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Zeros of solutions of certain second order linear differential equation ☆

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Abstract

In this paper, we investigate the exponent of convergence of the zero-sequence of solutions of the differential equation

$$f'' + \left(Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}\right) f = 0, \tag{1.3}$$

where $P_1(z)$, $P_2(z)$, $P_3(z)$ are polynomials of degree $n \geqslant 1$, Q_1, Q_2, Q_3 are entire functions of order less than n.

Keywords: Linear differential equation; Entire function; Zero

1. Introduction and results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g., see [1,2]). We will use the notation $\sigma(f)$ to denote the order of growth of meromorphic function $f(z)$, $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of $f(z)$.

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For second order linear differential equation

$$f'' + \left(e^{P_1(z)} + e^{P_2(z)} + Q_0(z)\right) f = 0, \tag{1.1}$$

where $P_1(z)$, $P_2(z)$ are non-constant polynomials

$$P_1(z) = \zeta_1 z^n + \cdots, \quad P_2(z) = \zeta_2 z^m + \cdots, \quad \zeta_1\zeta_2 \neq 0 \ (n, m \in N),$$

and $Q_0(z)$ is an entire function of order less than $\max\{n, m\}$. If $e^{P_1(z)}$ and $e^{P_2(z)}$ are linearly independent, K. Ishizaki and K. Tohge have studied the exponent of convergence of the zero-sequence of solutions of (1.1) and obtain the following results in [3,4].

Theorem A. [3] *Suppose that $n = m$, and that $\zeta_1 \neq \zeta_2$ in* (1.1). *If $\frac{\zeta_1}{\zeta_2}$ is non-real, then for any solution $f \not\equiv 0$ of* (1.1), *we have $\lambda(f) = \infty$.*

Theorem B. [4] *Suppose that $n = m$, and that $\frac{\zeta_1}{\zeta_2} = \rho > 0$ in* (1.1). *If $0 < \rho < \frac{1}{2}$ or $Q_0(z) \equiv 0$, $\frac{3}{4} < \rho < 1$, then for any solution $f \not\equiv 0$ of* (1.1), *we have $\lambda(f) \geqslant n$.*

Thus a natural question is: what condition on Q_0 when $\sigma(Q_0) = n$ can we get the same results as Theorems A and B? In this paper, we investigate the exponent of convergence of the zero-sequence of solutions of the equation

$$f'' + \left(Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}\right) f = 0. \tag{1.2}$$

Furthermore, we assume that $e^{P_1(z)}$, $e^{P_2(z)}$ and $e^{P_3(z)}$ are linearly independent and obtain the following results which improve the results of K. Ishizaki and K. Tohge.

Theorem 1. *Let $Q_1(z)$, $Q_2(z)$, $Q_3(z)$ be entire functions of order less than n, and $P_1(z)$, $P_2(z)$, $P_3(z)$ be polynomials of degree $n \geqslant 1$,*

$$P_1(z) = \zeta_1 z^n + \cdots, \qquad P_2(z) = \zeta_2 z^n + \cdots, \qquad P_3(z) = \zeta_3 z^n + \cdots,$$

where ζ_1, ζ_2, ζ_3 are complex numbers.

(i) *If $\frac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < \frac{1}{2}$, then for any solution $f \not\equiv 0$ of* (1.2), *we have $\lambda(f) = \infty$.*
(ii) *If $0 < \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < 1$, then for any solution $f \not\equiv 0$ of* (1.2), *we have $\lambda(f) \geqslant n$.*

Corollary 1. *Let $Q(z)$ be entire function of order less than n, suppose that $P_1(z)$, $P_2(z)$, $P_3(z)$, ζ_1, ζ_2, ζ_3 satisfy the hypotheses of Theorem* 1.

(i) *If $\frac{\zeta_1}{\zeta_2}$ is non-real, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < \frac{1}{2}$, then for any solution $f \not\equiv 0$ of the equation*

$$f'' + \left(e^{P_1(z)} + e^{P_2(z)} + Q e^{P_3(z)}\right) f = 0, \tag{1.3}$$

we have $\lambda(f) = \infty$.
(ii) *If $0 < \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $0 < \lambda = \frac{\zeta_3}{\zeta_2} < 1$, then for any solution $f \not\equiv 0$ of* (1.3), *we have $\lambda(f) \geqslant n$.*

2. Notations and some lemmas

To prove the theorem, we need some notations and a series of lemmas. Let $P_j(z)$ $(j = 1, 2, 3)$ be polynomials of degree $n \geqslant 1$, $P_j(z) = (\alpha_j + i\beta_j) z^n + \cdots$, $\alpha_j, \beta_j \in R$. Define

$$\delta(P_j,\theta)=\delta_j(\theta)=\alpha_j\cos n\theta-\beta_j\sin n\theta,\quad \theta\in[0,2\pi)\ (j=1,2,3),$$
$$S_j^+=\{\theta\mid\delta_j(\theta)>0\},\quad S_j^-=\{\theta\mid\delta_j(\theta)<0\}\quad (j=1,2,3).$$

Let $f(z), a(z)$ be meromorphic functions in the plane and satisfy

$$T(r,a)=o\{T(r,f)\},$$

except possibly for a set of r having finite linear measure, we call that $a(z)$ is a small function to $f(z)$ (see [1]).

