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## Tiling space and slabs with acute tetrahedra

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### Abstract

We show it is possible to tile three-dimensional space using only tetrahedra with acute dihedral angles. We present several constructions to achieve this, including one in which all dihedral angles are less than  $74.21^\circ$ , and another which tiles a slab in space.

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### 1. Problem definition

Triangulations of two and three-dimensional domains find numerous applications in scientific computing, computer graphics, solid modeling and medical imaging. Most of these applications impose a quality constraint on the elements of the triangulation. Among the most popular quality criteria for elements [5] are the aspect ratio (circumradius over inradius), the minimum dihedral angle, and the radius-edge ratio (circumradius over shortest edge). However, many other quality criteria have been considered, including maximum dihedral angle. Bern et al. for instance, studied nonobtuse triangulations [3,6], where domains are meshed with simplices having no obtuse angles. In this paper, we consider a slightly stronger quality constraint: all the dihedral angles in the mesh are forced to be acute (strictly less

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than  $90^\circ$ ). Although acuteness seems only slightly stronger than nonobtuseness, this problem turns out to be considerably harder than the nonobtuse triangulation problem, as we observe below in Section 3.

**Definition.** An angle is *acute* if it is strictly less than a right angle ( $\frac{\pi}{2} = 90^\circ$ ). A simplex is *acute* if all its (interior) dihedral angles are acute. A triangulation is *acute* if all of its simplices are acute.

**Problem 1.** Given a domain  $\Omega$ , compute an acute triangulation of  $\Omega$ .

There has been extensive work on the two-dimensional version of this problem, for the special cases where the domain  $\Omega$  is a triangle, square, quadrilateral, or a finite point set [6,11,26,33,34,36]. We review those results in Section 3. In three-dimensional space, however, although the problem was posed as early as 1991 [40] almost nothing has been known about acute triangulations before now. To the best of our knowledge, even the following relaxed form of the problem, where the input domain is the entire space, had not been addressed in the literature.

**Problem 2.** Is it possible to tile three-dimensional Euclidean space using acute tetrahedra?

We present an affirmative answer to this question, by several different constructions. The two-dimensional analog of this problem has a trivial positive answer: congruent copies of any single triangle will tile the plane. However, this idea does not extend to three dimensions, as the regular tetrahedron (for instance) cannot tile space. All tetrahedra known to tile space have right angles, as further discussed in Section 3.

We started this research on acute triangulations because of a method developed for space-time meshing which required an acute base mesh. This and our other motivations are discussed in Section 2. Section 3 surveys previous research in acute triangulations. Section 4 investigates what acuteness means for a three-dimensional simplex and gives a comparison of acute and Delaunay triangulations. Constructions tiling three-dimensional space, and hence solving Problem 2, are given in Section 5. The paper concludes in Section 6 with a quality assessment of these constructions and directions for future research.

## 2. Motivation

We were originally motivated to study acute triangulations by the space-time meshing algorithm of Üngör and Sheffer [46]. This *tent-pitcher* algorithm was designed to discretize space-time domains into meshes that obey a certain cone constraint, which requires all faces in the mesh to have smaller slopes than the cones that define the domain of influence imposed by the numerical (engineering) problem. (For instance, we might require simply that all faces make at most a  $45^\circ$  angle with the horizontal.) Because there is then a well-defined direction of information flow across element boundaries, such meshes enable the use of very efficient element-by-element methods (including space-time discontinuous Galerkin methods) to solve a wide variety of numerical problems, for instance in elastodynamics. The tent-pitching algorithm starts with a space mesh of the two- or three-dimensional input domain and constructs the space-time mesh using an advancing front approach. The algorithm is known to generate a valid space-time mesh if the initial space mesh is an acute triangulation [46], but may fail if there is an obtuse angle or even a right angle.

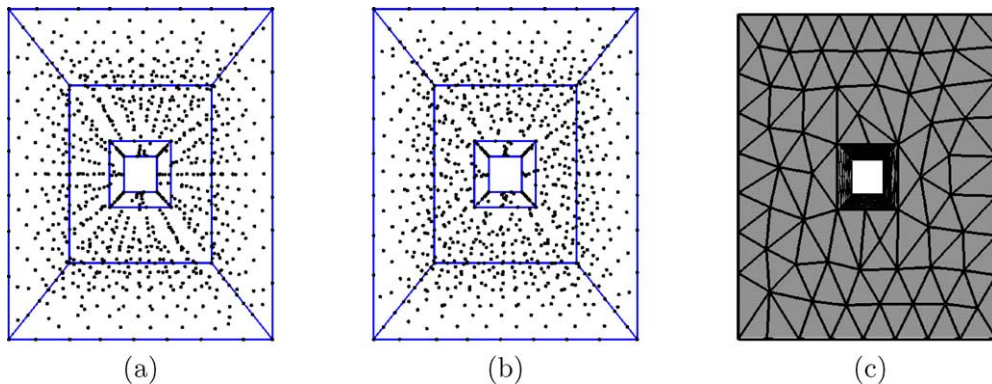


Fig. 1. An almost regular triangulation of a cube with a hole (A. Fuchs); (a) the point set of a body-centered cubic (BCC) lattice overlaid with the domain; (b) the adjusted point set; (c) the conforming Delaunay triangulation.

Later, Erickson et al. [17] proposed an improved version of the tent-pitching algorithm. By removing the acute angle requirement, the new space-time algorithm works over arbitrary spatial domains. However, there is a loss of efficiency (more elements are required) whenever there is a nonacute angle.

Thus the study of Problem 1 is motivated by current space-time meshing algorithms. But even a solution to Problem 2 is useful, since it leads to a better understanding of the acute triangulation problem for more general input domains, and it also finds some direct applications in mesh generation.

Spatial tilings of high quality have been used for designing meshing algorithms: Fuchs [24], Field and Smith [18,20], Naylor [38] and Molino et al. [37] built meshes by overlaying standard tilings onto the given polyhedral domain. They used tilings known at the time, such as Sommerville constructions (Fig. 5) and subdivided cubes (Fig. 7), which we discuss in Section 3.2. Their approach has three steps, illustrated in Fig. 1:

- (a) Overlay the chosen tiling with the given domain. The main challenge in this step is finding the right scaling, location and orientation for the tiling so that it matches the domain boundary as closely as possible.
- (b) Adjust the points to get a better fit. For this purpose, one of the standard smoothing techniques [10,19,23] can be used. Alternatively, Fuchs [24] suggested minimizing a function which penalizes configurations that produce irregular vertices.
- (c) Construct the mesh by computing the conforming Delaunay triangulation of the adjusted point set and the domain boundary.

Fuchs [24] reports good performance of his experiments when he used the second Sommerville construction (Fig. 5(b)) as the space tiling. (This tiling is the Delaunay triangulation of the body-centered cubic lattice.) The dihedral angles of his mesh in Fig. 1(c) range between  $7.6^\circ$  and  $168.2^\circ$ . However, most of the angles (here and also in his meshes of similar geometric domains) cluster around  $60^\circ$  and  $90^\circ$ , which are exactly the dihedral angles of the BCC tetrahedron in the input tiling. Some of the constructions we propose in Section 5 are considerably better in terms of dihedral angles and also other quality measures. Our new constructions can find immediate use to improve the results of this previous research [18,20,24,37,38] on tiling-based meshing.

