



## Note

# On perfect hashing of numbers with sparse digit representation via multiplication by a constant

Maurizio Monge\*

*Scuola Normale Superiore di Pisa - Piazza dei Cavalieri, 7 - 56126 Pisa, Italy*

## ARTICLE INFO

### Article history:

Received 6 June 2010

Received in revised form 26 February 2011

Accepted 6 March 2011

Available online 9 April 2011

### Keywords:

Magic multiplier

Hash function

Bitboard

Schur function

## ABSTRACT

Consider the set of vectors over a field having non-zero coefficients only in a fixed sparse set and multiplication defined by convolution, or the set of integers having non-zero digits (in some base  $b$ ) in a fixed sparse set. We show the existence of an optimal (or almost-optimal, in the latter case) 'magic' multiplier constant that provides a perfect hash function which transfers the information from the given sparse coefficients into consecutive digits. Studying the convolution case we also obtain a result of non-degeneracy for Schur functions as polynomials in the elementary symmetric functions in positive characteristic.

© 2011 Elsevier B.V. All rights reserved.

## 1. Introduction and motivation

Suppose  $n > 0$ , and let  $D = \{d_0, d_1, \dots, d_{n-1}\}$  be a set of indices such that  $0 = d_0 < d_1 < \dots < d_{n-1}$ . For a field  $F$  and  $N = d_{n-1} + 1$ , let  $F^N$  be the standard vector space with basis  $e_j$  for  $0 \leq j < N$  equipped with the convolution multiplication

$$(a * b)_i = \sum_{\substack{j+k=i \\ 0 \leq j, k < N}} a_j b_k,$$

and let  $F[D]$  be the subspace spanned by  $e_{d_0}, \dots, e_{d_{n-1}}$ , which is formed by the vectors with non-zero coefficients only for the indices  $d_i$  for  $0 \leq i \leq n-1$ . Similarly given a positive integer  $b$  let  $\mathbb{Z}_b[D]$  be the set of integers that can be written as  $\sum_{i=0}^{n-1} a_i b^{d_i}$  for some  $a_i \in \{0, 1, \dots, b-1\}$ , i.e. the set of numbers such that their base- $b$  representation only contains non-zero digits in positions that belong to the set  $D$ . We study the existence of constants that can be used as multipliers to transfer the information stored in the sparse digits of an element of  $F[D]$  or of  $\mathbb{Z}_b[D]$  into a smallest possible set of consecutive digits, providing a perfect hash function.

Motivation for this kind of question is provided by a technique used by many state-of-the-art chess playing programs [2,8], based on the concept of 'bitboards', that are numbers whose base-2 representation is interpreted as occupancy information of some kind, or more generally for storing 0–1 information for each square, having previously established a correspondence between a range of digits and the squares on the board. The technique in question, known under the name of 'magic bitboards', is a quick way to generate all possible attacks for sliding pieces, such as rooks and bishops. The bitboard containing occupancy information for all pieces is transformed with a bitwise-and to a bitboard whose only digits that may be different from zero are those corresponding to possible obstructions on the path of the sliding piece. This information about the obstructions, which is stored in a small set of sparse digits, is then mapped via a multiplication by a 'magic number' to a set of consecutive digits, which is then used as the index in a lookup table to recover pre-calculated information about the possible attacks.

\* Fax: +39 050 563513.

E-mail addresses: [maurizio.monge@sns.it](mailto:maurizio.monge@sns.it), [maurizio.monge@gmail.com](mailto:maurizio.monge@gmail.com).

While in the case of chess programs a database of very efficient multipliers has already been computed and is publicly available, we investigate the existence of multipliers that provide perfect hashing functions in a more general setting. In the convolution case we provide an optimal result, which shows that it is possible to transfer the information stored in any number of sparse digits into the same number of consecutive digits, and which incidentally provides a result about values of Schur functions as polynomials in elementary symmetric functions. On the other hand, in the case of base- $b$  integers it is not always possible to have a multiplier providing a hash into the same number of digits ( $D = \{0, 1, 2, 4, 6\}$  providing the smallest counterexample for  $b = 2$ , as can be checked with a simple computer program), and we provide a linear estimate of the number of consecutive digits that are required to ensure the existence of such a map.