Lemma 1. [1] *Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n(z)P(z)=Q(z),\tag{2.1}$$

where $P(z)$, $Q(z)$ are differential polynomials in $f(z)$ with small coefficients and the degree of $Q(z)$ is at most n, then

$$m\{r,P(z)\}=S(r,f),\quad \text{as } r\to+\infty.\tag{2.2}$$

Lemma 2. [5] *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f)=\sigma<\infty$, $\Gamma=\{(k_1,j_1),\dots,(k_m,j_m)\}$ be a finite set of distinct pairs of integers which satisfy $k_i>j_i\geqslant 0$ for $i=1,\dots,m$. And let $\varepsilon>0$ be a given constant, then there exists a set $E\subset[0,2\pi)$ which has linear measure zero, such that if $\varphi\in[0,2\pi)\setminus E$, there is a constant $R_1=R_1(\varphi)>1$, such that for all z satisfying $\arg z=\varphi$ and $|z|=r>R_1$ and for all $(k,j)\in\Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\sigma-1+\varepsilon)}.\tag{2.3}$$

Lemma 3. [6] *Suppose that $P(z)=(\alpha+\beta i)z^n+\cdots$ (α,β are real numbers, $|\alpha|+|\beta|\neq 0$) is a polynomial with degree $n\geqslant 1$, that $A(z)$ ($\not\equiv 0$) is an entire function with $\sigma(A)<n$. Set $g(z)=A(z)e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. Then for any given $\varepsilon>0$, there exists a set $H_1\subset[0,2\pi)$ that has the linear measure zero, such that for any $\theta\in[0,2\pi)\setminus(H_1\cup H_2)$, there is $R>0$ such that for $|z|=r>R$, we have*:

(i) *If $\delta(P,\theta)>0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}<|g(re^{i\theta})|<\exp\{(1+\varepsilon)\delta(P,\theta)r^n\};\tag{2.4}$$

(ii) *If $\delta(P,\theta)<0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}<|g(re^{i\theta})|<\exp\{(1-\varepsilon)\delta(P,\theta)r^n\},\tag{2.5}$$

where $H_2=\{\theta\in[0,2\pi);\ \delta(P,\theta)=0\}$ is a finite set.

Remark. The lemma also holds when $A(z)$ is a meromorphic function with $\sigma(A)<n$.

Lemma 4. [7] *Let $f(z)$ be an entire function of order $\sigma(f)=\alpha<+\infty$. Then for any given $\varepsilon>0$, there is a set $E\subset[1,\infty)$ that has finite linear measure and finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E$, we have*

$$\exp\{-r^{\alpha+\varepsilon}\}\leqslant|f(z)|\leqslant\exp\{r^{\alpha+\varepsilon}\}.\tag{2.6}$$

Lemma 5. *Let $P_j(z)$ $(j=1,2,3)$ be polynomials of degree $n \geqslant 1$,*

$$P_1(z)=\zeta z^n+B_1(z), \qquad P_2(z)=\rho_1\zeta z^n+B_2(z), \qquad P_3(z)=\rho_2\zeta z^n+B_3(z),$$

where $\zeta=\alpha+i\beta$, $\alpha,\beta\in R$, $|\alpha|+|\beta|\neq 0$, $0<\rho_1<1$, $0<\rho_2<1$, $B_1(z)$, $B_2(z)$, $B_3(z)$ are polynomials of degree at most $n-1$. Let $Q_1(z)\not\equiv 0$, $Q_2(z)$, $Q_3(z)$ be entire functions of order less than n, then for any given $\varepsilon>0$, there exist a set E with finite linear measure and a constant ξ $(n-1<\xi<n)$ such that

$$m\big(r, Q_1e^{P_1}+Q_2e^{P_2}+Q_3e^{P_3}\big)\geqslant(1-\varepsilon)m\big(r,e^{P_1}\big)+O\big(r^{\xi}\big), \quad r\to\infty\ (r\notin E). \tag{2.7}$$

Proof. By definition, for sufficiently large r, we have

$$m\big(r,e^{P_1}\big)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\big|e^{P_1(re^{i\theta})}\big|\,d\theta=\frac{1}{2\pi}\int_{S_1^+}\log^+\big|e^{P_1(re^{i\theta})}\big|\,d\theta=\frac{|\zeta|r^n}{\pi}+O\big(r^{n-1}\big). \tag{2.8}$$

If $\theta\in S_1^-$, then $\delta(P_j,\theta)<0$ $(j=1,2,3)$, by Lemmas 3 and 4, for any given $\varepsilon>0$ and for sufficiently large r, we have

$$\big|Q_1e^{P_1(re^{i\theta})}+Q_2e^{P_2(re^{i\theta})}+Q_3e^{P_3(re^{i\theta})}\big|\leqslant\sum_{j=1}^{3}\exp\big\{(1-2\varepsilon)\delta(P_j,\theta)r^n\big\}\leqslant 1. \tag{2.9}$$

If $\theta\in S_1^+$, since $0<\rho_1<1$, $0<\rho_2<1$, by Lemmas 3 and 4, there exists a set E with finite linear measure, for any given $\varepsilon>0$ and for sufficiently large r, we have

$$\begin{aligned}
&\big|Q_1+Q_2e^{P_2(re^{i\theta})-P_1(re^{i\theta})}+Q_3e^{P_3(re^{i\theta})-P_1(re^{i\theta})}\big|\\
&\quad\geqslant|Q_1|-\big|Q_2e^{P_2(re^{i\theta})-P_1(re^{i\theta})}\big|-\big|Q_3e^{P_3(re^{i\theta})-P_1(re^{i\theta})}\big|\\
&\quad\geqslant\frac{1}{2}\exp\big\{-r^{\sigma(Q_1)+\varepsilon}\big\}\geqslant\exp\big\{-r^{\xi}\big\}\quad(r\notin E),
\end{aligned}\tag{2.10}$$

where $\sigma(Q_1)<\xi<n$.