### 3. Background

#### 3.1. Acute and nonobtuse triangulations

There has been considerable research [1,3,5,7,15] on the nonobtuse triangulation problem, which imposes a slightly weaker constraint than the acute triangulation problem. Angles in a nonobtuse triangulation are less than or equal to  $90^\circ$ . Bern et al. [3] showed that any  $d$ -dimensional point set of size  $n$  can be triangulated with  $O(n^{\lceil d/2 \rceil})$  simplices, none of which has any obtuse dihedral angles. However, they also proved that a similar bound depending only on  $n$  and  $d$ , and not on the geometry of the input points, is not possible if all angles are required to be at most  $90^\circ - \varepsilon$ . This indicates that the acute triangulation problem is much more challenging than nonobtuse triangulation. To appreciate this difference, consider the two problems for a square domain in two dimensions. A single diagonal cuts a square into two nonobtuse triangles, as in Fig. 2(a). Finding an acute triangulation, however, can be a challenging recreational math problem.

Lindgren [33] showed that at least eight triangles, as in Fig. 2(b), are needed. Later, Cassidy and Lord [11] showed that for any  $n \geq 10$  (but not for  $n = 9$ ) there is an acute triangulation with exactly  $n$  triangles. Fig. 2(c) shows the solution with ten triangles. We can use the maximum angle in a triangulation as a quality measure. The triangulations in Fig. 2(b), (c) can be realized with maximum angles about  $85^\circ$  and  $80.3^\circ$ , respectively. Eppstein [16] improved this angle to  $72^\circ$  using fourteen acute triangles, as shown in Fig. 2(d). Using Euler's formula, Eppstein also showed that any acute triangulation of a square must have an interior vertex of valence five, implying that  $72^\circ$  is the best possible. It is unknown whether there is a triangulation achieving this with fewer than fourteen triangles.

The acute triangulation problem has been studied for other simple polygons as well. Gardner [26] asked the question for triangles. Manheimer proved that seven acute triangles are necessary and sufficient to subdivide a nonobtuse triangle [36]. Recently, Maehara [34] showed that an arbitrary quadrilateral can be tiled by 10 (but perhaps not by any fewer) acute triangles. Gerver [27] considered the problem of finding triangulations with a stricter upper bound (between  $72^\circ$  and  $60^\circ$ ) on their angles, and gave necessary conditions for a polygonal domain to have such a triangulation. If we restrict ourselves to two-dimensional point sets, a solution to the acute triangulation problem is given by Bern et al. [6]. Their approach starts with a quadtree, and replaces the squares by tiles with protrusions and indentations. Fig. 3 shows sample tiles together with an acute triangulation resulting from their algorithm.

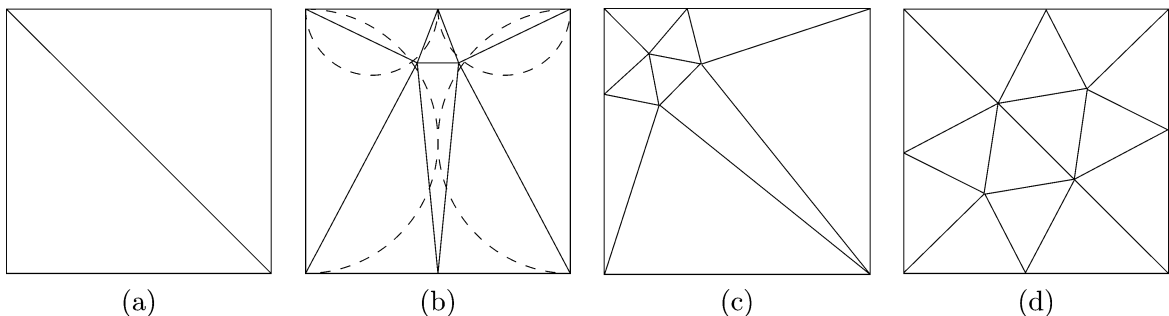


Fig. 2. (a) Nonobtuse triangulation of a square; (b) a square meshed with eight acute triangles; (c) a square meshed with ten acute triangles; (d) triangulation where maximum angle is  $72^\circ$ .

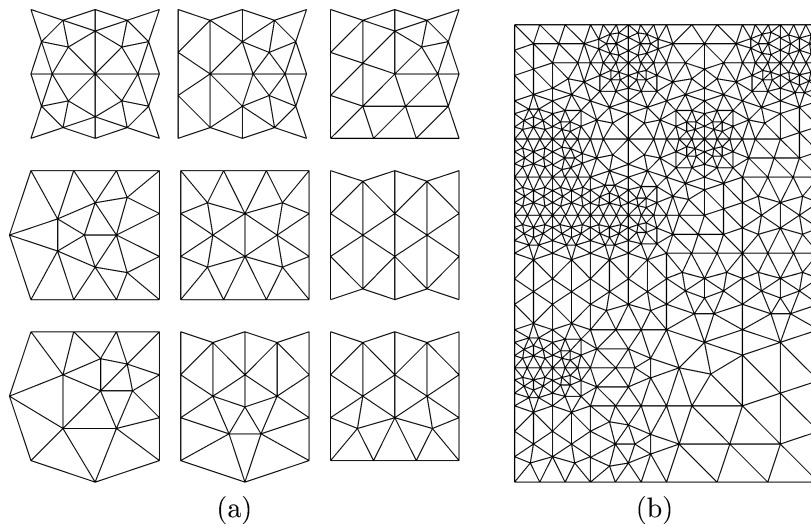


Fig. 3. Acute triangulation gadgets and their use on a point set (M. Bern and D. Eppstein [6]).

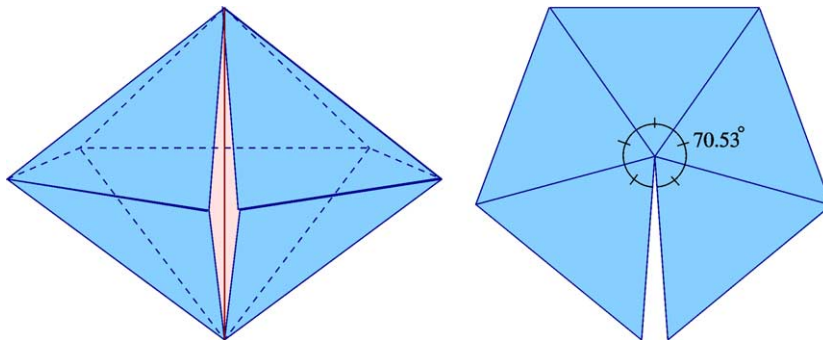


Fig. 4. The regular tetrahedron does not tile space.