While this kind of hashing cannot be directly compared to universal hashing (see [9,10,5,11] in particular) because of its more restricted scope, it is still possible to compare the results about its effectiveness, and this is done below. See also [7].

**2. The convolution case**

In this section we consider the convolution case. The operation of convolution multiplication with a fixed vector  $(a_0, \dots, a_{N-1})$  can be expressed by a lower triangular Toeplitz matrix  $A = (a_{i-j})_{0 \leq i, j < N}$ , where we have put  $a_i = 0$  for negative  $i$  (the full convolution with a vector  $(a_{-N+1}, \dots, a_{N-1})$  is expressed by a general Toeplitz matrix, but as shown below, we can restrict to the class of lower triangular matrices). In the same way, when restricting the output to a set of coefficients we shall consider the matrix formed by the corresponding selection of rows of  $A$ . We will show that for a good choice of the  $a_i$  the matrix formed by the last  $n$  rows of  $A$  defines a one-to-one function from  $F[D]$  to  $F^n$ .

The operation of taking the convolution with a fixed vector, or multiplying by a Toeplitz matrix, is also equivalent to multiplication by a polynomial in a ring of polynomials, and its properties as a hash function are well known, as well as fast algorithms; see [4,1]. Note also that taking the convolution with a vector with entries in the set  $\{0, 1\}$  is actually obtained by addition of selected entries.

**Theorem 1.** *For every set  $D$  of cardinality  $n$ , there exists a lower triangular Toeplitz matrix such that its last  $n$  rows define a one-to-one function from  $F[D]$  to  $F^n$ , and furthermore its entries can be taken in the set  $\{0, 1\}$ .*

**Proof.** Let  $\delta = N - n$ , and for a vector  $(a_0, \dots, a_{N-1})$  that is going to be determined let  $A$  be the associated rectangular Toeplitz matrix  $(a_{\delta+i-j})_{\substack{0 \leq i < n \\ 0 \leq j < N}}$  corresponding to the selection of the last  $n$  coefficients after taking the convolution. Let  $B$  be the  $n \times n$  matrix obtained from  $A$  by selecting the columns with indices  $d_0, \dots, d_{n-1}$ , which defines the linear map on the basis of  $F[D]$  formed by the  $e_{d_i}$ :

$$B = (a_{\delta+i-d_k})_{\substack{0 \leq i < n \\ 0 \leq k < n}} = \begin{pmatrix} a_{\delta-d_0} & a_{\delta-d_1} & \cdots & a_{\delta-d_{n-1}} \\ a_{\delta+1-d_0} & a_{\delta+1-d_1} & \cdots & a_{\delta+1-d_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\delta+n-1-d_0} & a_{\delta+n-1-d_1} & \cdots & a_{\delta+n-1-d_{n-1}} \end{pmatrix}.$$

We will now consider a sequence of  $k \times k$  minors, for  $k = 1, \dots, n$ , where each minor will contain the previous one, and inductively change some of the  $a_i$  ensuring at the  $k$ th step that the determinant of the  $k$ th minor is non-zero, while leaving unchanged the coefficients of the minors considered in the previous steps.

Let the  $k$ -minor  $B_k$  be obtained taking the first  $k$  rows, and a range of columns  $r_k, r_k + 1, \dots, r_k + k - 1$ , where  $0 \leq r_k \leq n - k$  is the biggest integer such that each column of  $B_k$  will contain at least one  $a_i$  with  $i \geq 0$ . It is an easy consequence of  $B$  being a selection of columns from the Toeplitz matrix  $A$  that the set of integers  $r_k$  satisfying the above condition is always non-empty, and that the set of columns selected for  $B_k$  is the same set as was selected for  $B_{k-1}$  with one column added either on the left or on the right.