By (2.8)–(2.10), we have

$$\begin{aligned}
&m\big(r,Q_1e^{P_1}+Q_2e^{P_2}+Q_3e^{P_3}\big)\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi}\log^+\big|Q_1e^{P_1(re^{i\theta})}+Q_2e^{P_2(re^{i\theta})}+Q_3e^{P_3(re^{i\theta})}\big|\,d\theta\\
&\quad=\frac{1}{2\pi}\int_{S_1^+}\log^+\big(\big|e^{P_1(re^{i\theta})}\big|\big|Q_1+Q_2e^{P_2(re^{i\theta})-P_1(re^{i\theta})}+Q_3e^{P_3(re^{i\theta})-P_1(re^{i\theta})}\big|\big)\,d\theta\\
&\quad=\frac{(1-\varepsilon)|\zeta|r^n}{\pi}-O\big(r^{\xi}\big)\quad(r\notin E).
\end{aligned}\tag{2.11}$$

By (2.8) and (2.11), we obtain (2.7). □

3. Proof of Theorem 1

Since $\zeta_3 = \lambda\zeta_2$, $\lambda > 0$, we have $S_2^+ = S_3^+$, $S_2^- = S_3^-$. We see that S_j^+ and S_j^- have n components S_{jk}^+ and S_{jk}^- respectively ($j = 1, 2, 3$; $k = 1, 2, \ldots, n$). Hence we write

$$S_j^+ = \bigcup_{k=1}^{n} S_{jk}^+, \quad S_j^- = \bigcup_{k=1}^{n} S_{jk}^- \quad (j = 1, 2, 3).$$

Furthermore, we define

$$D_{12} = \{\theta \in S_1^+ \cap S_2^+ \colon\ \delta_1(\theta) > (2\lambda + 2)\delta_2(\theta)\},$$
$$D_{21} = \left\{\theta \in S_1^+ \cap S_2^+ \colon\ \delta_2(\theta) > \frac{\lambda + 1}{\lambda}\delta_1(\theta)\right\}.$$

(i) Let $f \not\equiv 0$ be a solution of (1.2). Suppose that $\lambda(f) < \infty$. Write $f = \pi e^h$, where π is the canonical product from zeros of f, and h is an entire function. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < \infty$. From (1.2), we get

$$(h')^2 = -h'' - 2\frac{\pi'}{\pi}h' - \frac{\pi''}{\pi} - Q_1 e^{P_1} - Q_2 e^{P_2} - Q_3 e^{P_3}. \tag{3.1}$$

Eliminating e^{P_1} from (3.1), set $\frac{Q_1'}{Q_1} + P_1' = R$, we have

$$2U_1 h' = -h''' + \left(R - 2\frac{\pi'}{\pi}\right)h'' + 2\left(R\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + R\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)'$$
$$+ \left(RQ_2 - Q_2' - Q_2 P_2'\right)e^{P_2} + \left(RQ_3 - Q_3' - Q_3 P_3'\right)e^{P_3}, \tag{3.2}$$

where

$$U_1 = h'' - \frac{1}{2}Rh'. \tag{3.3}$$

Eliminating e^{P_2} from (3.1), set $\frac{Q_2'}{Q_2} + P_2' = T$, we obtain

$$2U_2 h' = -h''' + \left(T - 2\frac{\pi'}{\pi}\right)h'' + 2\left(T\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + T\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)'$$
$$+ \left(TQ_1 - Q_1' - Q_1 P_1'\right)e^{P_1} + \left(TQ_3 - Q_3' - Q_3 P_3'\right)e^{P_3}, \tag{3.4}$$

where

$$U_2 = h'' - \frac{1}{2}Th'. \tag{3.5}$$

Set $\max\{\sigma(Q_1), \sigma(Q_2), \sigma(Q_3)\} < \xi_1 < \xi_2 < \xi_3 < n$. Then we get

$$T(r, Q) = m(r, Q) \leqslant r^{\xi_1}, \qquad \left|Q\left(re^{i\theta}\right)\right| \leqslant \exp\{r^{\xi_1}\}$$

for sufficiently large r and for any $\theta \in [0, 2\pi)$.

We apply Lemma 1 to (3.1), for any given $\varepsilon > 0$

$$T(r, h') = m(r, h')$$
$$\leqslant m\left(r, \frac{\pi''}{\pi}\right) + m\left(r, \frac{\pi'}{\pi}\right) + m\left(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}\right) + S(r, h')$$
$$\leqslant O\left(r^{n+\varepsilon}\right) + S(r, h'),$$

which implies $\sigma(h') \leqslant n$. It follows from (3.3) and (3.5) that $\sigma(U_1) \leqslant n$ and $\sigma(U_2) \leqslant n$ respectively.

First we show that there exists a set $E_0 \subset [0, 2\pi)$, $m(E_0) = 0$ such that if $\theta \in S_2^- \setminus E_0$, then

$$\left|U_1\left(re^{i\theta}\right)\right| \leqslant O\left(e^{r^{\xi_2}}\right), \quad \text{as } r \to \infty. \tag{3.6}$$

In the case $|h'(re^{i\theta})| < 1$, from (3.3) we have

$$\left|U_1\left(re^{i\theta}\right)\right| \leqslant \left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| + \frac{1}{2}\left|R\left(re^{i\theta}\right)\right|. \tag{3.7}$$

If $|h'(re^{i\theta})| \geqslant 1$, then from (3.2), we get

$$\begin{aligned}
\left|2U_1\left(re^{i\theta}\right)\right| \leqslant{}& \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(\left|R\left(re^{i\theta}\right)\right| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\
&+ 2\left(\left|R\left(re^{i\theta}\right)\right|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\
&+ \left|R\left(re^{i\theta}\right)\right|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\
&+ \left(\left|R\left(re^{i\theta}\right)Q_2\left(re^{i\theta}\right)\right| + \left|Q_2'\left(re^{i\theta}\right)\right| + \left|Q_2\left(re^{i\theta}\right)P_2'\left(re^{i\theta}\right)\right|\right)\left|e^{P_2(re^{i\theta})}\right| \\
&+ \left(\left|R\left(re^{i\theta}\right)Q_3\left(re^{i\theta}\right)\right| + \left|Q_3'\left(re^{i\theta}\right)\right| + \left|Q_3\left(re^{i\theta}\right)P_3'\left(re^{i\theta}\right)\right|\right)\left|e^{P_3(re^{i\theta})}\right| \\
\leqslant{}& O\left(e^{r^{\xi_2}}\right), \quad \text{as } r \to \infty.
\end{aligned} \tag{3.8}$$

Since Q and h' are of finite order, by Lemma 2, (3.7) and (3.8), we obtain (3.6).

