{Tr} e^{A}_\lambda} > \frac{\text{Tr}((B - A)(e^{A}_\lambda)^{1-\lambda})}{\text{Tr} e^{A}_\lambda},
\]

and the result follows. \( \square \)

Remark 1. Replacing in Corollary 3.1, \( A, B \) by \( e^{A}_\lambda \) and \( e^{B}_\lambda \), respectively, we obtain

\[
\text{Tr}((e^{A}_\lambda)^{1-\lambda}(A - B)) > \text{Tr} e^{A}_\lambda - \text{Tr} e^{B}_\lambda.
\]

Since \( \text{Tr} e^{A}_\lambda > 0 \) we get

\[
\frac{\text{Tr} e^{B}_\lambda}{\text{Tr} e^{A}_\lambda} > 1 + \frac{\text{Tr}((B - A)(e^{A}_\lambda)^{1-\lambda})}{\text{Tr} e^{A}_\lambda}.
\]

Having in mind that \( e^x > 1 + x \) for any real \( x \), Corollary 3.3 improves this estimate.

Next, we present a parallel result to Theorem 2.2.

Theorem 3.2. Let \( \lambda \in (0, 1) \). The following statements hold and are mutually equivalent:

(i) \( \frac{\text{Tr} e^{B}_\lambda}{\text{Tr} e^{A}_\lambda} > \exp_{\lambda} \left( \frac{\text{Tr}((B - A)(e^{A}_\lambda)^{1-\lambda})}{\text{Tr} e^{A}_\lambda} \right) \) for \( (A, B) \in \mathcal{J}^* \);

(ii) \( \ln_{\lambda} \text{Tr} e^{B}_\lambda > \text{Tr} A^{1-\lambda}B^\lambda + S_{\lambda}(A) \), whenever \((A, B) \in \mathcal{J}^*, \text{Tr} A = 1 \) and \( \sigma(A) \subset \mathbb{R}^+ \).
(iii) \( \frac{1}{n} D_{\lambda}(\text{Tr} A) I_n, (\text{Tr} B) I_n < D_{\lambda}(A, B) \), whenever \((A, B) \in \mathcal{J}^+, \sigma(A), \sigma(B) \subset \mathbb{R}^+ \);
(iv) \( D_{\lambda}(A, B) > 0 \), whenever \((A, B) \in \mathcal{J}^+, \text{Tr} A = \text{Tr} B \) and \( \sigma(A), \sigma(B) \subset \mathbb{R}^+ \).

Proof. We proved (i) in Corollary 3.3.

(i) \( \Rightarrow \) (ii): Under the hypothesis, \( \ln_2 A \) is a \( J \)-Hermitian and \( J \)-unitarily diagonalizable matrix with eigenvalues \( \left( \alpha^j_j \right)/n, j = 1, \ldots, n \), such that \( \left( \alpha^j_1 \right)/n \geq \cdots \geq \left( \alpha^j_n \right)/n \) belong to \( \sigma^+ (\ln_2 A), \left( \alpha^j_{r+1} \right)/n \geq \cdots \geq \left( \alpha^j_{n} \right)/n \) belong to \( \sigma^- (\ln_2 A) \) and satisfy \( \left( \alpha^j_1 \right)/n \geq \left( \alpha^j_{r+1} \right)/n \). Thus, replacing \( A \) by \( \ln_2 A \) in (i) and recalling that \( \text{Tr} A = 1 \), we get

\[
\text{Tr} e^{\beta} > \exp_{\lambda} \text{Tr} (B - \ln_2 A) A^{1-\lambda} = \exp_{\lambda} (\text{Tr} (A^{1-\lambda} B) + S_{\lambda}(A)),
\]

and so (ii) follows.

(ii) \( \Rightarrow \) (iii): Assume the eigenvalues of \( A, B \) are positive. It is easy to see that

\[
S_{\lambda} \left( \frac{A}{\text{Tr} A} \right) = \frac{S_{\lambda}(A)}{(\text{Tr} A)^{1-\lambda}} + \ln_2 \text{Tr} A.
\]

Thus, replacing in (ii) \( A \) and \( B \) by \( A/\text{Tr}(A) \) and \( \ln_2 B \), respectively, we get

\[
\ln_2 \text{Tr} B > \frac{\text{Tr}(A^{1-\lambda} \ln_2 B) + S_{\lambda}(A)}{(\text{Tr} A)^{1-\lambda}} + \ln_2 \text{Tr} A.
\]

Taking into account that \( \text{Tr} A > 0 \) we find

\[
-\frac{1}{n} D_{\lambda}(\text{Tr} A) I_n, (\text{Tr} B) I_n = (\text{Tr} A)^{1-\lambda} (\ln_2 \text{Tr} B - \ln_2 \text{Tr} A) \geq \text{Tr}(A^{1-\lambda} \ln_2 B) + S_{\lambda}(A) = -D_{\lambda}(A, B).
\]

(iii) \( \Rightarrow \) (iv): Considering \( \text{Tr} A = \text{Tr} B \) in (iii), then (iv) is trivially obtained.