Let us begin the induction putting  $a_0 = 1$  and  $a_i = 0$  for all  $i \neq 0$ . For  $k = 1$ , also change  $a_{\delta-d_{r_1}}$  to be equal to 1. Let now  $k$  be  $> 1$ . When the columns of  $B_k$  are those of  $B_{k-1}$  plus one column on the right, we have that the matrix  $B_k = \begin{pmatrix} B_{k-1} & 0 \\ * & a_0 \end{pmatrix}$  is block lower triangular, with one block equal to  $B_{k-1}$ , and the other block being formed by the element  $a_0 = 1$ , and  $B_k$  is non-singular. On the other hand, when one column is added on the left,  $B_k$  is of the form  $B_k = \begin{pmatrix} * & B_{k-1} \\ a_\ell & * \end{pmatrix}$ , where  $\ell = \delta + k + 1 - d_{r_k}$  is the biggest index appearing in  $B_k$  (in fact, the indices are decreasing while moving right along a row or up along a column). Consequently, considering the Laplace expansion of the determinant of  $B_k$  along the first column

$$\det B_k = \sum_{i=1}^{k-1} a_{\ell-k+i} \cdot C_{i,1}(k) + a_\ell \cdot \det B_{k-1} \tag{1}$$

where for each  $i, j$ ,  $C_{i,j}(k)$  is the  $i, j$  cofactor of the matrix  $B_k$ , we can select an appropriate value for  $a_\ell$  which makes the determinant non-zero, while changing only the bottom left entry of  $B_k$ . Repeating this step up to  $k = n$  we have the theorem.  $\square$

**Remark 1.** Since no division is involved, the above proof works in any ring with 1, but it does not ensure that the resulting matrix  $B$  is invertible, only that it has non-zero determinant. Alternatively, it is possible to allow general  $a_i$ , and solve (1)

inductively in  $a_\ell$ , to ensure that  $\det B_k = 1$  at each step, obtaining that  $B$  is invertible because the determinant is an invertible element of the ring.

If  $F$  is a local ring (i.e. a ring with only one maximal ideal) with maximal ideal  $M$ , we can still take the  $a_i$  in  $\{0, 1\}$ , and make all the determinants of the  $B_k$  invertible: consider (1) and call  $S$  the sum; since in a local ring the invertible elements are precisely those not in  $M$  and we assume  $\det B_{k-1}$  to be invertible, we cannot have both  $S \in M$  and  $S + \det B_{k-1} \in M$  or we would also have  $\det B_{k-1} \in M$ , and hence putting  $a_\ell$  equal to either 0 or 1 we obtain that  $\det B_k \notin M$ , i.e. that it is invertible.

This has a practical consequence: take for instance  $F$  to be the set of integers modulo  $2^k$ ; it is a local ring and hence we have the existence of a magic multiplier with entries in the set  $\{0, 1\}$ , and taking the convolution is actually the addition of selected entries.

It is possible to observe that the Schur functions that are well known in algebraic combinatorics (see [3,6]) for their combinatorial properties and connections with the characters of the symmetric group can be expressed as a determinant of a special matrix having the elementary symmetric functions as coefficients via the Jacobi–Trudi identity (also known as the “determinant formula”), which has the same form as the transpose of the matrix  $B$  considered above. In particular, if  $s_\lambda$  is the Schur function associated with the partition  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\lambda'$  is the conjugate partition, and  $e_i$  is the  $i$ th elementary symmetric function for  $i \geq 0$ , we have the formula

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i, j \leq n}$$

expressing the Schur function  $s_\lambda$  as a uniquely determined polynomial in the  $e_i$ , having integral coefficients. Since the set  $D$  considered in Theorem 1 is arbitrary, for each partition of length  $n$  we chose  $D$  ensuring that  $\delta - d_{k-1} = \lambda'_k - k$  for  $1 \leq k \leq n$ , making the matrix  $B$  equal to the transpose of above matrix evaluated with  $e_i = a_i$  for all  $i$ . It follows that the polynomials expressing the  $s_\lambda$  in terms of the  $e_i$  assume a non-zero value when the  $e_i$  are replaced with appropriate values in the field  $F$  (note that this is not true for a general polynomial over a field, as the example  $x^p - x$  over  $\mathbb{F}_p$  shows). Hence we have:

**Theorem 2.** For a partition  $\lambda$ , consider the Schur function  $s_\lambda$  as a polynomial in the elementary symmetric functions  $e_i$ , considered as indeterminates. Then it takes a non-zero value after substitution of the variables  $e_i$  with appropriate elements of the field  $F$ , which moreover can be taken in the set  $\{0, 1\}$ .