We note that there exist $\bar{\theta}_j$ $(j = 1, 2, 3)$ satisfying $\delta_j(\theta) = 0$ on the rays $\arg z = \bar{\theta}_j + \frac{q\pi}{n}$, where $q = 0, \ldots, 2n-1$, which form $2n$ sectors of opening $\frac{\pi}{n}$ respectively, thus we may assume that $\bar{\theta}_j \in [0, \frac{\pi}{n})$. Since $\zeta_2 = \lambda\zeta_3$, $\lambda > 0$, we have $\bar{\theta}_2 = \bar{\theta}_3$. Write $\bar{\theta}_{jq} = \bar{\theta}_j + \frac{q\pi}{n}$, $j = 1, 2$, if there are some integers q_1 and q_2 such that $\bar{\theta}_{1q_1} = \bar{\theta}_{2q_2}$, then $\bar{\theta}_1 - \bar{\theta}_2 + (q_1 - q_2)\frac{\pi}{n} = 0$, we have that $\tan n\bar{\theta}_j = \frac{\alpha_j}{\beta_j}$, $j = 1, 2$. Which gives

$$0 = \tan\left(n\bar{\theta}_1 - n\bar{\theta}_2 + (q_1 - q_2)\pi\right) = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\alpha_2 + \beta_1\beta_2}.$$

This contradicts the assumption that $\frac{\zeta_1}{\zeta_2}$ is non-real. Hence we see that each component of S_1^+ and S_2^+ contains a component of $S_1^+ \cap S_2^+$. The boundaries of the components of $S_1^+ \cap S_2^+$ are some of the rays $\arg z = \bar{\theta}_{jq}$, we fix a component of $S_1^+ \cap S_2^+$, say S^*. We may write

$$S^* = \left\{\theta \in S_1^+ \cap S_2^+\colon\ \theta_1^* < \theta < \theta_2^*,\ \delta_1\left(\theta_1^*\right) = \delta_2\left(\theta_2^*\right) = 0\right\}$$

or

$$S^* = \left\{\theta \in S_1^+ \cap S_2^+\colon\ \theta_2^* < \theta < \theta_1^*,\ \delta_1\left(\theta_1^*\right) = \delta_2\left(\theta_2^*\right) = 0\right\}.$$

Since every component of S_1^+ and S_2^+ is of opening $\frac{\pi}{n}$, the rays $\arg z = \theta_1^*$ and $\arg z = \theta_2^*$ are contained in S_2^+ and S_1^+ respectively. We treat the first case, the proof of the second case can be obtained similarly. Hence there exist $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\left\{\theta\colon\ \theta_1^* < \theta < \theta_1^* + \eta_1\right\} \subset D_{21}, \qquad \left\{\theta\colon\ \theta_2^* - \eta_2 < \theta < \theta_2^*\right\} \subset D_{12}.$$

Hence there exists a $\theta \in (S_{2k}^{+} \cap D_{12}) \setminus E_0$ for any $k = 1, 2, \ldots, n$. Set $0 < (2\lambda + 2)\delta_2 < \sigma_2 < \sigma_1 < \delta_1$, $0 < \varepsilon_1 < 1 - \frac{\sigma_1}{\delta_1}$, $0 < \varepsilon_2 < \frac{\sigma_2}{2\delta_2} - 1$, $0 < \varepsilon_3 < \frac{\sigma_2}{2\lambda\delta_2} - 1$. By Lemma 3, we have

$$\begin{aligned}
&\left|Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + Q_3 e^{P_3(re^{i\theta})}\right| \\
&\quad \geqslant \left|Q_1 e^{P_1(re^{i\theta})}\right| \left|1 - \left|\frac{Q_2}{Q_1} e^{P_2(re^{i\theta}) - P_1(re^{i\theta})}\right| - \left|\frac{Q_3}{Q_1} e^{P_3(re^{i\theta}) - P_1(re^{i\theta})}\right|\right| \\
&\quad \geqslant e^{(1-\varepsilon_1)\delta_1 r^n}\bigl(1 - o(1)\bigr) \\
&\quad \geqslant e^{\sigma_1 r^n}\bigl(1 - o(1)\bigr), \quad \text{as } r \to \infty.
\end{aligned} \tag{3.9}$$

We assume that there exists an unbounded sequence $\{r_m\}$ such that $0 < |h'(r_m e^{i\theta})| \leqslant 1$. From (3.1), (3.9) and Lemma 2, we get for an $N_1 \in N$

$$\begin{aligned}
e^{\sigma_1 r_m^n}\bigl(1 - o(1)\bigr) &\leqslant 1 + \left|\frac{h''(r_m e^{i\theta})}{h'(r_m e^{i\theta})}\right| + 2\left|\frac{\pi'(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| + \left|\frac{\pi''(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| \\
&\leqslant r_m^{N_1}, \quad \text{as } m \to \infty,
\end{aligned}$$

which is absurd. Hence we may assume that $|h'(re^{i\theta})| \geqslant 1$ for sufficiently large r. It follows from (3.1) and Lemma 2, for an $N_2 \in N$

$$\begin{aligned}
&\left|Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + Q_3 e^{P_3(re^{i\theta})}\right| \\
&\quad \leqslant \left|h'\bigl(re^{i\theta}\bigr)\right|^2 \left(1 + \left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|\right) \\
&\quad \leqslant \left|h'\bigl(re^{i\theta}\bigr)\right|^2 \bigl(1 + O\bigl(r^{N_2}\bigr)\bigr), \quad \text{as } r \to \infty.
\end{aligned} \tag{3.10}$$

Thus, we obtain for sufficiently large r

$$\left|h'\bigl(re^{i\theta}\bigr)\right| \geqslant e^{\frac{1}{2}\sigma_2 r^n}. \tag{3.11}$$

