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Duality Theory for Grothendieck Categories and Linearly Compact Rings

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INTRODUCTION

The problem of this paper is to give concrete descriptions of the dual categories of Grothendieck categories, and also to construct explicit duality functors. A short, but rather unprecise description of the main results of this paper is the following

THEOREM A. *A category $\mathfrak{A}$ is a Grothendieck category iff it is dual to the category* $\mathrm{STC}(R)$ *of strict complete topologically coherent R-left modules over some strict complete topologically left coherent and linearly compact ring R.* ||

Here a topological R-left module X is called topologically coherent if it admits a basis of neighborhoods of 0 consisting of submodules X' such that X/X' is coherent in the category of discrete topological R-modules.

The main new definition is that of linear compactness (more general than in the literature): An R-module X is called algebraically linearly compact if for each decreasing family $(X_i\ ;\ i \in I)$, I directed, of finitely generated submodules X_i of X the intersection $\bigcap_i X_i$ is again finitely generated and the canonical map $X \to \lim X/X_i$ is surjective. A topological R-left module X is called topologically linearly compact if it has a basis of neighborhoods of 0 consisting of submodules X' such that X/X' is algebraically linearly compact.

A complete topologically coherent R-module is called strict (Jan-Erik Roos suggests the term "proper") if each topologically coherent closed submodule Y of X, such that $(X/Y)_{\mathrm{dis}}$ (X/Y with the discrete topology) is coherent, is open. Strictness is a notion of a technical nature which is necessary to insure that the relevant categories of topological modules are Abelian.

As a consequence of Theorem A one sees that a category $\mathfrak{A}$ is a locally noetherian Grothendieck category iff it is dual to the category of complete topologically coherent R-left modules over some complete topologically left coherent and coperfect ring R. An R-left module X is called (algebraically) left *coperfect* if it satisfies the descending chain condition on finitely generated submodules. It is easy to see that in this case all complete topologically coherent R-modules are strict. The above described important special case of Theorem A has been proven by Jan-Erik Roos in [11], and indeed Roos' paper inspired me to the more comprehensive result of this paper.

The first two sections of this paper contain preliminary material on Grothendieck categories and topological modules. The main references for these are the paper by J.-E. Roos (*loc. cit.*) and Peter Gabriel's thèse ([5]). Indeed both Roos' and my results are generalizations of the duality theory for locally finite Grothendieck categories in [5].

The third and fourth sections contain the first main theorem of this paper. Let $\mathfrak{A}$ be a Grothendieck category. Choose a full and skeletal-small subcategory $\mathfrak{N}$ of $\mathfrak{A}$ which generates $\mathfrak{A}$ and such that subobjects, quotient objects, and finite coproducts of objects in $\mathfrak{N}$ are again in $\mathfrak{N}$. Choose an injective cogenerator E of $\mathfrak{A}$ such that each $N \in \mathfrak{N}$ can be embedded into some E^k, $k \in \mathbb{N}$. The groups $\mathfrak{A}(A, E)$, $A \in \mathfrak{A}$, are left $\mathfrak{A}(E, E)$-modules in the natural manner. On $\mathfrak{A}(A, E)$ there is a unique topology [the $(\mathfrak{N}, E)$-topology] which makes $\mathfrak{A}(A, E)$ a topological group and has the subgroups $\mathfrak{A}(A/N, E)$ $N \subseteq A$, $N \in \mathfrak{N}$, as basis of neighborhoods of 0.

THEOREM B. *$\mathfrak{A}$, $\mathfrak{N}$, E, topologies as above.*

(1) *The ring $\mathfrak{A}(E, E)$ is a strict complete topologically left coherent and linearly compact ring.*

(2) *The functor $A \rightsquigarrow \mathfrak{A}(A, E)$ defines an equivalence*

$$\mathfrak{A}^{op} \to \mathrm{STC}(\mathfrak{A}(E, E)).$$

Theorem B proves the first half of theorem A in a precise manner, in particular a concrete duality functor is given.

The fifth and sixth sections prove the other half of Theorem A, namely

THEOREM C. *Let R be topologically left linearly compact ring. The category $\mathrm{STC}(R)^{op}$ is a Grothendieck category.*

In Section 5 I describe a more general procedure on how to construct co-Grothendieck categories from linearly compact categories of modules.

In Section 7 I prove a topological Morita theorem describing the relation between two strict complete topologically coherent and linearly compact rings R and S if the categories $\mathrm{STC}(R)$ and $\mathrm{STC}(S)$ are equivalent.

Section 8 is devoted to examples. I develop duality theories for module categories and spectral categories ([6]). In particular, I characterize by leftsided conditions those rings R which are right self-injective and have the property that each right ideal is the annihilator of finitely many elements in R (generalized right quasi-Frobenius rings, see e.g., [10]).

The last Section describes a representation of Grothendieck categories by functor categories.

It seems to me that the theorems B and C will have many other applications, e.g., in the theory of modules and rings.

Notation. I use the following abbreviations:

(1) iff = if and only if

(2) w.l.o.g. = without loss of generality

(3) w.r.t. = with respect to

(4) lim = inverse limit, limit

(5) colim = direct limit, colimit

(6) || denotes the end of a proof

(7) ker resp. coker denote kernel resp. cokernel of a morphism.