Krotov and Křížek [31] studied refinement methods to subdivide a nonobtuse tetrahedral partition into another finer one. Unfortunately, they called the resulting triangulations *acute type* instead of *nonobtuse* even though  $90^\circ$  dihedral angles were ubiquitous in them. Another related work is by Hangan, Itoh and Zamfirescu [29,30] who studied acute surface triangulations of certain special shapes such as a cube, sphere and icosahedron.

### 3.2. Acute and nonobtuse tilings

Aristotle claimed that regular tetrahedra could meet five-to-an-edge to tile space, and this claim was repeated over the centuries (see [39]). This of course is false, because the dihedral angle of a regular tetrahedron is not  $72^\circ$  but  $\arccos \frac{1}{3} \approx 70.53^\circ$ . Fig. 4 shows the small gap left when five tetrahedra are placed around an edge.

There are, however, tetrahedral shapes which can tile space. Sommerville [42] found four such tetrahedra, shown in Fig. 5. Four decades later, Davies [13] and Baumgartner [2] independently

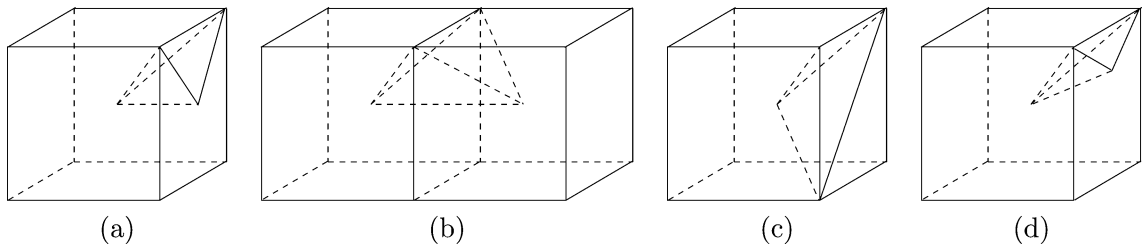


Fig. 5. Sommerville tetrahedra. The first tetrahedron is half of the third. The fourth tetrahedron is one fourth of the second.

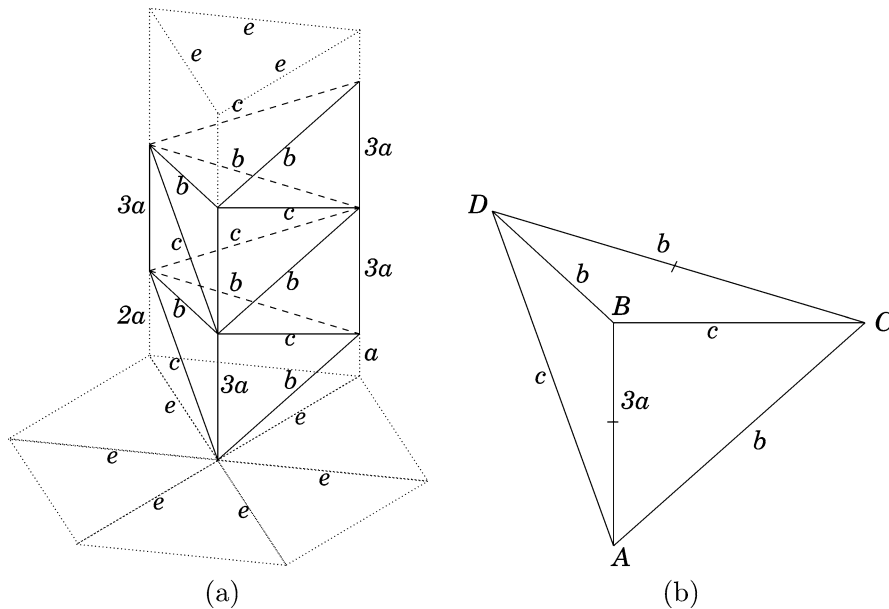


Fig. 6. Family of space tilings. The edges with length  $c$  have valence four. Hence some of the dihedral angles along these edges must be nonacute.

rediscovered three of the Sommerville tetrahedra; Baumgartner also found a new example. Goldberg [28] surveyed the list of all known space-tiling tetrahedra, and found three infinite families, including the one shown in Fig. 6(a).

The construction of this family is based on a tiling of the plane by equilateral triangles of side-length  $e$ . The infinite prism over each triangle is filled with tetrahedra whose sides are  $3a, b, b, b, c, c$ , where  $b^2 = a^2 + e^2$  and  $c^2 = 4a^2 + e^2$ , as shown in Fig. 6. Since the ratio  $a/e$  is arbitrary, there is a continuous family of tetrahedral space-fillers of this type. Goldberg’s two other families can be derived simply by cutting these tetrahedra into two congruent pieces, either by the triangle  $CDM_{AB}$ , or by the triangle  $ABM_{CD}$ , where  $M_{AB}$  and  $M_{CD}$  are the midpoints of the segments  $\overline{AB}$  and  $\overline{CD}$ , respectively. Notice that the second type of cut (through  $ABM_{CD}$ ) results in nonconforming triangulations. Whether the list of space-tiling tetrahedra is complete or not is still an open problem [14,39]. None of the known space-tiling tetrahedra is acute (although several are nonobtuse). In fact, the tilings all contain edges of valence four. (In Goldberg’s family, these are the edges of length  $c$ .)

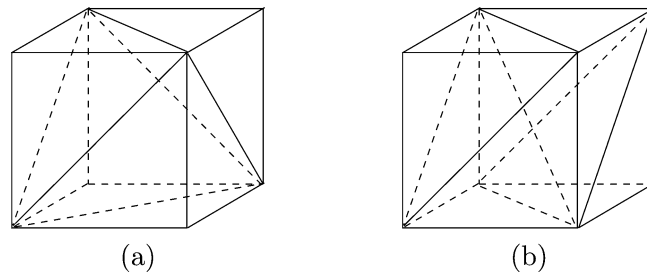


Fig. 7. Cube subdivided into (a) 5 or (b) 6 tetrahedra.

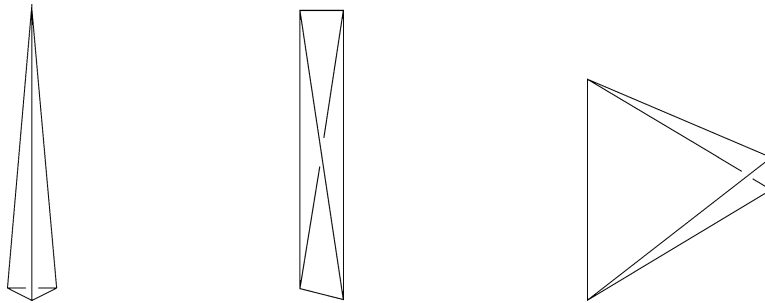


Fig. 8. Low-quality tetrahedra that can be acute: spire, splinter and wedge.