From Lemma 2, (3.2) and (3.11), we get

$$\begin{aligned}
\left|2U_1\bigl(re^{i\theta}\bigr)\right| \leqslant{}& \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(\left|R\bigl(re^{i\theta}\bigr)\right| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\
&+ 2\left(\left|R\bigl(re^{i\theta}\bigr)\right|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\
&+ \left|R\bigl(re^{i\theta}\bigr)\right|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\
&+ \left(\left|R\bigl(re^{i\theta}\bigr)Q_2\bigl(re^{i\theta}\bigr)\right| + \left|Q_2'\bigl(re^{i\theta}\bigr)\right| + \left|Q_2\bigl(re^{i\theta}\bigr)P_2'\bigl(re^{i\theta}\bigr)\right|\right)\left|\frac{e^{P_2(re^{i\theta})}}{h'(re^{i\theta})}\right| \\
&+ \left(\left|R\bigl(re^{i\theta}\bigr)Q_3\bigl(re^{i\theta}\bigr)\right| + \left|Q_3'\bigl(re^{i\theta}\bigr)\right| + \left|Q_3\bigl(re^{i\theta}\bigr)P_3'\bigl(re^{i\theta}\bigr)\right|\right)\left|\frac{e^{P_3(re^{i\theta})}}{h'(re^{i\theta})}\right| \\
\leqslant{}& O\bigl(r^{N_2}\bigr) + \bigl(1 + o(1)\bigr)\exp\left\{\left(\delta_2(1 + \varepsilon_2) - \frac{\sigma_2}{2}\right)r^n\right\} \\
&+ \bigl(1 + o(1)\bigr)\exp\left\{\left(\lambda\delta_2(1 + \varepsilon_3) - \frac{\sigma_2}{2}\right)r^n\right\}, \quad \text{as } r \to \infty.
\end{aligned} \tag{3.12}$$

Since $\delta_2(1+\varepsilon_2)-\frac{\sigma_2}{2}<0$, $\lambda\delta_2(1+\varepsilon_3)-\frac{\sigma_2}{2}<0$, it gives that for an $N_3\in N$ and sufficiently large r,

$$\left|U_1\left(re^{i\theta}\right)\right|\leqslant r^{N_3}. \tag{3.13}$$

Now we fix a $\gamma(=\gamma_{2k})\in(S_{2k}^{+}\cap D_{12})\setminus E_0$, $k=1,2,\dots,n$. Then we find $\gamma_1,\gamma_2\in S_2^{-}\setminus E_0$, $\gamma_1<\gamma<\gamma_2$ such that $\gamma-\gamma_1<\frac{\pi}{n}$, $\gamma_2-\gamma<\frac{\pi}{n}$. We first show that for any θ, $\gamma_1\leqslant\theta\leqslant\gamma$, we have

$$\left|U_1\left(re^{i\theta}\right)\right|\leqslant O\left(e^{r^{\xi_3}}\right),\quad \text{as } r\to\infty. \tag{3.14}$$

Write $\gamma-\gamma_1=\frac{\pi}{n+\tau_1}$, $\tau_1>0$, since $\sigma(U_1)\leqslant n$, we have that $|U_1(re^{i\theta})|\leqslant e^{r^{n+\tau_2}}$, $0<\tau_2<\tau_1$ for sufficiently large r. Set $g(z)=U_1(z)/\exp((ze^{-\frac{\gamma+\gamma_1}{2}i})^{\zeta_3})$, then $g(z)$ is regular in the region $\{z\colon\ \gamma_1\leqslant\arg z\leqslant\gamma\}$. Since $\gamma_1\leqslant\arg z=\theta\leqslant\gamma$, $\gamma-\gamma_1<\frac{\pi}{n}$, we infer that $\cos(\arg((ze^{-\frac{\gamma+\gamma_1}{2}i})^{\zeta_3}))\geqslant K$ for some $K>0$. In fact,

$$-\frac{\pi}{2}<-\frac{\pi\xi_3}{2n}\leqslant-\xi_3\frac{\gamma-\gamma_1}{2}\leqslant\arg\left(\left(ze^{-\frac{\gamma+\gamma_1}{2}i}\right)^{\zeta_3}\right)\leqslant\xi_3\frac{\gamma-\gamma_1}{2}\leqslant\frac{\pi\xi_3}{2n}<\frac{\pi}{2}.$$

Hence for $\gamma_1<\theta<\gamma$,

$$\left|g\left(re^{i\theta}\right)\right|\leqslant\left|\frac{U_1(re^{i\theta})}{e^{Kr^{\xi_3}}}\right|\leqslant O\left(e^{r^{n+\tau_2}}\right),\quad \text{as } r\to\infty.$$

It follows from (3.6) and (3.13) that for some $M>0$, as $r\to\infty$

$$\left|g\left(re^{i\gamma_1}\right)\right|\leqslant\frac{O(e^{r^{\xi_2}})}{e^{Kr^{\xi_3}}}\leqslant M$$

and

$$\left|g\left(re^{i\gamma}\right)\right|\leqslant\frac{O(r^{N_3})}{e^{Kr^{\xi_3}}}\leqslant M.$$

By the Phragmen–Lindelöf theorem, we obtain (3.14). Similarly we see that (3.14) holds for $\gamma<\theta<\gamma_2$. Hence we conclude that (3.14) holds for any $\theta\in[0,2\pi)$.