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I. Preliminary Notions for Categories

This paragraph serves as a dictionary for the later sections of this paper. I have collected several notions which have been defined and exploited by several authors, have been commonly used, but never appeared in a book. I also prove several easy propositions which are used later on.

Skeletal-small categories. A category $\mathfrak{N}$ is called *skeletal-small* if it is equivalent to a small category, i.e., if the skeleton $\mathrm{Sk}(\mathfrak{N})$ of $\mathfrak{N}$ is a set and not just a class. The skeleton of $\mathfrak{N}$ is a chosen representative system of the isomorphism classes of $\mathfrak{N}$. I shall use skeletal-small categories like small categories. In particular, if $\mathfrak{N}$ is a skeletal-small and $\mathfrak{A}$ any category one obtains the category $\mathfrak{A}^{\mathfrak{N}}$ of all functors form $\mathfrak{N}$ to $\mathfrak{A}$.

Cofinal functors. Let $F : X \to Y$ be a functor from a small (or skeletal-small) category X to a category Y. The functor F is called *cofinal* if the following relations hold: If $G : Y \to A$ is another functor, then $\mathrm{colim}_Y G$

exists iff $\operatorname{colim}_X GF$ exists, and if both these colimits exist the canonical morphism

$$\operatorname{colim}_X GF \to \operatorname{colim}_Y G$$

is an isomorphism.

It is known (for a proof see e.g., [9], Lemma 2.1) that a functor $F : X \to Y$ is cofinal iff all fibers y/F, $y \in Y$, of F are connected. The fiber y/F has the objects (x, f) where $x \in X$ and $f : y \to Fx$, and the corresponding morphisms. A category C is called *connected* if it is not empty and if any two objects c and d of C can be joined by a sequence of morphisms

$$c = c_0 \to c_1 \leftarrow c_2 \to c_3 \cdots \leftarrow c_n = d$$

in C.

A small (or skeletal-small) full subcategory X of Y is called cofinal if the injection functor is cofinal.

A category Y is called *filtered from above* or *directed* (*upwards*) if it satisfies the following three conditions:

(1) Y is not empty.

(2) For each $y_1, y_2 \in Y$ there is a $y \in Y$ with $Y(y_1, y) \neq \emptyset \neq Y(y_1, y)$.

(3) Each diagram $y_1 \rightrightarrows y_2$ in Y can be completed to a commutative diagram

$$y_1 \rightrightarrows y_2 \to y \text{ in } Y.$$

LEMMA 1.1. *Let X be a small, full subcategory of a directed category Y. Then X is cofinal in Y iff for each $y \in Y$ there is an $x \in X$ and a morphism $y \to x$. If X is cofinal in Y then X is also directed.*

Proof. The conditions mean that the categories y/X, $y \in Y$, are nonempty. It is easy to see that they are indeed connected since Y is filtered from above. ||

The dual notion of cofinal is coinitial.

Codense functors. A functor $F : X \to Y$ is called *codense* ([12], p. 80) if for each $y \in Y$ the relation

$$y = \operatorname{colim}_{F/y} P$$

holds where $P : F/y \to Y : (Fx \xrightarrow{f} y) \rightsquigarrow Fx$ is the canonical projection from F/y to Y. The universal cone from P to y is given by

$$P(Fx \xrightarrow{f} y) = Fx \xrightarrow{f} y.$$

If X is skeletal-small a functor $F : X \to Y$ is codense iff the functor

$$Y \to \mathrm{Set}^{X^{op}} : y \rightsquigarrow Y(-, y)|_X$$

is a full embedding (see e.g., [12], Lemma 1.7). This result is originally due to Lambek. The dual notion of codense is dense.

Grothendieck categories. A *Grothendieck category* is an Abelian category $\mathfrak{A}$ which admits arbitrary coproducts, has a family of generators and satisfies the following condition (AB5)([7], ch. 1): If $A \in \mathfrak{A}$, if A' is a subobject of A, and if $(A_i\,;\, i \in I)$, I directed, is an increasing family of subobjects of A then

$$\bigcup_{i\in I} (A' \cap A_i) = A' \cap \Big(\bigcup_{i\in I} A_i\Big).$$

I use the symbols $\bigcup$ resp. $\bigcap$ to denote the supremum resp. infimum in a lattice. The condition (AB5) on $\mathfrak{A}$ is equivalent with the exactness of the colimit functors

$$\operatorname{colim}_X : \mathfrak{A}^X \to \mathfrak{A}$$

where X is a directed small category ([7], ch. 1).

Finitely closed generating subcategories of Grothendieck categories. A class of objects (= full subcategory) $\mathfrak{N}$ of an Abelian category $\mathfrak{A}$ with coproducts is said to generate $\mathfrak{A}$ if for each $A \in \mathfrak{A}$ there is a family $(N_i\,;\, i \in I)$ of objects of $\mathfrak{N}$ and an exact sequence

$$\coprod (N_i\,;\, i \in I) \to A \to 0.$$

If $\mathfrak{N}$ is skeletal-small then $\mathfrak{N}$ generates $\mathfrak{A}$ iff the skeleton of $\mathfrak{N}$ is a set of generators of $\mathfrak{A}$.