### 3. The arithmetic case

The case of base- $b$  digits of integers seems to be much more difficult, and we give a linear bound on the number of digits required to ensure the existence of an appropriate multiplier. Let  $\mathbb{Z}_{(b)}$  be the set of rational numbers that can be written as  $r/b^k$  for some integers  $r, k$ , or equivalently that have a finite base- $b$  expansion. For such an  $a = \sum_{i < M} a_i b^i \in \mathbb{Z}_{(b)}$ , define  $\llbracket a \rrbracket_{k,m}$  as the  $m$ -tuple  $(a_{k+m-1}, \dots, a_{k+1}, a_k) \in \{0, 1, \dots, b-1\}^m$ . We can now state:

**Theorem 3.** Let  $D = \{d_0, \dots, d_{n-1}\}$  be a set of indices  $0 = d_0 < \dots < d_{n-1}$  having cardinality  $n$ . Then for

$$m = \lceil \log_b((2b-1)^n - 1) \rceil$$

there exist a multiplier  $\mu \in \mathbb{Z}$  and a  $k \in \mathbb{N}$  such that the map from  $\mathbb{Z}_b[D]$  to  $\{0, 1, \dots, b-1\}^m$  defined by  $a \mapsto \llbracket a \cdot \mu \rrbracket_{k,m}$  is injective. Furthermore, if  $D$  contains some consecutive integers and is formed by the union of the integral intervals  $\{c_i, c_i + 1, \dots, c_i + \ell_i - 1\}$  for  $i = 1, \dots, k$  and  $\ell_i \geq 1$ , we can take

$$m = \left\lceil \log_b \left( \prod_{i=1}^k (2b^{\ell_i} - 1) - 1 \right) \right\rceil.$$

It is possible to compare this estimate with what can be obtained using universal hashing: when a hashing function is randomly chosen in a universal class (i.e.  $h(x) = h(y)$  with probability at most  $1/b^m$ , which can be done when the output is formed by at least  $m$  digits; see [11]) we have that the probability of  $h$  being one-to-one on  $\mathbb{Z}_b[D]$  is at least  $1 - \binom{b^n}{2} b^{-m}$ , and we deduce the existence of a good hash function when  $m$  is  $\geq 2n$ . The above result is sharper because it just requires  $m$  to be about  $n \log_b(2b-1)$ , which is always smaller than  $2n$  and its ratio with  $n$  approaches 1 as  $b$  grows.

**Proof.** We will prove the second estimate, as the first one can be obtained by taking  $n$  intervals of length  $\ell_i = 1$ . Each element in  $\mathbb{Z}_b[D]$  can be written as

$$a = \sum_{i=1}^k \left( \sum_{j=0}^{\ell_i-1} a_{ij} b^j \right) \cdot b^{c_i} = \sum_{i=1}^k A_i b^{c_i},$$

with  $0 \leq A_i \leq b^{\ell_i} - 1$ , for each  $1 \leq i \leq k$ . Consequently the difference of two elements  $a, a' \in \mathbb{Z}_b[D]$  can be written as

$$a - a' = \sum_{i=1}^k (A_i - A'_i) b^{c_i},$$

where  $-b^{\ell_i} + 1 \leq A_i - A'_i \leq b^{\ell_i} - 1$ , for each  $1 \leq i \leq k$ . In particular the number  $\Delta$  of positive differences of two elements of  $\mathbb{Z}_b[D]$  is at most

$$\frac{\prod_{i=1}^k (2b^{\ell_i} - 1) - 1}{2}.$$

For any real number  $r$ , let  $\mathbb{T}_r = \mathbb{R}/r\mathbb{Z}$ , and let  $\pi_r : \mathbb{R} \rightarrow \mathbb{T}_r$  be the projection map. For  $z \in \mathbb{Z} \setminus \{0\}$  the map  $\beta : \mathbb{T}_r \rightarrow \mathbb{T}_r$  given by multiplication by  $z$  is measure-preserving, i.e. for each measurable  $X \subseteq \mathbb{T}_r$  the measure of  $\beta^{-1}(X)$  is equal to the measure of  $X$ .