In the following, we need to proof for any $\theta\in[0,2\pi)$

$$\left|U_2\left(re^{i\theta}\right)\right|\leqslant O\left(e^{r^{\xi_3}}\right),\quad \text{as } r\to\infty. \tag{3.15}$$

By recalling the previous reasoning, we can also obtain that there exists a set $E_1\subset[0,2\pi)$, $m(E_1)=0$ such that if $\theta\in S_1^{-}\cap S_2^{-}\setminus E_1$, then

$$\left|U_2\left(re^{i\theta}\right)\right|\leqslant O\left(e^{r^{\xi_2}}\right),\quad \text{as } r\to\infty. \tag{3.16}$$

By the similar proof in front, there exists a $\theta\in(S_{1k}^{+}\cap D_{21})\setminus E_1$ for any $k=1,2,\dots,n$. Set $0<(2\lambda+2)\delta_1<2\lambda\delta_2<\sigma_4<\sigma_3<\delta_2$, $0<\varepsilon_4<1-\frac{\sigma_3}{\delta_2}$, $0<\varepsilon_5<\frac{\sigma_4}{2\delta_1}-1$, $0<\varepsilon_6<\frac{\sigma_4}{2\lambda\delta_2}-1$. By Lemma 3, we have

$$\begin{aligned}
&\left|Q_1e^{P_1(re^{i\theta})}+Q_2e^{P_2(re^{i\theta})}+Q_3e^{P_3(re^{i\theta})}\right|\\
&\quad\geqslant\left|Q_2e^{P_2(re^{i\theta})}\right|\left|1-\left|\frac{Q_1}{Q_2}e^{P_1(re^{i\theta})-P_2(re^{i\theta})}\right|-\left|\frac{Q_3}{Q_2}e^{P_3(re^{i\theta})-P_2(re^{i\theta})}\right|\right|\\
&\quad\geqslant e^{(1-\varepsilon_4)\delta_2r^n}\left(1-o(1)\right)\\
&\quad\geqslant e^{\sigma_3r^n}\left(1-o(1)\right),\quad \text{as } r\to\infty.
\end{aligned} \tag{3.17}$$

We assume that there exists an unbounded sequence $\{r_m\}$ such that $0 < |h'(r_m e^{i\theta})| \leqslant 1$. From (3.1), (3.17) and Lemma 2, we get for an $N_4 \in N$

$$e^{\sigma_3 r_m^n}\big(1-o(1)\big) \leqslant 1 + \left|\frac{h''(r_m e^{i\theta})}{h'(r_m e^{i\theta})}\right| + 2\left|\frac{\pi'(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| + \left|\frac{\pi''(r_m e^{i\theta})}{\pi(r_m e^{i\theta})}\right| \leqslant r_m^{N_4}, \quad \text{as } m \to \infty.$$

This is absurd. Hence we may assume that $|h'(re^{i\theta})| \geqslant 1$ for sufficiently large r. It follows from (3.1) and Lemma 2, for an $N_5 \in N$

$$\begin{aligned}&\big|Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + Q_3 e^{P_3(re^{i\theta})}\big| \\ &\quad\leqslant \big|h'\big(re^{i\theta}\big)\big|^2\left(1 + \left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right|\right) \\ &\quad\leqslant \big|h'\big(re^{i\theta}\big)\big|^2\big(1 + O\big(r^{N_5}\big)\big), \quad \text{as } r \to \infty. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we obtain for sufficiently large r

$$\big|h'\big(re^{i\theta}\big)\big| \geqslant e^{\frac{1}{2}\sigma_4 r^n}. \tag{3.19}$$

It follows from (3.4) and (3.19) that

$$\begin{aligned}\big|2U_2\big(re^{i\theta}\big)\big| &\leqslant \left|\frac{h'''(re^{i\theta})}{h'(re^{i\theta})}\right| + \left(\big|T\big(re^{i\theta}\big)\big| + 2\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|\right)\left|\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right| \\ &\quad + 2\left(\big|T\big(re^{i\theta}\big)\big|\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^2\right) \\ &\quad + \big|T\big(re^{i\theta}\big)\big|\left|\frac{\pi''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi'''(re^{i\theta})}{\pi(re^{i\theta})}\right| + \left|\frac{\pi''(re^{i\theta})\pi'(re^{i\theta})}{\pi(re^{i\theta})^2}\right| \\ &\quad + \big(\big|T\big(re^{i\theta}\big)Q_1\big(re^{i\theta}\big)\big| + \big|Q_1'\big(re^{i\theta}\big)\big| + \big|Q_1\big(re^{i\theta}\big)P_1'\big(re^{i\theta}\big)\big|\big)\left|\frac{e^{P_1(re^{i\theta})}}{h'(re^{i\theta})}\right| \\ &\quad + \big(\big|T\big(re^{i\theta}\big)Q_3\big(re^{i\theta}\big)\big| + \big|Q_3'\big(re^{i\theta}\big)\big| + \big|Q_3\big(re^{i\theta}\big)P_3'\big(re^{i\theta}\big)\big|\big)\left|\frac{e^{P_3(re^{i\theta})}}{h'(re^{i\theta})}\right| \\ &\leqslant O\big(r^{N_5}\big) + \big(1+o(1)\big)\exp\left\{\left(\delta_1(1+\varepsilon_5) - \frac{\sigma_4}{2}\right)r^n\right\} \\ &\quad + \big(1+o(1)\big)\exp\left\{\left(\lambda\delta_2(1+\varepsilon_6) - \frac{\sigma_4}{2}\right)r^n\right\}, \quad \text{as } r \to \infty. \end{aligned} \tag{3.20}$$

Since $\delta_1(1+\varepsilon_5) - \frac{\sigma_4}{2} < 0$, $\lambda\delta_2(1+\varepsilon_6) - \frac{\sigma_4}{2} < 0$, it gives that for an $N_6 \in N$ and sufficiently large r,

$$\big|U_2\big(re^{i\theta}\big)\big| \leqslant r^{N_6}. \tag{3.21}$$

Now we fix a $\gamma'(=\gamma'_{2k}) \in (S_{2k}^+ \cap D_{12}) \setminus E_1$, $k = 1, 2, \ldots, n$. Then we find $\gamma_3, \gamma_4 \in S_1^- \cap S_2^- \setminus E_1$, $\gamma_3 < \gamma' < \gamma_4$ such that $\gamma' - \gamma_3 < \frac{\pi}{n}$, $\gamma_4 - \gamma' < \frac{\pi}{n}$. By the same reasoning as in proof of (3.14), for any $\gamma_3 \leqslant \theta \leqslant \gamma_4$, we have

$$\big|U_2\big(re^{i\theta}\big)\big| \leqslant O\big(e^{r^{\xi_3}}\big), \quad \text{as } r \to \infty. \tag{3.22}$$

Hence we conclude that (3.15) holds for any $\theta \in [0, 2\pi)$.

