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Fixed point theorems and convergence theorems for some generalized nonexpansive mappings

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Abstract

We introduce some condition on mappings. The condition is weaker than nonexpansiveness and stronger than quasinonexpansiveness. We present fixed point theorems and convergence theorems for mappings satisfying the condition.

Keywords: Nonexpansive mapping; Fixed point; Convergence theorem

1. Introduction

A mapping T on a subset C of a Banach space E is called a *nonexpansive mapping* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty in the case when E is uniformly convex and C is bounded, closed and convex; see Browder [3]. See also [1,2,10,13] and others.

Very recently, in order to characterize the completeness of underlying metric spaces, Suzuki introduced a weaker notion of contractions and proved the following theorem.

Theorem 1. *(See [21].) Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leqslant r \leqslant 2^{-1/2}, \\ (1 + r)^{-1} & \text{if } 2^{-1/2} \leqslant r < 1. \end{cases}$$

Then for a metric space (X, d), the following are equivalent:

(i) *X is complete.*
(ii) *There exists $r \in (0, 1)$ such that every mapping T on X satisfying the following has a fixed point*:
- *$\theta(r)\, d(x, Tx) \leqslant d(x, y)$ implies $d(Tx, Ty) \leqslant r\, d(x, y)$ for all $x, y \in X$.*

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Theorem 1 is meaningful because contractions do not characterize the metric completeness while Caristi and Kannan mappings do; see [4,14,17]. Since $\lim_{r\to 1-0}\theta(r) = 1/2$, it is very natural to consider the following condition.

Definition. Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy *condition* (C) if

$$\text{(C)} \qquad \frac{1}{2}\|x - Tx\| \leqslant \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leqslant \|x - y\|$$

for all $x, y \in C$.

The condition is weaker than nonexpansiveness and stronger than quasinonexpansiveness (see Propositions 1 and 2 below). In this paper, we present fixed point theorems and convergence theorems for mappings satisfying condition (C).

2. Preliminaries

In this section, we give some preliminaries.

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let E be a Banach space. E is said to have the *Opial property* [15] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit z,

$$\liminf_{n\to\infty}\|x_n - z\| < \liminf_{n\to\infty}\|x_n - y\|$$

for all $y \in E$ with $y \neq z$. All Hilbert spaces, all finite dimensional Banach spaces and ℓ^p $(1 \leqslant p < \infty)$ have the Opial property. See also [6,11]. E is said to be *strictly convex* if

$$\|x + y\| < 2$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that E is said to be *uniformly convex in every direction* (*UCED*, for short) if for $\varepsilon \in (0, 2]$ and $z \in E$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that

$$\|x + y\| \leqslant 2\big(1 - \delta(\varepsilon, z)\big)$$

for all $x, y \in E$ with $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and $x - y \in \{tz\colon t \in [-2, -\varepsilon] \cup [+\varepsilon, +2]\}$. E is said to be *uniformly convex* if E is UCED and

$$\inf\big\{\delta(\varepsilon, z)\colon \|z\| = 1\big\} > 0$$

for all $\varepsilon \in (0, 2]$. It is obvious that uniform convexity implies UCED, and UCED implies strictly convexity. We know that every separable Banach space can be equivalently renormed so that it is UCED. See [9,16] and others.

UCED is characterized as follows:

Lemma 1. *(See [16].) For a Banach space E, the following are equivalent*:

(i) *E is UCED.*
(ii) *If sequences $\{u_n\}$ and $\{v_n\}$ in E satisfy $\lim_n\|u_n\| = 1$, $\lim_n\|v_n\| = 1$, $\lim_n\|u_n + v_n\| = 2$ and $\{u_n - v_n\} \subset \{tw\colon t \in \mathbb{R}\}$ for some $w \in E$ with $\|w\| = 1$, then $\lim_n\|u_n - v_n\| = 0$ holds.*

Using Lemma 1, we can prove the following.

Lemma 2. *For a Banach space E, the following are equivalent*:

(i) *E is UCED.*
(ii) *If $\{x_n\}$ is a bounded sequence in E, then a function f on E defined by $f(x) = \limsup_n\|x_n - x\|$ is strictly quasiconvex, that is,*

$$f\big(\lambda x + (1 - \lambda)y\big) < \max\big\{f(x), f(y)\big\} \tag{1}$$

for all $\lambda \in (0, 1)$ and $x, y \in E$ with $x \neq y$.