A full subcategory $\mathfrak{N}$ of an Abelian category $\mathfrak{A}$ is called *finitely closed* if the subobjects, quotient objects, and finite coproducts of objects in $\mathfrak{N}$ are again in $\mathfrak{N}$. If $\mathfrak{N}$ is a finitely closed subcategory of the Abelian category $\mathfrak{A}$ then $\mathfrak{N}$ is Abelian and the injection $\mathfrak{N} \to \mathfrak{A}$ is exact.

LEMMA 1.2. *Let $\mathfrak{A}$ be a Grothendieck category and $\mathfrak{N}$ a full, skeletal-small, and finitely closed subcategory of $\mathfrak{A}$. The following assertions are equivalent*:

(1) *$\mathfrak{N}$ generates $\mathfrak{A}$.*

(2) *$\mathfrak{N}$ is codense in $\mathfrak{A}$.*

If (1) and (2) are satisfied and $A \in \mathfrak{A}$ then the category $\mathfrak{N}/A$ is filtered from above and the full subcategory $(\mathfrak{N}/A)' := \{N \subseteq A,\, N \in \mathfrak{N}\}$ of $\mathfrak{N}/A$ is a small, cofinal subcategory of $\mathfrak{N}/A$.

Proof. (2) $\Rightarrow$ (1): This implication is obvious because of the canonical epimorphisms

$$\coprod (N;\, N \in \operatorname{Sk}(\mathfrak{N}),\, (N \to A) \in \mathfrak{N}/A) \to \operatorname{colim}_{\mathfrak{N}/A} N \cong A.$$

Here $A \in \mathfrak{A}$, and $\operatorname{Sk}(\mathfrak{N})$ denotes the skeleton of $\mathfrak{N}$.

(1) $\Rightarrow$ (2): (a) A straight forward calculation shows that $\mathfrak{N}/A$ is filtered from above for each $A \in \mathfrak{A}$ whenever $\mathfrak{N}$ is a full, finitely closed subcategory of $\mathfrak{A}$.

(b) By Lemma 1.1 the category $(\mathfrak{N}/A)'$ is cofinal in $\mathfrak{N}/A$. For if $N \xrightarrow{f} A$ is any object in $\mathfrak{N}/A$, then f factorizes as

$$N \xrightarrow{f'} \operatorname{Im} f \longrightarrow A,$$

and f' is a morphism in $\mathfrak{N}/A$ from

$$(N \xrightarrow{f} A) \quad \text{to} \quad (\operatorname{Im} f \longrightarrow A) \in (\mathfrak{N}/A)'.$$

(c) Since $\mathfrak{N}$ generates $\mathfrak{A}$ and is finitely closed one has

$$A = \bigcup (N; (N \subseteq A) \in (\mathfrak{N}/A)').$$

But $(\mathfrak{N}/A)'$ is filtered from above, hence $A \cong \operatorname{colim}_{(\mathfrak{N}/A)'} N$. Since $(\mathfrak{N}/A)'$ is cofinal in $\mathfrak{N}/A$ there results

$$A \cong \operatorname{colim}_{\mathfrak{N}/A} N, \qquad \text{i.e. } \mathfrak{N} \text{ is codense in } \mathfrak{A}. \quad \|$$

Lemma 1.3. *Let $\mathfrak{A}$ be a Grothendieck category and $\mathfrak{N}_0$ a class of objects of $\mathfrak{A}$. Then there is a smallest full and finitely closed subcategory $\langle \mathfrak{N}_0 \rangle$ of $\mathfrak{A}$ containing $\mathfrak{N}_0$. If $\mathfrak{N}_0$ is skeletal-small so is $\langle \mathfrak{N}_0 \rangle$.*

Proof. By induction I define a sequence of full subcategories $\mathfrak{N}_n$ of $\mathfrak{A}$. For $n = 0$ one starts with the given class $\mathfrak{N}_0$. If $n \geqslant 1$, then $\mathfrak{N}_n$ is the full subcategory of $\mathfrak{A}$ consisting of all subobjects, quotient objects, or finite coproducts of objects in $\mathfrak{N}_{n-1}$. Let $\langle \mathfrak{N}_0 \rangle := \bigcup_n \mathfrak{N}_n$. Obviously $\langle \mathfrak{N}_0 \rangle$ is the smallest full, finitely closed subcategory of $\mathfrak{A}$ containing $\mathfrak{N}_0$. If $\mathfrak{N}_0$ is skeletal-small, then so is $\langle \mathfrak{N}_0 \rangle$. This follows from the fact that in a Grothendieck category the subobjects of a given object form a set. $\|$

As a corollary one obtains the

Proposition 1.4. *Let $\mathfrak{A}$ be a Grothendieck category and $\mathfrak{G}$ a set of generators of $\mathfrak{A}$. Then $\langle \mathfrak{G} \rangle$ is a skeletal-small, full, finitely closed and generating subcategory of $\mathfrak{A}$.* $\|$

Proposition 1.5. *If $\mathfrak{A}$ is a Grothendieck category then there are a full, skeletal-small, finitely closed and generating subcategory $\mathfrak{N}$ of $\mathfrak{A}$ and an injective cogenerator E of $\mathfrak{A}$ such that for each $N \in \mathfrak{N}$ there is a short exact sequence*

$$0 \to N \to E^k, \qquad k \in \mathbb{N}.$$

Proof. Take $\mathfrak{N} := \langle \mathfrak{G} \rangle$ as in the preceding proposition, and for E an injective envelope of $\coprod (N; N \in \mathrm{Sk}(\mathfrak{N}))$. ||

Given a Grothendieck category $\mathfrak{A}$ there are in general many pairs $(\mathfrak{N}, E)$ of a full, skeletal-small, finitely closed and generating subcategory $\mathfrak{N}$ of $\mathfrak{A}$ and an injective cogenerator E such that each $N \in \mathfrak{N}$ can be embedded into some E^k, $k \in \mathbb{N}$. For special $\mathfrak{A}$'s however, special pairs $(\mathfrak{N}, E)$ are naturally associated with $\mathfrak{A}$.