By (3.3) and (3.5), we have

$$U_1 - U_2 = \frac{1}{2}h'(T - R). \tag{3.23}$$

Since $\sigma(Q_j) < \xi_2 < \xi_3$ $(j = 1, 2, 3)$ and the theorem on the logarithmic derivatives, by (3.1), (3.23)

$$\begin{aligned} m\bigl(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}\bigr) &\\ \leqslant 2m(r, h') + O(\log r) &\\ \leqslant 2m(r, U_1 - U_2) + O(\log r) \leqslant O\bigl(r^{\xi_3}\bigr), \quad \text{as } r \to \infty. \end{aligned} \tag{3.24}$$

Since $\frac{\zeta_1}{\zeta_2}$ is non-real, $S_1^+ \cap S_2^-$ contains an interval $I = [\varphi_1, \varphi_2]$ satisfying $\min_{\theta \in I} \delta_1(\theta) = s > 0$. By Lemma 3, there exists an $R(I)$ (> 0) such that for any $\theta \in I$ and $r \geqslant R(I)$,

$$\bigl|Q_1 e^{P_1(re^{i\theta})}\bigr| \geqslant \exp\bigl((1-\varepsilon)\delta_1 r^n\bigr),$$
$$\bigl|Q_2 e^{P_2(re^{i\theta})}\bigr| \leqslant \exp\bigl((1-\varepsilon)\delta_2 r^n\bigr),$$

and

$$\bigl|Q_3 e^{P_3(re^{i\theta})}\bigr| \leqslant \exp\bigl((1-\varepsilon)\lambda\delta_2 r^n\bigr).$$

Hence, we have

$$\begin{aligned} m\bigl(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}\bigr) &\\ \geqslant \int_{\varphi_1}^{\varphi_2} \log^+\bigl|Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + Q_3 e^{P_3(re^{i\theta})}\bigr|\, d\theta &\\ \geqslant \int_{\varphi_1}^{\varphi_2} \bigl(1 - o(1)\bigr) \log^+\bigl|Q_1 e^{P_1(re^{i\theta})}\bigr|\, d\theta &\\ \geqslant \int_{\varphi_1}^{\varphi_2} \bigl(1 - o(1)\bigr)(1-\varepsilon) s r^n \, d\theta &\\ \geqslant \bigl(1 - o(1)\bigr)(1-\varepsilon) s r^n (\varphi_2 - \varphi_1), \quad \text{as } r \to \infty. \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25) and recalling that $\xi_3 < n$, we get a contradiction. Hence, $\lambda(f) = \infty$.

(ii) Let $f \not\equiv 0$ be a solution of (1.2). Write $f = \pi e^h$, suppose that $\lambda(f) < n$. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < n$. Eliminating e^{P_1} from (3.1), we have

$$\begin{aligned} 2Uh' = -h''' + \left(R - 2\frac{\pi'}{\pi}\right)h'' + 2\left(R\frac{\pi'}{\pi} - \left(\frac{\pi'}{\pi}\right)'\right)h' + R\frac{\pi''}{\pi} - \left(\frac{\pi''}{\pi}\right)' &\\ + \bigl(RQ_2 - Q_2' - Q_2 P_2'\bigr)e^{P_2} + \bigl(RQ_3 - Q_3' - Q_3 P_3'\bigr)e^{P_3}, \end{aligned} \tag{3.26}$$

where

$$U = h'' - \frac{1}{2}Rh'. \tag{3.27}$$

From (3.26) and (3.27), we get

$$C_1(z)h' = C_0(z),$$

where

$$C_0(z) = -U' + \frac{1}{2}RU - 2\frac{\pi'}{\pi}U + R\frac{\pi''}{\pi} - \frac{\pi'''}{\pi} + \frac{\pi''\pi'}{\pi^2} + \left(RQ_2 - Q_2' - Q_2P_2'\right)e^{P_2} + \left(RQ_3 - Q_3' - Q_3P_3'\right)e^{P_3}, \tag{3.28}$$

$$C_1(z) = 2U + \frac{1}{2}R' - \frac{1}{4}R^2 - R\frac{\pi'}{\pi} + 2\frac{\pi''}{\pi} - 2\left(\frac{\pi'}{\pi}\right)^2. \tag{3.29}$$

If $C_0(z) \not\equiv 0$, $C_1(z) \not\equiv 0$, by Nevanlinna's first fundamental theorem, we obtain

$$T(r, h') \leqslant T(r, C_0) + T(r, C_1) + o(1).$$

Set $\max\{\sigma(Q_1), \sigma(Q_2), \sigma(Q_3), \lambda(f)\} < \xi_2 < \xi_3 < n$, from (3.1), we obtain

$$T\left(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + Q_3e^{P_3(z)}\right) \leqslant 2T(r, h') + O(\log r). \tag{3.30}$$

By Lemma 5, we have

$$m\left(r, Q_1e^{P_1(z)} + Q_2e^{P_2(z)} + Q_3e^{P_3(z)}\right) \geqslant (1-\varepsilon)m\left(r, e^{P_1}\right) + O\left(r^{\xi_3}\right), \quad r \to \infty\ (r \notin E), \tag{3.31}$$

where E has finite linear measure. From (3.30) and (3.31), we obtain

$$T(r, h') \geqslant \frac{1-\varepsilon}{2}T\left(r, e^{P_1}\right) + O\left(r^{\xi_3}\right), \quad r \to \infty\ (r \notin E). \tag{3.32}$$