Proof. We first show that (i) implies (ii). We assume (i). Fix $\lambda \in (0,1)$ and $x, y \in E$ with $x \neq y$. We have

$$f\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda f(x) + (1-\lambda) f(y) \tag{2}$$

because f is convex. We consider the following three cases:

- $f(x) \neq f(y)$;
- $f(x) = f(y)$ and $\lambda = 1/2$;
- $f(x) = f(y)$ and $\lambda \neq 1/2$.

In the first case, we have

$$\lambda f(x) + (1-\lambda) f(y) < \max\big\{f(x), f(y)\big\}.$$

(1) follows from this and (2). In the second case, arguing by contradiction, we assume that (1) does not hold. Then from (2), we have

$$f\big((1/2)x + (1/2)y\big) = f(x) = f(y) =: \alpha.$$

We choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $\lim_j \|x_{n_j} - (1/2)x - (1/2)y\| = \alpha$. Since

$$\begin{aligned}
\alpha &= \lim_{j\to\infty} \big\|x_{n_j} - (1/2)x - (1/2)y\big\| \\
&\leqslant \liminf_{j\to\infty} \big((1/2)\|x_{n_j} - x\| + (1/2)\|x_{n_j} - y\|\big) \\
&\leqslant (1/2)\liminf_{j\to\infty} \|x_{n_j} - x\| + (1/2)\limsup_{j\to\infty} \|x_{n_j} - y\| \\
&\leqslant (1/2)\limsup_{j\to\infty} \|x_{n_j} - x\| + (1/2)\limsup_{j\to\infty} \|x_{n_j} - y\| \\
&\leqslant (1/2) f(x) + (1/2) f(y) = \alpha,
\end{aligned}$$

we have $\lim_j \|x_{n_j} - x\| = \alpha$. Similarly we can prove $\lim_j \|x_{n_j} - y\| = \alpha$. It follows from $x \neq y$ that $\alpha > 0$. Put $u_j = (x_{n_j} - x)/\alpha$ and $v_j = (x_{n_j} - y)/\alpha$. Then we have $\lim_j \|u_j\| = 1$, $\lim_j \|v_j\| = 1$, $\lim_j \|u_j + v_j\| = 2$ and $u_j - v_j = (y - x)/\alpha$ for $j \in \mathbb{N}$. However, $\lim_n \|u_n - v_n\| = (y-x)/\alpha \neq 0$, which contradicts Lemma 1. Therefore (1) holds. In the third case, if $0 < \lambda < 1/2$, then we have

$$\begin{aligned}
f\big(\lambda x + (1-\lambda)y\big) &= f\big(2\lambda\big((1/2)x + (1/2)y\big) + (1-2\lambda)y\big) \\
&\leqslant 2\lambda f\big((1/2)x + (1/2)y\big) + (1-2\lambda) f(y) \\
&< 2\lambda \max\big\{f(x), f(y)\big\} + (1-2\lambda) f(y) \\
&= \max\big\{f(x), f(y)\big\}.
\end{aligned}$$

Similarly we can prove (1) in the case of $1/2 < \lambda < 1$. We next show that (ii) implies (i). We assume (ii). We suppose $\{u_n\}$ and $\{v_n\}$ are sequences in E, $w \in E$ and $\{t_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_n \|u_n\| = 1$, $\lim_n \|v_n\| = 1$, $\lim_n \|u_n + v_n\| = 2$, $\|w\| = 1$ and $u_n - v_n = t_n w$ for $n \in \mathbb{N}$. Arguing by contradiction, we assume that $\limsup_n \|u_n - v_n\| > 0$ holds. Then since $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $\{t_{n_j}\}$ converges to some $\tau \neq 0$. Define a continuous convex function f on E by

$$f(x) = \limsup_{j\to\infty} \|v_{n_j} - x\|.$$

It is obvious that

$$\begin{aligned}
&f(0) = 1, \\
&f(-\tau w) = \limsup_{j\to\infty} \|v_{n_j} + \tau w\| = \limsup_{j\to\infty} \|u_{n_j} - t_{n_j} w + \tau w\| = 1 \quad \text{and} \\
&f\big((1/2)0 + (1/2)(-\tau w)\big) = (1/2)\limsup_{j\to\infty} \|v_{n_j} + v_{n_j} + \tau w\| = (1/2)\limsup_{j\to\infty} \|u_{n_j} - t_{n_j} w + v_{n_j} + \tau w\| = 1.
\end{aligned}$$

Thus, f is not strictly quasiconvex, which contradicts (ii). Therefore we obtain $\lim_n \|u_n - v_n\| = 0$. By Lemma 1, E is UCED. $\square$

The following lemma is proved by Goebel and Kirk. See also [18–20].