Since it seems likely that there is no tiling of space by congruent acute tetrahedra, we will now consider tilings with several shapes of tetrahedra. There are now many more ways to fill space, for instance by subdividing the cube into five or six tetrahedra as in Fig. 7. These tilings also, of course, have  $90^\circ$  dihedral angles, and so are nonobtuse but not acute.

There are many results (like minmax and maxmin angle results) known about optimality of Delaunay triangulations in the plane. But these do not extend to three dimensions, and little is yet known about optimum triangulations in space. Thus it is not surprising that the constructions of acute triangulations do not easily extend from two to three dimensions. It is remarkable that acute triangulations of space can be constructed at all.

#### 4. Acute tetrahedra

An acute tetrahedron does not necessarily have high quality in terms of either aspect ratio or radius-edge ratio. Low-quality tetrahedra have been classified into nine types [12], and three of these (the spire, splinter and wedge in Fig. 8) can have all their dihedral angles acute. However, fortuitously, the tetrahedra in our constructions are mostly quite close to regular, and are high-quality for use in mesh generation and numerical simulations.

##### 4.1. Acuteness test

By definition, a tetrahedron is acute if each of its six dihedral angles is less than  $90^\circ$ .

**Lemma 1.** Consider an edge  $ab$  of a tetrahedron  $abcd$ , and let  $\Pi$  denote projection to a plane normal to  $ab$ . The dihedral angle along  $ab$  is acute if and only if  $\Pi(a) = \Pi(b)$  lies strictly outside the circle with diameter  $\Pi(c)\Pi(d)$ .

**Proof.** The dihedral angle along  $ab$  is by definition the angle  $\angle \Pi(c)\Pi(a)\Pi(d)$ ; the lemma follows from standard plane geometry (Thales’ theorem, see Fig. 9(c)).  $\square$

This lemma can be applied to each of the edges of a tetrahedron. We now examine some alternate criteria for acuteness.

**Lemma 2.** A tetrahedron is acute if and only if the orthogonal projection of each vertex onto the plane of the opposite facet lies strictly inside that facet. An acute tetrahedron has acute facets, but not every tetrahedron with acute facets is itself acute.

**Proof.** Suppose the projection  $p$  of a vertex  $d$  is not inside the opposite triangle  $\Delta abc$ . Then  $p$  lies in one of the six other regions of the plane determined by the edges of  $\Delta abc$ , as depicted in Fig. 9(a). Then, the dihedral angle along Any extended edge of  $\Delta abc$  that separates this region from the triangle must then be an edge whose dihedral angle in the original tetrahedron is nonacute. (If the projection is on an extended edge, then the corresponding dihedral angle is exactly  $90^\circ$ .) Conversely, if the dihedral angle along edge  $ab$  is nonacute, then  $d$  projects outside  $\Delta abc$ , as in Fig. 9(b).

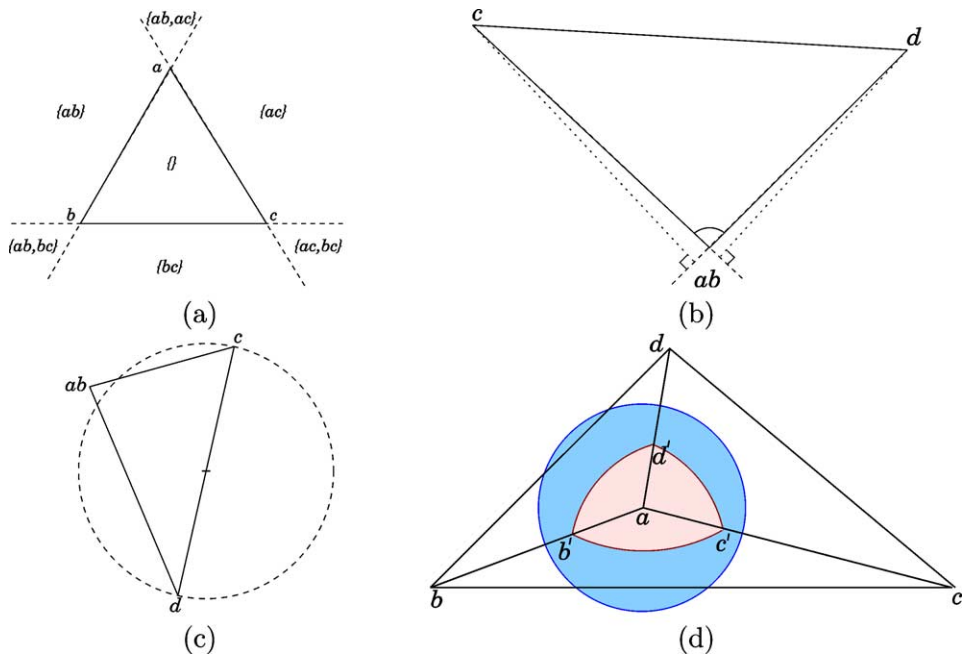


Fig. 9. Acuteness tests: (a) if vertex  $d$  projects outside  $\Delta abc$ , the label shows which edges have obtuse dihedral angles; (b) if the dihedral angle along  $ab$  is obtuse, both  $c$  and  $d$  project outside their opposite triangles; (c) Thales’ theorem says that the angle at  $ab$  is acute exactly when it lies outside the circle with diameter  $cd$ ; (d) if the vertex  $d$  projects inside the triangle  $abc$  then the face angle  $\angle bac$  is smaller than the dihedral angle on  $ad$ .



We prove the second statement in contrapositive form, while noting that nonacute sliver tetrahedra can have acute face angles. Suppose tetrahedron  $abcd$  has a nonacute face angle  $\angle bac$ ; we will show the tetrahedron is nonacute. If the projection of  $d$  onto  $\triangle abc$  is not in the interior we are done by the first part of the lemma. Otherwise, we claim the dihedral angle along  $ad$  is larger than  $\angle bac$  and thus is nonacute. To check the claim, remember the spherical dual law of cosines (see [45]):

$$\cos d' = -\cos b' \cos c' + \sin b' \sin c' \cos \angle bac,$$

where  $b'$ ,  $c'$  and  $d'$  are the dihedral angles along the edges  $ab$ ,  $ac$  and  $ad$ , respectively. (See Fig. 9(d).) Assuming  $b', c' < \frac{\pi}{2}$ , this gives  $\cos d' < \cos \angle bac$  as desired.  $\square$

#### 4.2. Acuteness of Delaunay triangulations

Given a set of vertices, the Delaunay triangulation is optimal in many ways. However, a Delaunay triangulation in any dimension can have obtuse angles. In this section, we investigate the converse, whether an acute triangulation is necessarily the Delaunay triangulation for its vertices. The answer is positive in the plane, but negative in three-space. The following appears to be folklore; it was asserted without proof by Bern et al. [6].