EXAMPLES 1.6. (1) If $\mathfrak{A}$ is a locally noetherian category, i.e., a Grothendieck category with a family of noetherian generators, one can take as $\mathfrak{N}$ the category of all noetherian objects in $\mathfrak{A}$. An appropriate choice for E is the coproduct of a representative system of indecomposable injective objects in $\mathfrak{A}$. (Compare [11], Section 3, Theorem 4).

(2) Let $\mathfrak{A}$ be a spectral category, i.e., a Grothendieck category in which every morphism splits ([6], 1.4), and let E be a generator of $\mathfrak{A}$. Then E is also an injective cogenerator of $\mathfrak{A}$. The full subcategory $\mathfrak{N}$ of $\mathfrak{A}$ of all subobjects (= direct factors) of some E^k, $k \in \mathbb{N}$, is skeletal-small, finitely closed and generates $\mathfrak{A}$. The pair $(\mathfrak{N}, E)$ satisfies the conditions of Proposition 1.5.

(3) Let R be a ring. The module R_R is a generator of mod R_R, and $\langle R_R \rangle =: \mathfrak{N}$ is the full subcategory of mod R_R of all submodules of finitely generated modules. This $\mathfrak{N}$ is a full, skeletal-small, finitely closed and generating subcategory of mod R_R. The following proposition answers for which injectives E the pair $(\mathfrak{N}, E)$ satisfies the conditions of Proposition 1.5

PROPOSITION 1.7. *Let R be a ring, $\mathfrak{N}$ the full subcategory of mod R_R of all submodules of finitely generated ones, and E an injective R-right module. The following assertions are equivalent:*

(1) *The module E is a cogenerator of mod R_R, and for each $N \in \mathfrak{N}$ there is an exact sequence*

$$0 \to N \to E^k, \qquad k \in \mathbb{N}.$$

(2) *For each right ideal $\mathfrak{a}$ of R there are finitely many elements $x_i \in E$, $i \in I$, such that $\mathfrak{a} = \bigcap_{i \in I} (0 : x)$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{a}$ be a right ideal of R. By (1) there is an exact sequence

$$0 \to R/\mathfrak{a} \to E^k, \qquad k \in \mathbb{N},$$

i.e., an exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow R \xrightarrow{f} E^k.$$

Let $f_1, \ldots, f_k$ be the k components of f. The linear maps f_i are of the form

$$f_i : R \to E : r \rightsquigarrow x_i r$$

where $x_i \in E$. Moreover, $\ker f_i = (0 : x_i)$. But then

$$\mathfrak{a} = \ker f = \bigcap_i \ker f_i = \bigcap_i (0 : x_i).$$

(2) $\Rightarrow$ (1): (a) I show first that E is a cogenerator. Let $A \xrightarrow{f} B$ be a nonzero linear map, and let bR be a nonzero cyclic submodule of $\operatorname{Im} f$. By (2) the right ideal $(0 : b)$ of R is of the form

$$(0 : b) = \bigcap_{i \in I} (0 : x_i)$$

where x_i, $i \in I$, are finitely many elements of E. One of the x_i, say x_1, is not 0 since b is not zero. There results the nonzero morphism

$$h : bR \cong R/(0 : b) \to E : (0 : b) + r \rightsquigarrow x_1 r$$

which can be extended to a nonzero morphism $k : B \to E$ with

$$k|_{bR} = h \neq 0. \quad \text{Since} \quad 0 \neq bR \subseteq \operatorname{Im} f$$

this implies that $kf \neq 0$. Hence E is a cogenerator.

(b) It is obviously enough to show that for each finitely generated R-right module N there is an exact sequence

$$0 \to N \to E^k, \qquad k \in \mathbb{N}.$$

I show this by induction on the number of generators of N. If $N = nR$ is cyclic, then

$$(0 : n) = \bigcap_{i \in I} (0 : x_i)$$

where x_i, $i \in I$, is a finite family of elements of E. There results the exact sequence

$$0 \longrightarrow (0 : n) \longrightarrow R \xrightarrow{f} E^I$$

where the i-th component of f is the map $R \to E : r \rightsquigarrow x_i r$. Hence the exact sequence

$$0 \longrightarrow N = nR \cong R/(0 : n) \xrightarrow{g} E^I$$

where g is induced from f. Now assume that N has k generators. Then there is an exact sequence

$$0 \to N' \to N \to N'' \to 0$$

where N' resp. N'' have $k-1$ resp. one generators. The modules N' resp. N'' can be embedded into $E^{k'}$ resp. $E^{k''}$ by the induction hypothesis. By [4], ch. 5, prop. 2.2, there results a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^{k'} \arrow[r] & E^{k'+k''} \arrow[r] & E^{k''} \arrow[r] & 0
\end{tikzcd}$$

which shows that N can also be embedded into some E^k, $k \in \mathbb{N}$. ||

(4) I consider a special case of (3). In the situation of (3) it is reasonable to ask when one may choose $E = R$.