(iv) \( \Rightarrow \) (i): Replacing \( A \) and \( B \) in (iv) by \( e^{\lambda} / \text{Tr} e^{\lambda} \) and \( e^{\beta} / \text{Tr} e^{\beta} \), respectively, we find (i). \( \square \)

4. Final remarks and open questions

The famous Golden–Thompson inequality [17,19,26,27] for Hermitian matrices \( H, K \) states that

\[
\text{Tr} (e^{H+K}) \leq \text{Tr} (e^H e^K).
\]

This inequality is a basic tool in quantum statistical mechanics and extensions to infinite dimension have extensive literature [3,24]. As the next example shows, the indefinite counterpart of this inequality is not valid. Let

\[
U = \begin{bmatrix} u & v \\ v & u \end{bmatrix},
\]

for \( u, v \in \mathbb{R} \) such that \( u^2 - v^2 = 1 \). Consider \( J = \text{diag}(1, -1) \) and

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = UA^\# U^* = \begin{bmatrix} 1 + u^2 & -uv \\ uv & 1 - v^2 \end{bmatrix}.
\]

Thus,

\[
\log A = \log 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \log B = U \log UA^\# = \log 2 \begin{bmatrix} u^2 & -uv \\ uv & -v^2 \end{bmatrix},
\]
being the spectrum of $\log A + \log B$ given by $\log 2\{1 + \sqrt{1 + v^2}, 1 - \sqrt{1 + v^2}\}$ and so
$$\text{Tr} \left( e^{\log A + \log B} \right) = 2 \left( 2^\sqrt{1 + v^2} + 2^{-\sqrt{1 + v^2}} \right), \quad \text{Tr} (AB) = 5 + v^2.$$  

For the $J$-Hermitian matrices $H = \log A$ and $K = \log B$, we have $\text{Tr}(e^{H+K}) > \text{Tr}(e^H)$ if $v \neq 0$ and so the Golden–Thompson inequality is not valid. This remarkable inequality relies on the following fact: if $H, K$ are Hermitian, then $i[H, K]$ is also Hermitian and unitarily diagonalizable, the matrix $-\{H, K\}^2$ is positive definite and so $-\text{Tr}([H, K]^2) \geq 0$ (cf. [17,27]). In the indefinite case, if $H, K$ are $J$-Hermitian, so it is $i[H, K]$ as well as $-\{H, K\}^2$. However, in the above example $i[H, K]$ is not $J$-unitarily diagonalizable and moreover $-\text{Tr}([H, K]^2) \leq 0$.

The following question arises:
(I) Under what conditions does $\text{Tr}(e^{H+K}) > \text{Tr}(e^H)$ hold for $H$ and $K$ $J$-Hermitian matrices?

We conjecture that the opposite Golden–Thompson inequality holds for the $2 \times 2$ case of $J = \text{diag}(1, -1)$-Hermitian matrices if their eigenvalues are positive and do not interlace. Our conjecture is based on numerical experiments.

If $A, B$ are positive semi-definite Hermitian matrices, then
$$\text{Tr} \left( A(\log A + \log B) \right) \leq \frac{1}{p} \text{Tr} \left( A \log \left( A^{p/2} B^p A^{p/2} \right) \right)$$  

for any $p > 0$ and the right-hand side of (6) converges decreasingly to the left-hand side as $p \downarrow 0$. This result was firstly proved by Hiai and Petz [18] and it was later strengthened by Ando and Hiai [1]. If $J = \text{diag}(1, -1)$ and $A, B \in M_2$ are $J$-Hermitian matrices in $\mathcal{J}$ such that $A, B$ and $A^{p/2} B^p A^{p/2}$, $p > 0$, have positive eigenvalues, then it can be proved that
$$\text{Tr} \left( A(\log A + \log B) \right) \geq \frac{1}{p} \text{Tr} \left( A \log \left( A^{p/2} B^p A^{p/2} \right) \right)$$  

and the right-hand side of (7) converges increasingly to the left-hand side as $p \downarrow 0$. It is remarkable that inequality (7) for $2 \times 2$ $J$-Hermitian matrices with positive eigenvalues is opposite to (6) for positive semi-definite matrices.

The following question takes place:
(II) Under what conditions on the $J$-Hermitian matrices $A, B \in M_n$ does (7) hold for any $p > 0$, $n \geq 3$?

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References

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