**Lemma 3.** *Any acute two-dimensional triangulation  $\mathcal{T}$  is Delaunay.*

**Proof.** Since  $\mathcal{T}$  is acute, the diametral circle of each edge is empty of other vertices. By definition, this means the edge is in the Gabriel graph [25] of the vertex set, which is a subgraph of the Delaunay triangulation. But since the edges of  $\mathcal{T}$  form a triangulation, it must be the entire Delaunay triangulation. See also Fig. 10(a).  $\square$

**Corollary.** *If an acute triangulation of a two-dimensional vertex set exists then it is unique.*

**Lemma 4.** *There is an acute triangulation  $\mathcal{T}$  in three dimensions which is not Delaunay.*

**Proof.** Consider a “cube corner” tetrahedron, and glue it to a copy of itself across the equilateral face. Then move the two corner vertices away from each other a tiny amount to make the tetrahedra

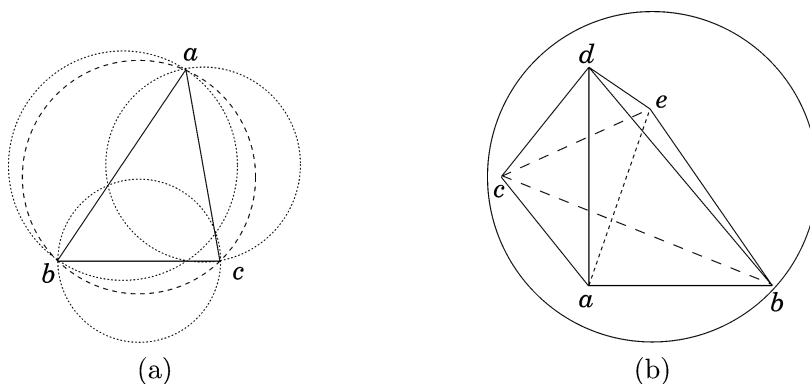


Fig. 10. (a) Acute triangles in the plane are Delaunay. An alternate proof uses the fact that the circumcircle is contained in the union of the three diametral circles around the edges. (b) An example of an acute triangulation in space which is not Delaunay.

acute. In coordinates, take a suitable small  $\varepsilon > 0$ , and let  $a = (-\varepsilon, -\varepsilon, -\varepsilon)$ ,  $b = (1, 0, 0)$ ,  $c = (0, 1, 0)$ ,  $d = (0, 0, 1)$  and  $e = (2/3 + \varepsilon, 2/3 + \varepsilon, 2/3 + \varepsilon)$ . The two tetrahedra  $abcd$  and  $bcde$  are acute, but the Delaunay triangulation of these five points consists of three tetrahedra:  $abce$ ,  $acde$  and  $abde$ , as in Fig. 10(b), because  $e$  is inside the circumsphere of  $abcd$ . Note that the three Delaunay tetrahedra are obtuse, having  $120^\circ$  dihedral angles along edge  $ae$ . The acute triangulation we started with is obtained by performing a 3-to-2 flip on the Delaunay triangulation.  $\square$

In two dimensions, a triangle is acute if and only if its circumcenter lies inside the triangle. Given an acute (Delaunay) triangulation, the dual Voronoi tessellation thus provides an orthogonal dual mesh whose nodes are inside the corresponding triangles. For this reason, Bossavit has suggested [8,9] that acute triangulations are useful in computational electromagnetics. In three dimensions, an acute tetrahedron can fail to contain its own circumcenter (as with the example in the lemma above), and conversely a tetrahedron that contains its circumcenter can fail to be acute (as with a symmetric sliver). Many of the triangulations we construct below are built from tetrahedra containing their circumcenters, but we have not investigated this problem in detail.

## 5. Constructions for acute tilings

### 5.1. TCP triangulations

Our first set of acute triangulations basically come from the crystallography literature. Here we survey some known results and refer the reader to [43,44] for more details. Chemists studying alloys of two transition metals have often found that since the two types of atoms are similar (but slightly different) in size, the Delaunay triangulation of their positions is built of nearly regular tetrahedra. These TCP (tetrahedrally close packed) structures were first described by Frank and Kasper [21,22] and have been studied extensively by the Shoemakers [41] among others.

A combinatorial definition of the TCP class was given by Sullivan [43]: A triangulation is called TCP if every edge has valence 5 or 6, and no triangle has two 6-valent edges. This definition includes all the chemically known TCP structures, but also allows some new structures [44] not yet seen in nature.

It is not hard to check that the definition allows exactly four types of vertex star in a TCP triangulation. Dually, the Voronoi cell around any vertex has one of the four combinatorial types shown in Fig. 11: these are the polyhedra with pentagonal and hexagonal faces but no adjacent hexagons. (It is interesting that these dual structures are seen in some other crystal structures: in some zeolites, silicon dioxide outlines the Voronoi edges, while in clathrates, water cages along the Voronoi skeleton trap large gas molecules.)

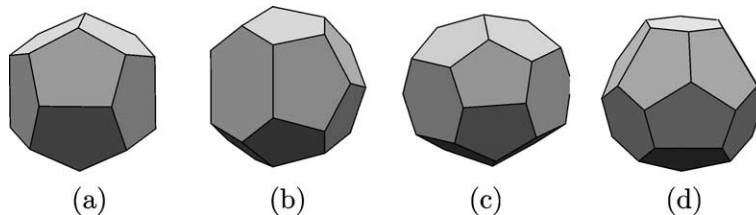


Fig. 11. Foam cells with pentagonal and hexagonal faces.

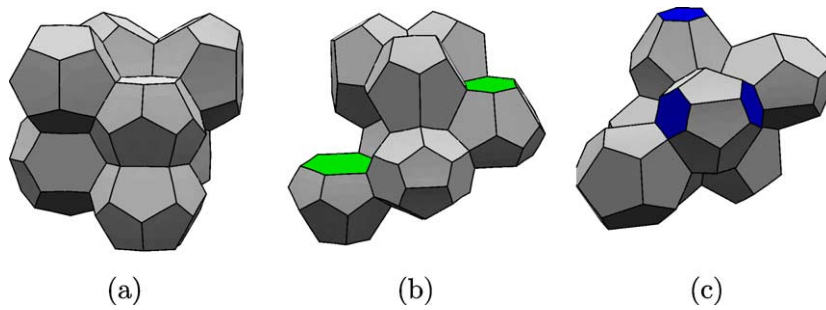


Fig. 12. The Voronoi cells for the three basic TCP structures, A15, Z and C15.

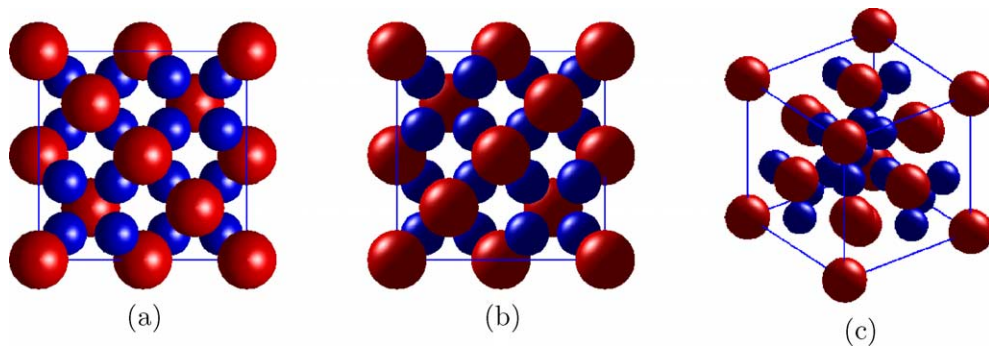


Fig. 13. The vertices of the C15 triangulation are at the centers of these balls.