COROLLARY 1.8. *R and $\mathfrak{N}$ as in Proposition 1.7. The following assertions are equivalent*:

(1) *The module R_R is an injective cogenerator of mod R_R, and for each $N \in \mathfrak{N}$ there is a short exact sequence*

$$0 \to N \to R^k, \qquad k \in \mathbb{N}.$$

(2) *The ring R is right self-injective, and for each richt ideal $\mathfrak{a}$ of R there are finitely many elements x_i, $i \in I$, in R with $\mathfrak{a} = \bigcap_{i \in I} (0 : x_i)$. Here $(0 : x_i) := \{r \in R; x_i r = 0\}$ denotes the right annihilator.* ||

A ring R is called a *generalized right quasi-Frobenius ring* if it satisfies the equivalent conditions of the preceding corollary. In [10] B. Osofsky has investigated rings R which are injective cogenerators in mod R_R, i.e., more general rings than those which I denote as generalized quasi-Frobenius rings. Her results should be compared with Theorem 8.5 of this paper.

Objects of finite type. Let $\mathfrak{A}$ be a Grothendieck category. An object $A \in \mathfrak{A}$ is of *finite type* if for each increasing family $(A_i ; i \in I)$ I directed, of subobjects A_i of A with $\bigcup_i A_i = A$ there is a $j \in I$ with $A_j = A$. Quotient objects and finite coproducts of objects of finite type are again such.

Coherent objects. An object A of a Grothendieck category $\mathfrak{A}$ is called *coherent* (see e.g., [11], Section 2, def. 1) if

(1) A is of finite type and if

(2) for each morphism $f : A' \to A$ with A' of finite type, the kernel of f is also of finite type.

LEMMA 1.9. *Let $\mathfrak{A}$ be a Grothendieck category and*

$$0 \to A' \to A \to A'' \to 0$$

an exact sequence in $\mathfrak{A}$. Then any two objects of this sequence are coherent iff all three are. ||

The proof is done by easy diagram chasing. It is well-known from the theory of modules and sheaves (see e.g., [2], Section 2, example 11 ff). The category Coh $\mathfrak{A}$ of all coherent objects of $\mathfrak{A}$ is a full, skeletal-small subcategory of $\mathfrak{A}$ and closed under finite limits and colimits and extensions in $\mathfrak{A}$. In particular, Coh $\mathfrak{A}$ is Abelian and the injection Coh $\mathfrak{A} \to \mathfrak{A}$ is exact.

COROLLARY 1.10. *$\mathfrak{A}$ a Grothendieck category. Assume that $\mathfrak{G}$ is a set of generators of $\mathfrak{A}$ and that each object in $\mathfrak{G}$ is of finite type.*

(1) *An object A of $\mathfrak{A}$ is of finite type iff there is a short exact sequence*

$$G_1 \coprod \cdots \coprod G_m \to A \to 0, \qquad G_i \in \mathfrak{G}.$$

(2) *An object A of $\mathfrak{A}$ is coherent iff it is of finite type and if for each morphism*

$$G_1 \coprod \cdots \coprod G_m \to A, \qquad G_i \in \mathfrak{G},$$

there is a short exact sequence

$$K_1 \coprod \cdots \coprod K_n \to G_1 \coprod \cdots \coprod G_m \to A, \qquad K_i \in \mathfrak{G}.$$

Proof. The assertion (1) is obvious. That the condition of (2) is necessary follows easily from (1) and the definition of coherence. Assume now that A satisfies the conditions in (2) and let $f : A' \to A$ be a morphism with A' of finite type. By (1) there is an epimorphism

$$e : G_1 \coprod \cdots \coprod G_m \to A'.$$

using condition (2) one obtains an exact sequence

$$K_1 \coprod \cdots \coprod K_n \longrightarrow G_1 \coprod \cdots \coprod G_m \xrightarrow{fe} A, \qquad K_i \in \mathfrak{G}.$$

There results a commutative diagram with exact rows

$$\begin{tikzcd} K_1 \coprod \cdots \coprod K_n \arrow[r] \arrow[d, "e'"] & G_1 \coprod \cdots \coprod G_m \arrow[r, "fe"] \arrow[d, "e"] & A \arrow[d, equal] \\ 0 \arrow[r] \ker f \arrow[r, "\text{inj}"] & A' \arrow[r, "f"] & A. \end{tikzcd}$$

Diagram chasing shows that e' is an epimorphism, hence $\ker f$ is of finite type. Thus A is coherent. ||

EXAMPLE 1.11. Applying the preceding corollary to $\mathfrak{A} = \operatorname{mod} R$, R a ring, and $\mathfrak{G} = \{R\}$ one sees that the usual definition of coherent module coincides with the categorical one.

II. PRELIMINARIES ON TOPOLOGICAL RINGS AND MODULES

All rings considered in this paper are supposed to have a multiplicative identity. If R is a ring an R-left module on which the identity of R operates as the identity transformation is simply called an R-module. The category of all R-modules is denoted by mod R, or by mod ${}_RR$ if I want to emphasize that R operates on the left. The R-right modules form the category mod R_R. For most of the material in this paragraph the papers [5], ch. 5, Section 2, and [11], Section 4 can be consulted.