Let now  $b^m$  be a power of  $b$  which is  $> 2\Delta$ . The measure in  $\mathbb{T}_{b^m}$  of the set  $U_z = \beta^{-1}(\pi_{b^m}([-1, 1]))$  of the 'bad'  $\lambda \in \mathbb{T}_{b^m}$  such that  $z\lambda \in \pi_{b^m}([-1, 1])$  is equal to  $2/b^m$ , supposing the measure of  $\mathbb{T}_{b^m}$  to be normalized to 1. Since the number of positive and non-zero differences is  $< b^m/2$ , and clearly  $U_{-z} = U_z$ , we have that the union of all the  $U_{a-a'}$  for all distinct  $a, a' \in \mathbb{Z}_p[D]$  cannot be all  $\mathbb{T}_{b^m}$ . Consequently since  $\mathbb{Z}_{(b)}$  is dense in  $\mathbb{R}$  there exists an element in  $\nu \in \mathbb{Z}_{(b)}$  which falls out of all the  $U_{a-a'}$  when reduced modulo  $b^m$ , the above union being a closed set. We have that  $a\nu$  and  $a'\nu$  differ by at least 1 after reduction modulo  $b^m$ , and hence the map  $a \mapsto \llbracket a \cdot \nu \rrbracket_{0,m}$  is injective.

Multiplying by the smallest power of  $b^k$  divisible by the denominator of  $\nu$  we obtain an integer  $\mu = \nu b^k$  with the required properties with respect to the map  $a \mapsto \llbracket a \cdot \mu \rrbracket_{k,m}$ .  $\square$

## Acknowledgements

We wish to thank Vincenzo Mantova for the time enjoyed discussing this and related questions. We also thank the reviewers for suggesting relevant references.

## References

- [1] I. Gohberg, V. Olshevsky, Complexity of multiplication with vectors for structured matrices, *Linear Algebra and its Applications* 202 (1994) 163–192.
- [2] P. Kannan, Magic move-bitboard generation in computer chess, Preprint 2007. Available at: [http://www.pradu.us/old/Nov27\\_2008/Buzz/research/magic/Bitboards.pdf](http://www.pradu.us/old/Nov27_2008/Buzz/research/magic/Bitboards.pdf).
- [3] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, New York, 1995.
- [4] Y. Mansour, N. Nisan, P. Tiwari, The computational complexity of universal hashing, in: *Proceedings of the Twenty-Second Annual ACM Symposium on Theory of Computing*, ACM, 2008, pp. 235–243.
- [5] R. Raman, Priority queues: small, monotone and trans-dichotomous, in: *Algorithms*, ESA '96, 1996, pp. 121–137.
- [6] B.E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer-Verlag, 2001.
- [7] M. Sauerhoff, P. Woelfel, Time-space tradeoff lower bounds for integer multiplication and graphs of arithmetic functions, in: *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, ACM, 2003, pp. 186–195.
- [8] S. Tannous, Avoiding rotated bitboards with direct lookup, *ICGA Journal* 30 (2) (2007) 85–91.
- [9] M. Thorup, Even strongly universal hashing is pretty fast, in: *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms*, 2000, pp. 496–497. Society for Industrial and Applied Mathematics.
- [10] M. Thorup, Y. Zhang, Tabulation based 4-universal hashing with applications to second moment estimation, in: *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 2004, pp. 615–624. Society for Industrial and Applied Mathematics.
- [11] P. Woelfel, Efficient strongly universal and optimally universal hashing, *Mathematical Foundations of Computer Science* 1999 (1999) 262–272.