All known TCP structures can be viewed as convex combinations of the three basic ones (called A15, Z and C15) shown in Fig. 12. There are many ways to understand these structures [43]. To construct A15, we can start with a BCC lattice. Its Delaunay triangulation is the Sommerville tiling shown in Fig. 5(b); since the edges have even valence the tetrahedra can be colored alternately black and white. If we take the BCC lattice together with the circumcenters of all black tetrahedra, we have the vertices of A15: their Delaunay tetrahedra are all now nearly regular. Similarly, the C15 structure arises from the diamond lattice by adding selected circumcenters, and the Z structure can be obtained similarly starting with hexagonal prisms.

The C15 structure (also known as the cubic Friauf–Laves phase) is shown in Fig. 13, where the red spheres are centered on a diamond lattice (FCC together with a certain translate) and the blue spheres are at selected circumcenters.

In any triangulation of space, the average dihedral angle multiplied by the average edge valence is exactly  $360^\circ$ . If a tiling could be made of regular tetrahedra, the average edge valence would thus be  $n_0 := 360^\circ / \arccos(\frac{1}{3}) \approx 5.1043$ . But by symmetry, the regular tetrahedron is a critical point for average dihedral angle, so any tiling made of nearly regular tetrahedra should have average valence quite close to  $n_0$ . Indeed, all known TCP structures have average valence between  $5\frac{1}{10}$  and  $\leq 5\frac{1}{9}$ , the values for C15 and A15.

Sullivan [44] has formalized a construction suggested by Frank and Kaspar for mixing the basic TCP structures. Start with any tiling of the plane by copies of an equilateral triangle and a square, like one of the four shown in Fig. 14. Suppose the side length of the square and triangle is 4. Mark black and

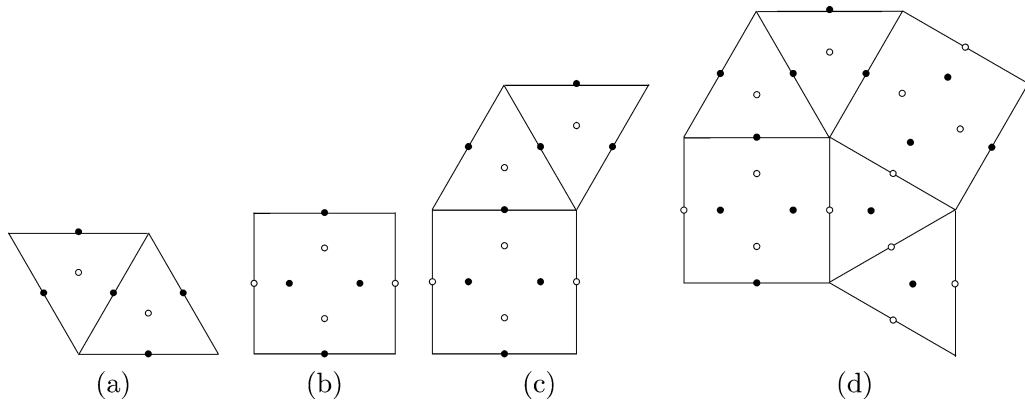


Fig. 14. Four simple periodic square/triangle tilings of the plane which lead to the TCP structures named (a) Z; (b) A15; (c)  $\sigma$ ; and (d) H.

white dots on the tilings as shown. (The dots are at edge midpoints, at triangle centers, and at distance 1 from the sides of the squares.) Then the vertices of the corresponding TCP structure are at heights  $4k - 1$  above the black dots, at heights  $4k + 1$  above the white dots, and at heights  $4k$  and  $4k + 2$  above the vertices of the square/triangle tiling. (Here  $k$  ranges over all integers.)

Again, in each case the TCP triangulation is simply the Delaunay triangulation of this periodic point set. See Fig. 16. The triangulations constructed in this way are all combinations of the A15 and Z structures. A more complicated variant of this construction [44] builds combinations of the Z and C15 structures, again starting from an arbitrary square/triangle tiling.

Especially in the mixed structures like  $\sigma$  and H, the particular geometry we have described here may differ slightly from that found in the actual crystals with the same combinatorics. Presumably, these slight adjustments do not affect the shapes of the tetrahedra very much. The quality figures we present below are measured using the exact geometry we have just described.

Sullivan's original interest in these structures was for the mathematical study of foam geometry. The Kelvin problem asks for the most efficient partition of space into unit-volume cells, that is, for the partition with least surface area. Lord Kelvin's suggested solution was a slightly relaxed form of the BCC Voronoi cells (truncated octahedra). But in 1994, Weaire and Phelan [47] discovered that a relaxed form of the Voronoi cells for the TCP structure A15 is more efficient than the Kelvin foam [32].

It is perhaps not surprising that TCP structures are related to foams: Plateau's rules for singularities in soap films minimizing their surface area imply that a foam is combinatorially dual to some triangulation, preferably one with nearly regular tetrahedra. It is an interesting question whether any triangulation meeting the combinatorial definition of TCP can be built with tetrahedra close to regular, but certainly for the known TCP structures this seems always to be the case. Thus our acute triangulations arising from this construction have high quality by almost all measures.

## 5.2. Icosahedral construction of the Z structure

An alternate construction for the TCP Z structure is inspired by the work of Field [18]. His tilings involved right-angled tetrahedra, but by selectively adjusting the point set, we obtain a tiling with only acute tetrahedra. A regular icosahedron can be subdivided into 20 acute (and nearly regular) tetrahedra

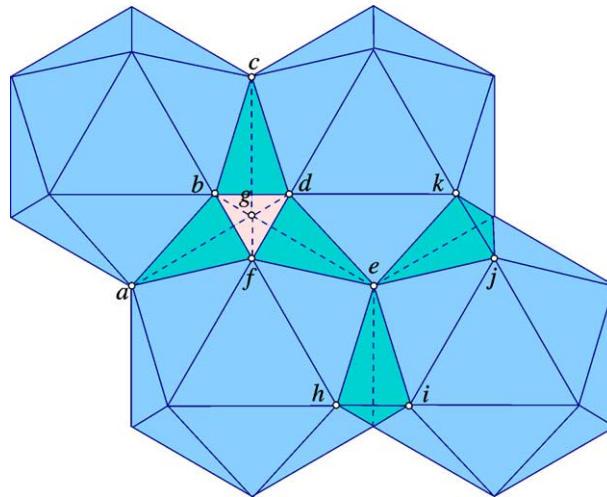


Fig. 15. Icosahedron construction.

simply by coning to the center point. We place icosahedra in a hexagonal lattice in the plane, each in the same orientation, touching edge to edge, as in Fig. 15.