Linear topological rings.

A *topological ring* is a ring R with a topology such that the addition and multiplication are continuous. A topological ring R is called *left linear topological* if it has a basis of neighborhoods of 0 consisting of left ideals ([5], ch. 5, Section 2). A basis of neighborhoods of 0 will simply be called a "basis" in this paper.

LEMMA 2.1. (1) *If R is a left linear topological ring the open left ideals of R satisfy the following conditions*:

(i) *If* $\mathfrak{a} \subseteq \mathfrak{b}$ *are left ideals and* $\mathfrak{a}$ *is open, then so is* $\mathfrak{b}$.

(i) *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are open left ideals, then so is* $\mathfrak{a} \cap \mathfrak{b}$.

(iii) *If* $\mathfrak{a}$ *is an open left ideal and* $r \in R$ *then the left ideal*

$$(\mathfrak{a} : r) = \{x \in R;\ xr \in \mathfrak{a}\}$$

is open.

(2) *If R is any ring and if $\mathfrak{J}$ is a set of left ideals of R satisfying*:

(i) *if* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{J}$ *there is a* $\mathfrak{c} \in I$ *such that* $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$,

(ii) *if* $\mathfrak{a} \in \mathfrak{J}$ *and* $r \in R$ *there is a* $\mathfrak{b} \in \mathfrak{J}$ *with* $(\mathfrak{a} : r) \supseteq \mathfrak{b}$,

then there is a unique left linear topology on R having $\mathfrak{J}$ as basis (*of neighborhoods of* 0).

The proof is obvious. See also [5], ch. 5, Section 2, and [11], Section 4, prop. 3. ||

Left linear topological R-modules

If R is a topological ring a *topological R-left*) *module* is an R-module M with a topology such that the addition of M and the multiplication $R \times M \rightarrow M$ are continuous. If the topology of R is left linear a topological R-module M is called left linear topological if it has a basis (of neighborhoods of 0) consisting of R-submodules.

LEMMA 2.2. (1) *If R is a left linear topological ring and M a linear topological R-module the open submodules of M satisfy the following conditions*:

(i) *If $M' \subseteq M''$ are submodules and M' is open, then so is M''.*

(ii) *If M' and M'' are open submodules, then so is $M' \cap M''$.*

(iii) *If N is an open submodule and if $x \in M$, then the left ideal*

$$(N : x) = \{r \in R;\ rx \in N\}$$

is open.

(2) *If R is a left linear topological ring, if M is an R-module and if $\mathfrak{M}$ is a set of submodules of M satisfying*:

(i) *If* $M_1, M_2 \in \mathfrak{M}$ *there is a* $M_3 \in \mathfrak{M}$ *with* $M_3 \subseteq M_1 \cap M_2$,

(ii) *if* $M_1 \in \mathfrak{M}$ *and* $x \in M$ *the left ideal* $(M_1 : x)$ *is open*,

then there is a unique left linear topology on M having $\mathfrak{M}$ as basis.

The proof is again obvious. ||

In particular, if R is a left linear topological ring then an R-module M is left linear topological with the discrete topology iff for each $x \in M$ the annihilator $(0 : x) = \{r \in R;\ rx = 0\}$ is open in R. One obtains the full subcategory Dis R of mod R of all discrete topological R-modules ([11], Section 4, prop. 3). The category Dis R is a closed subcategory of mod R, i.e., it is closed under subobjects, quotient objects, and coproducts ([5], p. 395). Given a ring R the function which assigns to each left linear topology $\mathfrak{T}$ on R the category Dis$(R, \mathfrak{T})$ is a bijection between the left linear topologies on R and the closed subcategories of mod R. ([G], p. 412).

Coherent objects in Dis R

LEMMA 2.3. *Let R be a left linear topological ring with a basis $\mathfrak{J}$ of open left ideals.*

(1) *A module $M \in$ Dis R is of finite type in* Dis R *iff it is of finite type in* mod R, *i.e., if it is R-finitely generated.*

(2) *The modules $R/\mathfrak{a}$, $\mathfrak{a} \in \mathfrak{J}$, form a family of generators of* Dis R *of finite type, in particular* Dis R *is a Grothendieck category.*

(3) *A module $M \in$ Dis R is coherent in* Dis R *is iff there is an epimorphism*

$$R/\mathfrak{a}_1 \coprod \cdots \coprod R/\mathfrak{a}_m \to M, \qquad \mathfrak{a}_i \in \mathfrak{J},$$

and if for each linear map

$$f : R/\mathfrak{a}_1 \coprod \cdots \coprod R/\mathfrak{a}_m \to M, \qquad \mathfrak{a}_i \in \mathfrak{J},$$

there is a short exact sequence

$$R/\mathfrak{b}_1 \coprod \cdots \coprod R/\mathfrak{b}_n \to R/\mathfrak{a}_1 \coprod \cdots \coprod R/\mathfrak{a}_m \to M, \qquad \mathfrak{b}_i \in \mathfrak{J}.$$

Proof. (1) The subobjects of M in Dis R are the same as those in mod R, and the injection functor Dis $R \to$ mod R preserves colimits. This implies the assertion.