Lemma 3. *(See Goebel and Kirk [8].) Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let λ belong to $(0,1)$. Suppose that $z_{n+1} = \lambda w_n + (1-\lambda) z_n$ and $\|w_{n+1} - w_n\| \leqslant \|z_{n+1} - z_n\|$ for all $n \in \mathbb{N}$. Then $\lim_n \|w_n - z_n\| = 0$.*

3. Basic properties

In this section, we discuss basic properties on condition (C).

We recall that a mapping T on a subset C of a Banach space E is called *quasinonexpansive* [5] if $\|Tx - z\| \leqslant \|x - z\|$ for all $x \in C$ and $z \in F(T)$. All nonexpansive mappings with a fixed point are quasinonexpansive.

The following propositions are obvious.

Proposition 1. *Every nonexpansive mapping satisfies condition* (C).

Proposition 2. *Assume that a mapping T satisfies condition* (C) *and has a fixed point. Then T is a quasinonexpansive mapping.*

Proof. Fix $z \in F(T)$ and $x \in C$. Since $(1/2)\|z - Tz\| = 0 \leqslant \|z - x\|$, we have $\|z - Tx\| = \|Tz - Tx\| \leqslant \|z - x\|$. $\square$

Example 1. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0 & \text{if } x \neq 3, \\ 1 & \text{if } x = 3. \end{cases}$$

Then T satisfies condition (C), but T is not nonexpansive.

Proof. If $x < y$ and $(x,y) \in ([0,3] \times [0,3]) \setminus ((2,3) \times \{3\})$, then $\|Tx - Ty\| \leqslant \|x - y\|$ holds. If $x \in (2,3)$ and $y = 3$, then

$$(1/2)\|x - Tx\| = x/2 > 1 > \|x - y\| \quad \text{and} \quad (1/2)\|y - Ty\| = 1 > \|x - y\|$$

hold. Thus, T satisfies condition (C). However, since T is not continuous, T is not nonexpansive. $\square$

Example 2. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0 & \text{if } x \neq 3, \\ 2 & \text{if } x = 3. \end{cases}$$

Then $F(T) \neq \emptyset$ and T is quasinonexpansive, but T does not satisfy condition (C).

Proof. It is obvious that $F(T) = \{0\} \neq \emptyset$ and T is quasinonexpansive. However, since

$$(1/2)\|3 - T3\| = 1/2 \leqslant 1 = \|3 - 2\| \quad \text{and} \quad \|T3 - T2\| = 2 > 1 = \|3 - 2\|$$

hold. Thus, T does not satisfy condition (C). $\square$

From the definition, we can prove the following lemmas.

Lemma 4. *Let T be a mapping on a closed subset C of a Banach space E. Assume that T satisfies condition* (C). *Then $F(T)$ is closed. Moreover, if E is strictly convex and C is convex, then $F(T)$ is also convex.*

Proof. Let $\{z_n\}$ be a sequence in $F(T)$ converging to some point $z \in C$. Since $(1/2)\|z_n - Tz_n\| = 0 \leqslant \|z_n - z\|$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{n\to\infty} \|z_n - Tz\| &= \limsup_{n\to\infty} \|Tz_n - Tz\| \\ &\leqslant \limsup_{n\to\infty} \|z_n - z\| = 0.\end{aligned}$$

That is, $\{z_n\}$ converges to Tz. This implies $Tz = z$. Therefore $F(T)$ is closed. Next, we assume that E is strictly convex and C is convex. We fix $\lambda \in (0, 1)$ and $x, y \in F(T)$ with $x \neq y$, and put $z := \lambda x + (1 - \lambda)y \in C$. Then we have

$$\begin{aligned}\|x - y\| &\leqslant \|x - Tz\| + \|y - Tz\| = \|Tx - Tz\| + \|Ty - Tz\| \\ &\leqslant \|x - z\| + \|y - z\| = \|x - y\|.\end{aligned}$$

From the strict convexity of E, there exists $\mu \in [0, 1]$ such that $Tz = \mu x + (1 - \mu)y$. Since

$$(1 - \mu)\|x - y\| = \|Tx - Tz\| \leqslant \|x - z\| = (1 - \lambda)\|x - y\|$$

and

$$\mu\|x - y\| = \|Ty - Tz\| \leqslant \|y - z\| = \lambda\|x - y\|,$$

we have $1 - \mu \leqslant 1 - \lambda$ and $\mu \leqslant \lambda$. These imply $\lambda = \mu$. Therefore we obtain $z \in F(T)$. □

Lemma 5. *Let T be a mapping on a subset C of a Banach space E. Assume that T satisfies condition* (C)*. Then for $x, y \in C$, the following hold*:

(i) $\|Tx - T^2x\| \leqslant \|x - Tx\|$.
(ii) *Either* $(1/2)\|x - Tx\| \leqslant \|x - y\|$ *or* $(1/2)\|Tx - T^2x\| \leqslant \|Tx - y\|$ *holds.*
(iii) *Either* $\|Tx - Ty\| \leqslant \|x - y\|$ *or* $\|T^2x - Ty\| \leqslant \|Tx - y\|$ *holds.*

Proof. (i) Follows from $(1/2)\|x - Tx\| \leqslant \|x - Tx\|$. (iii) Follows from (ii). Let us prove (ii). Arguing by contradiction, we assume that

$$(1/2)\|x - Tx\| > \|x - y\| \quad \text{and} \quad (1/2)\big\|Tx - T^2x\big\| > \|Tx - y\|.$$

Then we have by (i)

$$\begin{aligned}\|x - Tx\| &\leqslant \|x - y\| + \|Tx - y\| \\ &< (1/2)\|x - Tx\| + (1/2)\big\|Tx - T^2x\big\| \\ &\leqslant \|x - Tx\|.\end{aligned}$$

This is a contradiction. Therefore we obtain the desired result. □

4. Convergence theorems

In this section, we give two convergence theorems for mappings with condition (C). We first prove the following lemmas, which play very important roles in this paper.

Lemma 6. *Let T be a mapping on a bounded convex subset C of a Banach space E. Assume that T satisfies condition* (C)*. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \lambda Tx_n + (1 - \lambda)x_n$$

for $n \in \mathbb{N}$, where λ is a real number belonging to $[1/2, 1)$*. Then*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0$$

holds.

Proof. From the assumption, we have

$$\frac{1}{2}\|x_n - Tx_n\| \leqslant \lambda\|x_n - Tx_n\| = \|x_n - x_{n+1}\|$$

for $n \in \mathbb{N}$. Hence

$$\|Tx_n - Tx_{n+1}\| \leqslant \|x_n - x_{n+1}\|$$

holds for $n \in \mathbb{N}$. So, by Lemma 3, we obtain the desired result. □

Lemma 7. *Let T be a mapping on a subset C of a Banach space E. Assume that T satisfies condition* (C). *Then*

$$\|x - Ty\| \leqslant 3\|Tx - x\| + \|x - y\|$$

holds for all $x, y \in C$.

Proof. By Lemma 5, either

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \text{or} \quad \left\|T^2x - Ty\right\| \leqslant \|Tx - y\|$$

holds. In the first case, we have

$$\|x - Ty\| \leqslant \|x - Tx\| + \|Tx - Ty\| \leqslant \|x - Tx\| + \|x - y\|.$$

In the second case, we have by Lemma 5

$$\begin{aligned}\|x - Ty\| &\leqslant \|x - Tx\| + \left\|Tx - T^2x\right\| + \left\|T^2x - Ty\right\| \\ &\leqslant 2\|x - Tx\| + \|Tx - y\| \\ &\leqslant 3\|x - Tx\| + \|x - y\|.\end{aligned}$$

Therefore we obtain the desired result in both cases. □

Using the above two lemmas, we can prove the following, which is connected with Ishikawa's convergence theorem [12].

Theorem 2. *Let T be a mapping on a compact convex subset C of a Banach space E. Assume that T satisfies condition* (C). *Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \lambda Tx_n + (1 - \lambda)x_n$$

for $n \in \mathbb{N}$, where λ is a real number belonging to $[1/2, 1)$. *Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 6, we have $\lim_n \|Tx_n - x_n\| = 0$. Since C is compact, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in C$ such that $\{x_{n_j}\}$ converges to z. By Lemma 7, we have

$$\|x_{n_j} - Tz\| \leqslant 3\|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - z\|$$

for all $j \in \mathbb{N}$. Therefore $\{x_{n_j}\}$ converges to Tz. This implies $Tz = z$. That is, z is a fixed point of T. By Proposition 2, we have

$$\|x_{n+1} - z\| \leqslant \lambda\|Tx_n - z\| + (1 - \lambda)\|x_n - z\| \leqslant \|x_n - z\|$$

for $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges to z. □

We next prove a convergence theorem connected with Edelstein and O'Brien's [7]. Before proving it, we give the following proposition.