This layer then gets repeated vertically, with each icosahedron sharing a horizontal face with the ones just above and below it. Our point set is then the vertices and centers of all the icosahedra. Its Delaunay triangulation, shown in Fig. 16(d), is combinatorially the TCP Z structure, but with slightly different geometry than that constructed before. The horizontal faces (seen head-on as equilateral triangles in Fig. 15) are shared by two icosahedra. Each other face separates an icosahedron from one of the four types of Delaunay tetrahedra that fill the gaps. There are two types of gaps. The deeper gaps are defined by the points  $a, b, c, d, e, f, g$  and  $g'$  (the mirror image of  $g$  with respect to the plane through  $b, d$  and  $f$ ); they are filled with two types of tetrahedra, e.g.,  $bdfg$  and  $abfg$ . The shallower gaps are defined by the points  $d, f, h, i, j, k, e$  and  $e'$  (the mirror image of  $e$  with respect to the plane through  $f, k$  and  $i$ ); they are filled with two types of tetrahedra, e.g.,  $ee'df$  and  $ee'fh$ .

Notice that in our triangulation we use only the vertices of the icosahedra. This differs from Field's construction [18] which introduces additional points to triangulate the shallower gaps.

### 5.3. An acute triangulation of a single slab

The acute triangulations we have described so far, though periodic, do not have any planar boundaries within them. Here we describe an acute triangulation of a slab (which can of course be repeated to fill all of space). We view this as partial progress towards the problem of triangulating an arbitrary domain, although it seems much harder to find an acute triangulation of a cube or even an infinite square prism. We triangulate the bottom half of the slab in the following eight steps. Let  $h$  be the height of the slab and  $\gamma = h/14.2$ .

1. Start with a grid of equilateral triangles of side length  $6\gamma$  on the base plane, as in Fig. 17(a).
2. Place a nearly regular tetrahedron (with height  $4\gamma$ ) over each triangle, as in Fig. 17(b).

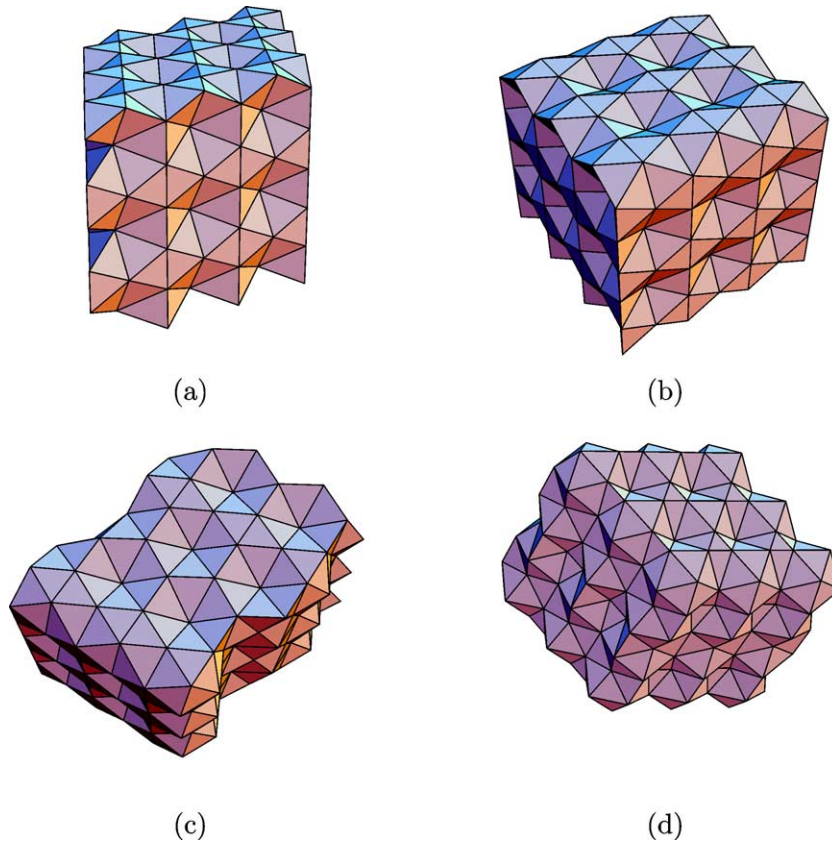


Fig. 16. Acute triangulations filling space. (a) The TCP structure Z (from a triangle tiling). (b) The TCP structure A15 (from a square tiling). (c) The TCP structure  $\sigma$ , a mixture of A15 and Z. (d) Icosahedron construction of Fig. 15.

3. Add a tetrahedron in the gap between each pair of adjacent tetrahedra, as in Fig. 17(c). The resulting surface has deep hexagonal dimples at the original vertices in the base plane.
4. Add six tetrahedra in each dimple, each with one vertex on the starting plane, two vertices at height  $4\gamma$ , and one new vertex at height  $4.6\gamma$  over the starting vertex, as in Fig. 17(d). Now we have a surface with shallow hexagonal bumps.
5. Place a vertex at height at  $7.1\gamma$  over the midpoint of the edge between each pair of adjacent bumps. Each such vertex and edge form a vertical triangle; let this separate two new tetrahedra whose fourth vertices are the two nearby bump vertices, as in Fig. 17(e). The surface is now covered by tall diamond-shaped bumps.
6. Place a tetrahedron between each adjacent pair of bumps, as in Fig. 17(f). We now have an alternating grid of medium-depth six-sided holes (over each of the shallow hexagonal bumps) and deep tetrahedral holes (over the points where three of the shallow hexagonal bumps meet).
7. Fill each tetrahedral hole, to form a surface alternating between six-sided holes and flat triangles, as in Fig. 17(g).