(2) For $M \in$ Dis R and $x \in M$ the left ideal $(0 : x)$ is open. Since $\mathfrak{J}$ is a basis there is a left ideal $\mathfrak{a}_x \subseteq (0 : x)$, $\mathfrak{a}_x \in \mathfrak{J}$.

There result the maps

$$R/\mathfrak{a}_x \to R/(0 : x) \to M,$$

and hence the epimorphism

$$\coprod_x R/\mathfrak{a}_x \to M$$

where x runs over all elements of M. This shows that the $R/\mathfrak{a}$, $\mathfrak{a} \in \mathfrak{J}$, form a family of generators of Dis R. They are of finite type by (1). (3) follows from (2) and Lemma 1.10 |

COROLLARY 2.4. *R as in preceding lemma. Assume that $M \in$* Coh(Dis R). *The finitely generated submodules of M are exactly the subobjects of M in* Coh(Dis R).

Proof. If M' is a finitely generated submodule of M then M' is of finite type in Dis R by the preceding lemma. Since M is coherent in Dis R also

M' is coherent in Dis R. If M' is a submodule of M and $M' \in \mathrm{Coh}(\mathrm{Dis}\, R)$ then M' is of finite type in Dis R, hence finitely generated. ||

The left linear topological ring R is called *topologically left coherent* if it admits a basis of left ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ is coherent. It is the same to say that Coh(Dis R) generates Dis R (see [11], Section 4, def. 3).

Complete topologically coherent modules

LEMMA 2.5. *Let X be a topological Abelian group which admits a basis of neighborhoods $\mathfrak{X}$ of 0 consisting of subgroups. Then X is complete (and Hausdorff) iff the canonical homomorphism*

$$X \xrightarrow{\mathrm{can}} \lim_{X' \in \mathfrak{X}} X/X' = : \hat{X}$$

is a bijection. If this is the case then can *is a homeomorphism if one equips $\hat{X} = \lim_{X' \in \mathfrak{X}} X/X'$ with the limit topology having the basis*

$$\ker(\hat{X} \xrightarrow{\mathrm{proj}} X/X'), \qquad X' \in \mathfrak{X}.$$

The proof is a special case of [3], Section 7, p. 86 ff. ||

Let now R be a left linear topological ring. A *topologically coherent R-module* is a left linear topological R-module X which admits a basis of neighborhoods of 0 consisting of submodules X' with $X/X' \in \mathrm{Coh}(\mathrm{Dis}\, R)$. Let CTC($R$) be the category of complete (and Hausdorff) topologically coherent R-modules. The morphisms of CTC(R) are the continuous linear maps.

A submodule Y of $X \in \mathrm{CTC}(R)$ is called *special open* if Y is open in X and $X/Y \in \mathrm{Coh}(\mathrm{Dis}\, R)$. A submodule Y of $X \in \mathrm{CTC}(R)$ is called *special closed* if it is closed in X and if Y with the induced topology lies in CTC(R). Remark here that if Y is closed in the complete module X, then Y is complete w.r.t. the induced topology. A module $X \in \mathrm{CTC}(R)$ is called *strict* if each special closed submodule Y of X with

$$X/Y \in \mathrm{Coh}(\mathrm{Dis}\, R)$$

is open in X. Let STC(R) be the full subcategory of CTC(R) of all strict modules. The modules $X \in \mathrm{Coh}(\mathrm{Dis}\, R)$ equipped with the discrete topology are obviously strict in CTC(R). One obtains the full inclusions

$$\mathrm{Coh}(\mathrm{Dis}\, R) \subset \mathrm{STC}(R) \subset \mathrm{CTC}(R).$$

If $_R R$ itself is an object in STC(R) then R is called a strict complete topologically left coherent ring.

III. The Duality Theorem for Grothendieck Categories

Let $\mathfrak{A}$ be a Grothendieck category and $(\mathfrak{N}, E)$ a pair of a full, skeletal-small, finitely closed and generating subcategory of $\mathfrak{A}$ together with an injective cogenerator E such that for each $N \in \mathfrak{N}$ there is a short exact sequence

$$0 \to N \to E^k, \qquad k \in \mathbb{N},$$

(see Proposition 1.5).

If $A \in \mathfrak{A}$, then $\mathfrak{A}(A, E)$ is a left $\mathfrak{A}(E, E)$-module, the scalar multiplication being given by composition. The functor

$$\mathfrak{A}(-, E) : \mathfrak{A}^{op} \to \operatorname{mod} \mathfrak{A}(E, E)$$

is a faithful and exact functor. If $A \in \mathfrak{A}$ and $B \subseteq A$ I identify $\mathfrak{A}(A/B, E)$ with its image under

$$\mathfrak{A}(A/B, E) \xrightarrow{\text{can}} \mathfrak{A}(A, E),$$

i.e.

$$\mathfrak{A}(A/B, E) = \{f : A \to E; f|_B = 0\}.$$

I also identify

$$\mathfrak{A}(A, E)/\mathfrak{A}(A/B, E)$$

with $\mathfrak{A}(B, E)$. In particular, if $B \subseteq E$, then $\mathfrak{A}(E/B, E)$ is a left ideal of $\mathfrak{A}(E, E)$.

