A decomposition of strongly unimodular matrices into incidence matrices of digraphs

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Received 8 June 1988
Revised 22 December 1989

Abstract


A new decomposition of strongly unimodular matrices is given.

1. Introduction—Building up a strongly unimodular matrix

A matrix $A$ is said to be totally unimodular (TU) if the determinant of each of its square submatrices is 0, ±1. A well-known theorem by Ghouila-Houri [3] asserts that a matrix $A$ is TU if and only if each submatrix $A'$ of $A$ has the following property: the rows of $A'$ can be split into two parts so that, for each column, the sums of the entries in each part differ at most by one. A matrix $A$ is strongly unimodular (SU) if (i) $A$ is TU and (ii) every matrix obtained from $A$ by setting a $±1$ entry to 0 is also TU. These matrices are also called 1-TU matrices in [5]. The notation of SU matrices was introduced in [2], where the following equivalent condition (for hypergraphs, i.e. 0, 1-matrices only) was also given: a matrix is SU iff all its bases are triangular. This characterization has been extended to 0, ±1-matrices in [5].
It is convenient to describe a \((0, \pm 1)\)-matrix \(A\) by means of an associated signed graph. The nodes \(r_i, i = 1, \ldots, m\), and \(c_j, j = 1, \ldots, n\), correspond to rows and columns of \(A\) respectively. A pair \(r_i c_j\) forms an edge if the entry \(a_{ij}\) is nonzero, and the weight (=the sign) of the edge \(r_i c_j\) is \(a_{ij}\). The edges weighted by +1, resp. -1, will be called \textit{positive}, resp. \textit{negative}. The weight of a cycle in the graph is defined as the sum of the weights of its edges. Notice that the weight of a cycle must be either 0 (mod 4) or 2 (mod 4) since the graph is bipartite. As usual, we will often not distinguish between a matrix and its graph. In particular, we say that a graph has a certain property if the matrix has it. An important subclass of SU matrices, introduced earlier in [7], are restricted unimodular matrices.

A \((0, \pm 1)\)-matrix \(A\) is called \textit{restricted unimodular} (RU) if the weight of each cycle in the associated graph is 0 (mod 4). It has been shown in [7] that every RU matrix can be built up by a series of certain operations starting from so-called basic RU matrices (see below). Further, it has been shown in [1] that SU matrices can be built up from RU matrices by means of one additional operation. Let us now recall these decompositions.

First, it is obvious that the membership of a matrix to RU and SU classes is invariant under multiplying a row or column by -1. We stress this obvious fact, since a suitable multiplication of a set of rows and columns by -1 can simplify the description of the other operations.

\textbf{Operation 0.} Multiplying a row or a column by -1.

\textbf{Lemma 1.1.} Both RU and SU properties are invariant under operation 0.

\textbf{Operation 1.} Let \(G_1\) and \(G_2\) be two graphs (on disjoint sets of nodes), and \(x_i\) be a node of \(G_i\) of degree at least 2 and such that all edges incident to \(x_i\) are positive, \(i = 1, 2\). Construct \(G\) as follows: Connect each neighbour of \(x_1\) to each neighbour of \(x_2\) by a positive edge, and delete nodes \(x_1\) and \(x_2\).

\textbf{Lemma 1.2 [1].} (i) \textit{If} \(G_1\) and \(G_2\) \textit{are SU, then} \(G\) \textit{constructed by Operation 1 is SU as well.}

(ii) \textit{If} \(G\) \textit{is a SU graph which is not RU, then} \(G\) \textit{can be obtained by Operations 0 and 1 from some SU graphs} \(G_1\) \textit{and} \(G_2\).

\textbf{Operation 2.} Let \(G_1\) and \(G_2\) be two graphs, and let \(x\) and \(y\) be nodes of \(G_1\) and \(G_2\) of degree 2. Assume that \(xx_2, yx_1\), and \(yy_2\) are positive edges, and \(xx_1\) is a negative edge. Construct \(G\) by deleting \(x\) and \(y\), and adding two positive edges \(x_1y_1\) and \(x_2y_2\).

\textbf{Operation 3 (1-\textit{sum}).} Construct \(G\) from \(G_1\) and \(G_2\) by identifying a node \(x_1\) of \(G_1\) with a node \(x_2\) of \(G_2\).
A \((0, \pm 1)\)-matrix is called basic RU if it is TU and if either the matrix or its transpose has exactly two nonzero entries per column. Let us recall a result by Heller and Tompkins [4].

**Lemma 1.3** [4]. Let \(A\) be a \((0, \pm 1)\)-matrix with exactly two nonzero entries per column. Then the following are equivalent:

(i) \(A\) is TU,

(ii) the rows of \(A\) can be split into two classes so that the nonzero entries of each column lie in the same class if and only if they have opposite sign.

It follows immediately from Lemma 1.3 that the notions of RU, SU and TU coincide for matrices with exactly two nonzero entries per column. Up to multiplication of some rows by \(-1\), these matrices are exactly the node–arc incidence matrices of digraphs.

**Lemma 1.4** [7]. (i) If \(G_1\) and \(G_2\) are RU, then a graph \(G\) constructed by Operation 2 or Operation 3 is RU as well.

(ii) Let \(G\) be RU but not basic RU. Then \(G\) can be obtained by means of Operations 0 and 2 or Operation 3 from some RU graphs \(G_1\) and \(G_2\).

Lemmas 1.1, 1.2 and 1.4 together yield a decomposition of SU matrices into basic RU matrices. In this paper, we give another type of decomposition (Theorem 2.2). As a by-product of this result, we prove in the last section that each SU matrix has the so-called ‘on-line balancing property’.