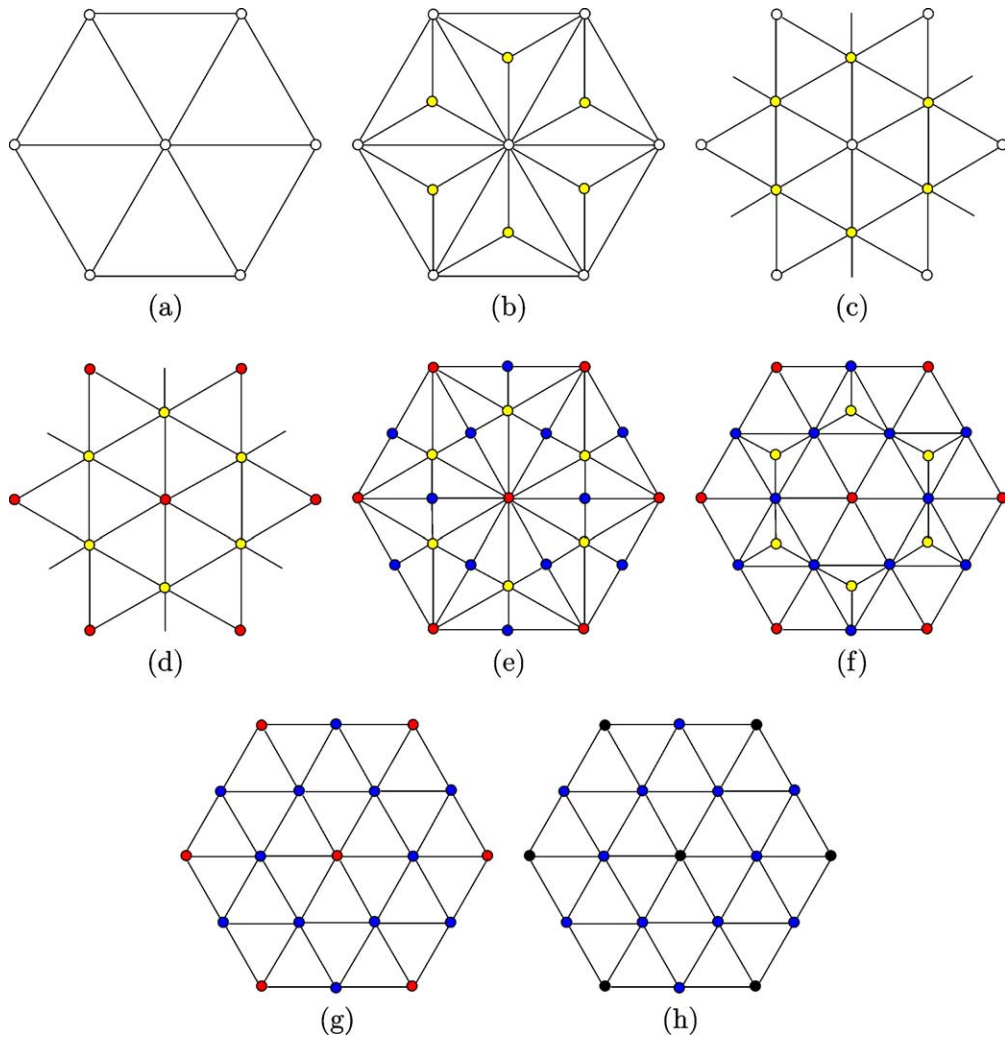


Fig. 17. Eight steps in filling a slab with acute tetrahedra. The nodes in the base plane are colored white; successive layers above that plane are then colored yellow, red, blue and black, in order.

8. Place six tetrahedra into each medium-height hexagonal hole to turn it into a medium-height hexagonal bump, as in Fig. 17(h). In order to make the bumps equal to the holes, the height of the new vertices is chosen as  $[2(7.1 - 4.6) + 4.6]\gamma = 9.6\gamma$ .

To complete the triangulation of the slab, we now repeat the first seven steps in reverse order. Any of the constructions given in this section serves to prove our main result:

**Theorem 1.** *It is possible to tile three-dimensional Euclidean space with acute tetrahedra.*

## 6. Conclusions

### 6.1. Evaluation of the constructions

We report in Table 1 the quality of the tetrahedra used in our constructions in comparison to the Sommerville tetrahedra, the cube subdivision tetrahedra and the regular tetrahedron. For each construction we list the radius-edge ratio of the best and the worst quality tetrahedra as well as the extreme dihedral angles. The radius-edge ratio of a tetrahedron is the ratio of its circumradius to the length of its shortest edge. Tetrahedra with smaller radius-edge ratio are preferable in many of the applications discussed in Section 1. Also, the smaller the largest dihedral angle and the larger the smallest dihedral angle, the better a construction is. A study of Table 1 indicates that our constructions are superior in quality to three of the four Sommerville tetrahedra and competitive with Sommerville II and with the cube subdivisions. Thus our constructions are quite suitable for mesh generation.

As far as the bounds on dihedral angles, the TCP structure C15 is the best of our constructions. Note that given any vertex  $v$  in any triangulation, there must be some edges of valence less than six incident to  $v$ . (Otherwise the cell around  $v$  in the dual complex would be a polyhedron whose faces all have at least six sides, contradicting Euler's formula.) Thus every triangulation must have some dihedral angles at least  $72^\circ$ ; our C15 structure comes close to this limit. Conversely, any acute triangulation of flat space must have some edge of valence at least six. (Otherwise each edge has valence exactly five, and the triangulation would be spherical. We expect in fact that the acuteness hypothesis here is unnecessary.) Thus there is some dihedral angle less than or equal to  $60^\circ$ ; our C15 structure exactly achieves this bound.

Table 1

The quality of the tetrahedra in our constructions (and of the regular tetrahedron) can be measured in terms of the radius-edge ratio and the extreme dihedral angles

Construction	Radius-edge ratio		Smallest dihedral angle		Largest dihedral angle	
	min	max	min	max	min	max
TCP Z from triangle tiling	.651	.737	53.13	67.37	73.89	77.07
TCP A15 from square tiling	.645	.707	53.13	67.79	73.39	78.46
TCP $\sigma$	.645	.737	53.13	67.79	73.39	78.46
TCP C15	.612	.711	60	70.53	70.53	74.20
TCP Z from icosahedra	.629	1.000	41.81	69.09	71.99	83.62
Slab	.636	.938	46.83	67.88	74.39	87.70
Sommerville I (Fig. 5(a))	1.118	1.118	45	45	90	90
Sommerville II (Fig. 5(b))	.645	.645	60	60	90	90
Sommerville III (Fig. 5(c))	.866	.866	45	45	120	120
Sommerville IV (Fig. 5(d))	1.581	1.581	30	30	131.81	131.81
Cube V (Fig. 7(a))	.612	.866	54.73	70.53	70.53	90
Cube VI (Fig. 7(b))	.866	.866	45	45	90	90
Regular tetrahedron	.612	.612	70.53	70.53	70.53	70.53



## 6.2. Future research

There are still some challenges left to make use of these tilings in real-life meshing techniques. A strategy is required to fit the tilings into a planar projection of the spatial domain. Malkevitch studies this problem in [35]. He describes the conditions for a polygon to be tiled by squares and equilateral triangles. Also, even though one of our constructions fits between two parallel planes in a slab, all of them have dimples (cavities on the surface) in most directions, making them not very suitable for meshing domains with flat surfaces.

Open problems related to this work include the following:

1. How efficiently can we test whether a three-dimensional point set forms the vertices of an acute triangulation, and find such a triangulation if it exists? The planar version of this problem is solved by Delaunay triangulation, but Lemma 4 provides a counter-example to this approach in three dimensions. Alternatively, one could use a triangulation algorithm that minimizes the maximum angle [4]. However, such an algorithm is yet to be developed for three-dimensional domains.
2. Are these constructions the best possible? For instance, which tiling of space with tetrahedra minimizes the maximum dihedral angle?
3. Is it possible to tile the space with congruent copies of some single acute tetrahedron?
4. Is it possible to subdivide a cube (or even an acute tetrahedron) into acute tetrahedra?
5. Can the gadget-based acute triangulation methods of [6] be extended to an algorithm for finding a strictly acute triangulation of any three-dimensional point set?

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