Proposition 3. *Let T be a mapping on a subset C of a Banach space E with the Opial property. Assume that T satisfies condition* (C). *If $\{x_n\}$ converges weakly to z and* $\lim_n \|Tx_n - x_n\| = 0$, *then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Proof. By Lemma 7, we have

$$\|x_n - Tz\| \leqslant 3\|Tx_n - x_n\| + \|x_n - z\|$$

for $n \in \mathbb{N}$ and hence

$$\liminf_{n\to\infty} \|x_n - Tz\| \leqslant \liminf_{n\to\infty} \|x_n - z\|.$$

From the Opial property, we obtain $Tz = z$. $\square$

Theorem 3. *Let T be a mapping on a weakly compact convex subset C of a Banach space E with the Opial property. Assume that T satisfies condition* (C). *Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \lambda T x_n + (1 - \lambda) x_n$$

for $n \in \mathbb{N}$, where λ is a real number belonging to $[1/2, 1)$. *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By Lemma 6, we have $\lim_n \|Tx_n - x_n\| = 0$. Since C is weakly compact, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in C$ such that $\{x_{n_j}\}$ converges weakly to z. By Proposition 3, we have z is a fixed point of T. As in the proof of Theorem 2, we can prove $\{\|x_n - z\|\}$ is a nonincreasing sequence. Arguing by contradiction, assume that $\{x_n\}$ does not converge to z. Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $w \in C$ such that $\{x_{n_k}\}$ converges weakly to w and $z \neq w$. We note $Tw = w$. From the Opial property,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z\| &= \lim_{j\to\infty} \|x_{n_j} - z\| < \lim_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\| \\ &= \lim_{k\to\infty} \|x_{n_k} - w\| < \lim_{k\to\infty} \|x_{n_k} - z\| = \lim_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. We obtain the desired result. $\square$

5. Existence theorems

In this section, we prove existence theorems of fixed points of mappings with condition (C). The following theorem directly follows from Theorems 2 and 3.

Theorem 4. *Let T be a mapping on a convex subset C of a Banach space E. Assume that T satisfies condition* (C). *Assume also that either of the following holds*:

- *C is compact*;
- *C is weakly compact and E has the Opial property.*

Then T has a fixed point.

We generalize a fixed point theorem due to Browder [3] and Göhde [10].

Theorem 5. *Let C be a weakly compact convex subset of a UCED Banach space E. Let T be a mapping on C. Assume that T satisfies condition* (C). *Then T has a fixed point.*

Proof. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = (1/2)Tx_n + (1/2)x_n$ for $n \in \mathbb{N}$. Then by Lemma 6, $\limsup_n \|Tx_n - x_n\| = 0$ holds. Define a continuous convex function f from C into $[0, \infty)$ by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|$$

for all $x \in C$. Since C is weakly compact and f is weakly lower semicontinuous, there exists $z \in C$ such that

$$f(z) = \min\{f(x) \colon x \in C\}.$$

Since

$$\|x_n - Tz\| \leqslant 3\|Tx_n - x_n\| + \|x_n - z\|$$

by Lemma 7, we have $f(Tz) \leqslant f(z)$. Since $f(z)$ is the minimum, $f(Tz) = f(z)$ holds. If $Tz \neq z$, then since f is strictly quasiconvex, we have

$$f(z) \leqslant f\left(\frac{z+Tz}{2}\right) < \max\{f(z), f(Tz)\} = f(z).$$

This is a contradiction. Hence $Tz = z$. $\square$

We finally prove the existence of common fixed points for families of mappings.

Theorem 6. *Let C be a weakly compact convex subset of a UCED Banach space E. Let S be a family of commuting mappings on C satisfying condition* (C). *Then S has a common fixed point.*

Proof. Let $T_1, T_2, \ldots, T_\ell \in S$. By Theorem 5, T_1 has a fixed point in C, that is, $F(T_1) \neq \emptyset$. By Lemma 4, $F(T_1)$ is closed and convex. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty, closed and convex for some $k \in \mathbb{N}$ with $1 < k \leqslant \ell$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \leqslant j < k$, since $T_k \circ T_j = T_j \circ T_k$, we have

$$T_k x = T_k \circ T_j x = T_j \circ T_k x,$$

thus $T_k x$ is a fixed point of T_j, which implies $T_k x \in A$. Therefore we obtain $T_k(A) \subset A$. By Theorem 5, T_k has a fixed point in A, that is,

$$A \cap F(T_k) = \bigcap_{j=1}^{k} F(T_j) \neq \emptyset.$$

Also, the set is closed and convex by Lemma 4. By induction, we obtain $\bigcap_{j=1}^{\ell} F(T_j) \neq \emptyset$. In other words, $\{F(T)\colon T \in S\}$ has the finite intersection property. Since C is weakly compact and $F(T)$ is weakly closed for every $T \in S$, we have $\bigcap_{T \in S} F(T) \neq \emptyset$. $\square$

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