Since $0 < \rho = \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $\zeta_3 = \lambda\zeta_2$, $0 < \lambda < 1$, we get

$$\delta(P_2, \theta) = \rho\delta(P_1, \theta), \qquad S_{1k}^+ = S_{2k}^+ = S_{3k}^+, \quad S_{1k}^- = S_{2k}^- = S_{3k}^- \quad (k = 1, \ldots, n).$$

By the same reasoning as in (3.7) and (3.8), we have

$$\left|U\left(re^{i\theta}\right)\right| \leqslant O\left(e^{r^{\xi_2}}\right), \quad \text{as } r \to \infty \tag{3.33}$$

for any $\theta \in S_1^- \setminus E_0$, $m(E_0) = 0$. Also by the same reasoning as in (3.9)–(3.13), we have

$$\left|U\left(re^{i\theta}\right)\right| \leqslant r^{N_3}, \quad \text{as } r \to \infty \tag{3.34}$$

for any $\theta \in S_1^+ \setminus E_0$, $m(E_0) = 0$. Since $\sigma(U) \leqslant n$, by the Phragmen–Lindelöf theorem, we have

$$\left|U\left(re^{i\theta}\right)\right| \leqslant O\left(e^{r^{\xi_3}}\right), \quad \text{as } r \to \infty \tag{3.35}$$

for any $\theta \in [0, 2\pi)$. In the following, we estimate $T(r, C_0)$ and $T(r, C_1)$.

$$T(r, C_0) \leqslant T\left(r, U' - \frac{1}{2}RU + 2\frac{\pi'}{\pi}U\right) + T\left(r, R\frac{\pi''}{\pi} - \frac{\pi'''}{\pi} + \frac{\pi''\pi'}{\pi^2}\right) + T\left(r, RQ_2 - Q_2' - Q_2P_2'\right) + T\left(r, e^{P_2}\right) + T\left(r, RQ_3 - Q_3' - Q_3P_3'\right) + T\left(r, e^{P_3}\right).$$

Since $\max\{\sigma(Q_1), \sigma(Q_2), \sigma(Q_3), \sigma(R), \sigma(\pi)\} < n$, we have

$$T(r, C_0) \leqslant T\left(r, e^{P_2}\right) + T\left(r, e^{P_3}\right) + O\left(r^{\xi_3}\right) = (1+\lambda)T\left(r, e^{P_2}\right) + O\left(r^{\xi_3}\right) \leqslant (1+\lambda)\rho T\left(r, e^{P_1}\right) + O\left(r^{\xi_3}\right), \quad \text{as } r \to \infty. \tag{3.36}$$

From (3.29) and (3.35), we have

$$T(r, C_1) \leqslant O\big(r^{\xi_3}\big), \quad \text{as } r \to \infty. \tag{3.37}$$

From (3.30), (3.32), (3.36) and (3.37), we get

$$\frac{1-\varepsilon}{2} T\big(r, e^{P_1}\big) + O\big(r^{\xi_3}\big) \leqslant T(r, h') \leqslant (1+\lambda)\rho T\big(r, e^{P_1}\big) + O\big(r^{\xi_3}\big), \quad r \to \infty\ (r \notin E). \tag{3.38}$$

Thus (3.38) implies

$$\left(\frac{1-\varepsilon}{2} - (1+\lambda)\rho - o(1)\right) T\big(r, e^{P_1}\big) \leqslant 0, \quad r \to \infty\ (r \notin E).$$

Since $0 < \rho = \frac{\zeta_2}{\zeta_1} < \frac{1}{4}$, $0 < \lambda < 1$, we get a contradiction. Hence $C_0(z) \equiv C_1(z) \equiv 0$. From (3.28), we obtain

$$\begin{aligned} &\big(RQ_2 - Q_2' - Q_2P_2'\big)e^{P_2} + \big(RQ_3 - Q_3' - Q_3P_3'\big)e^{P_3} \\ &\quad = U' - \frac{1}{2}RU + 2\frac{\pi'}{\pi}U - R\frac{\pi''}{\pi} + \frac{\pi'''}{\pi} - \frac{\pi''\pi'}{\pi^2}. \end{aligned} \tag{3.39}$$

We assume that $(RQ_2 - Q_2' - Q_2P_2')e^{P_2} + (RQ_3 - Q_3' - Q_3P_3')e^{P_3} \not\equiv 0$, if $(RQ_2 - Q_2' - Q_2P_2')e^{P_2} + (RQ_3 - Q_3' - Q_3P_3')e^{P_3} \equiv 0$, we have

$$e^{P_2 - P_3} = \frac{Q_3' + Q_3P_3' - RQ_3}{RQ_2 - Q_2' - Q_2P_2'}.$$

Since $\zeta_3 = \lambda\zeta_2$, $0 < \lambda < 1$, by a simple order consideration, this is a contradiction.

From (3.39), by Lemma 5, we obtain

$$\begin{aligned} &(1-\varepsilon)T\big(r, e^{P_2}\big) + O\big(r^{\xi_3}\big) \\ &\quad \leqslant T\big(r, RQ_2 - Q_2' - Q_2P_2'\big)e^{P_2} + \big(RQ_3 - Q_3' - Q_3P_3'\big)e^{P_3} \\ &\quad \leqslant T\left(r, U' - \frac{1}{2}RU\right) + T(r, U) + T(r, R) + T\left(r, \frac{\pi'}{\pi}\right) + T\left(r, \frac{\pi''}{\pi}\right) \\ &\qquad + T\left(r, \frac{\pi'''}{\pi}\right) + o(1) \\ &\quad \leqslant O\big(r^{\xi_3}\big), \quad r \to \infty\ (r \notin E). \end{aligned} \tag{3.40}$$

From (3.40), we have $\sigma(e^{P_2}) < \xi_3 < n$, we get a contradiction. Hence $\lambda(f) \geqslant n$.

Proof of Corollary 1. By the same reasoning as in Theorem 1, we can complete the proof. □

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