2. Main result

We first establish a lemma, which was proved in [2] for 0,1-matrices only.

**Lemma 2.1.** If \(A\) is an SU matrix, then there exists a non-empty subset \(S\) of rows of \(A\) such that every column of \(A\) with at least 2 nonzero entries has either 0 or 2 nonzero entries in \(S\).

**Proof.** For simplicity, we refer to the property stated in the conclusion of the lemma as ‘Property 2.1.’ Let \(A\) be an SU matrix of size \(m \times n\). We proceed by induction on \(m \times n\). According to the results summarized in the previous section, it is sufficient to prove that Property 2.1 holds for basic RU matrices, and that, if the property holds for two graphs \(G_1, G_2\) and all their induced subgraphs, then it also holds for the graph obtained from \(G_1, G_2\) by Operations 0, 1, 2 or 3.

(i) Basic RU matrices. Property 2.1 trivially holds if \(A\) has two nonzero entries per column. So, assume that \(A\) has two nonzero entries per row. Let us define an auxiliary graph \(H\) as follows: let \(A_1\) be a matrix obtained from \(A\) by
replacing each ‘−1’ by ‘+1’. Then, \( H \) is the graph such that \( A_1 \) is its \( E \times V \) incidence matrix (thus rows of \( A_1 \) correspond to edges of \( H \)). If \( H \) is not connected, then we conclude by induction. Else, if \( H \) has a cycle, then the rows of \( A \) corresponding to the edges of the cycle define a suitable set \( S \). If \( H \) is a tree, then choose a path between two arbitrary leaves of \( H \). Then, the set \( S \) formed by the rows of \( A \) corresponding to the edges of this path satisfies property 2.1.

(ii) Operation 0 preserves Property 2.1. Trivial.

(iii) Operation 2 preserves Property 2.1. Let \( G_1, G_2 \) be as in the definition of operation 2 and \( A_1, A_2 \) be the corresponding matrices, with \( S_1 \) and \( S_2 \) the subsets of rows satisfying Property 2.1. Assume without loss of generality that \( x \) is a column vertex and \( y \) is a row vertex. Now if \( S_1 \) (resp. \( S_2 \)) does not contain both \( x_1 \) and \( x_2 \) (resp. \( y \)) then \( S_1 \) (resp. \( S_2 \)) satisfies Property 2.1 with respect to \( A \). If \( S_1 \) contains both \( x_1 \) and \( x_2 \) and \( S_2 \) contains \( y \) then \( S = S_1 \cup S_2 \) satisfies Property 2.1 for \( A \).

(iv) Operation 3 preserves Property 2.1. With the same notation as above, assume first that both \( x_1 \) and \( x_2 \) are column-vertices.
- If \( x_1 \) (resp. \( x_2 \)) contains either 0 or 2 nonzero entries from \( S_1 \) (resp. \( S_2 \)), then \( S_1 \) (resp. \( S_2 \)) satisfies Property 2.1 with respect to \( A \).
- If \( x_1 \) contains exactly one nonzero entry from \( S_1 \), and \( x_2 \) contains exactly one nonzero entry from \( S_2 \), then \( S = S_1 \cup S_2 \) satisfies the required property with respect to \( A \).

A similar argument works when both \( x_1 \) and \( x_2 \) are row-vertices.

(v) Operation 1 preserves Property 2.1. With the same notation as before, assume without loss of generality that \( x_2 \) is a column of \( A_2 \). Then, \( S_2 \) satisfies the required property with respect to \( A \) (this is because \( x_2 \) has degree at least 2 in \( G_2 \)). This concludes the proof.  

**Theorem 2.2.** If \( A \) is an SU matrix, then there exists a partition \((S_1, \ldots, S_k)\) of the rows of \( A \) with the following properties:

(i) every column of \( A \) has 0, 1 or 2 nonzero entries in each \( S_i \), for \( i = 1, \ldots, k \);

(ii) if a column has exactly one nonzero entry in some \( S_i \), then all its entries in \( S_{i+1}, \ldots, S_k \) are zeroes.

**Proof.** The result follows directly from an iterated application of Lemma 2.1.  

**Remark.** This gives another decomposition of strongly unimodular matrices into (almost) basic RU matrices.
3. Application: On-line balancing

Lovász asked in [6] whether the theorem of Ghouila-Houri mentioned in the first paragraph of this paper can be strengthened as follows.

Let us say that a matrix $A$ is on-line balanced, if there are row-multipliers $s_1, \ldots, s_m \in \{-1, 1\}$ and a permutation $\pi$ of the rows such that

$$\sum_{i=1}^k s_{\pi(i)} a_{\pi(i)j} \leq 1 \quad \text{for every } k = 1, \ldots, m \text{ and } j = 1, \ldots, n$$

(where $m$ and $n$ are the number of rows and columns of $A$ respectively).

Clearly, if each submatrix of a matrix $A$ can be on-line balanced, then $A$ is TU. Lovász asked whether the converse is also true, i.e., whether every TU matrix can be on-line balanced. We here give a positive answer for SU matrices. The question is still open for arbitrary TU matrices.

**Theorem 3.1.** Every strongly unimodular matrix can be on-line balanced.

**Proof.** This follows directly from Theorem 2.2. To get an on-line balancing of $A$, take its rows in the order induced by the partition given in Theorem 2.2, and use the Heller–Tompkins characterization (Lemma 1.3) for each partition class $S_i$, $i = 1, \ldots, k$. $\Box$

**References**