Lemma 3.1. *There is a unique left linear topology on $\mathfrak{A}(E, E)$ having the ideals*

$$\mathfrak{A}(E/N, E), \qquad N \subseteq E, \qquad N \in \mathfrak{N},$$

as basis (of nbh. of 0).

Proof. (1) If $N_i \subseteq E$, $N_i \in \mathfrak{N}$, $i = 1, 2$, then by the exactness of $\mathfrak{A}(-, E)$ one has

$$\mathfrak{A}(E/N_1, E) \cap \mathfrak{A}(E/N_2, E) = \mathfrak{A}(E/N_1 + N_2, E).$$

The right side is again of the desired form since

$$N_1 + N_2 \subseteq E, \qquad N_1 + N_2 \in \mathfrak{N}.$$

(2) Let $f \in \mathfrak{A}(E, E)$ and $N \subseteq E$, $N \in \mathfrak{N}$.

Then

$$\begin{aligned}(\mathfrak{A}(E/N, E) : f) &= \{r : E \to E; rf|_N = 0\} \\ &= \{r : E \to E; r|_{f(N)} = 0\} \\ &= \mathfrak{A}(E/f(N), E).\end{aligned}$$

Since $\mathfrak{N}$ is closed under epimorphic image one obtains $f(N) \in \mathfrak{N}$. Hence

$$\mathfrak{A}(E/f(N), E)$$

is again of the desired form.

The results (1) and (2) and Lemma 2.1 show the lemma.

This topology depends on $(\mathfrak{N}, E)$, hence is called the $(\mathfrak{N}, E)$-topology on $\mathfrak{A}(E, E)$. Given $(\mathfrak{N}, E)$ this topology is always meant if $\mathfrak{A}(E, E)$ is considered as a topological space. Next the $\mathfrak{A}(E, E)$-modules $\mathfrak{A}(A, E)$ are made into topological ones.

LEMMA 3.2. *For $A \in \mathfrak{A}$ there is a unique $\mathfrak{A}(E, E)$-left linear topology on $\mathfrak{A}(A, E)$ having the submodules $\mathfrak{A}(A/N, E)$, $N \subseteq A$, $N \in \mathfrak{N}$, as basis, If $a : A \to A'$ is a morphism, then $\mathfrak{A}(a, E) : \mathfrak{A}(A', E) \to \mathfrak{A}(A, E)$ is a continuous linear map (w.r.t. the above defined topologies).*

Proof. The first part of the lemma is proven as in Lemma 3.1. If $a : A \to A'$ is a morphism, then

$$\mathfrak{A}(a, E)^{-1}(\mathfrak{A}(A/N, E)) = \mathfrak{A}(A'/a(N), E).$$

This follows from the exactness of $\mathfrak{A}(-, E)$. If $N \in \mathfrak{N}$, then $a(N) \in \mathfrak{N}$, hence $\mathfrak{A}(a, E)$ is continuous. $\|$

Given $(\mathfrak{N}, E)$ the preceding topology on $\mathfrak{A}(A, E)$ is always meant if $\mathfrak{A}(A, E)$ is considered as a topological space. The two lemmas show that

$$A \rightsquigarrow \mathfrak{A}(A, E)$$

is a faithful functor from $\mathfrak{A}^{op}$ to the category of all left linear topological $\mathfrak{A}(E, E)$-modules.

PROPOSITION. 3.3. *$\mathfrak{A}$, $\mathfrak{N}$, E as above. The functor $A \rightsquigarrow \mathfrak{A}(A, E)$ induces an equivalence*

$$\mathfrak{N}^{op} \to \mathrm{Coh}(\mathrm{Dis}\, \mathfrak{A}(E, E)).$$

This proposition can be considered as a sharpening of B. Mitchell's full embedding theorem ([8], p. 151) for the category $\mathfrak{N}$.

Proof. (1) I show first that for $N \in \mathfrak{N}$ and $A \in \mathfrak{A}$ the group homomorphism

$$\mathfrak{A}(A, N) \to \hom(\mathfrak{A}(N, E), \mathfrak{A}(A, E)) : a \rightsquigarrow \mathfrak{A}(a, E)$$

is bijective. Here hom denotes the $\mathfrak{A}(E, E)$-linear maps. The group homomorphism is injective since E is a cogenerator. Let, then,

$$f : \mathfrak{A}(N, E) \to \mathfrak{A}(A, E) \quad \text{be} \quad \mathfrak{A}(E, E)\text{-linear}.$$

By assumption on $(\mathfrak{N}, E)$ there is a finite family $(a_i\ ;\ i \in I)$ of morphisms

$$a_i : N \to E =: E_i$$

such that the morphism,

$$a : N \to \prod (E_i\ ;\ i \in I) = E^I$$

with components a_i, is a monomorphism. Since E is a cogenerator a can be extended to a short exact sequence

$$(*) \qquad 0 \longrightarrow N \xrightarrow{a} \prod (E_i\ ;\ i \in I) \xrightarrow{b} \prod (E_j\ ;\ j \in J)$$

where $(E_j\ ;\ j \in J)$ is some family of copies of E. Since I is finite one has

$$\mathfrak{A}\left(\prod_i E_i\,,\prod_j E_j\right) \cong \prod_{i,j} \mathfrak{A}(E_i\,, E_j).$$

Let

$$b_{ji} : E_i = E \to E_j = E, \qquad i \in I, \qquad j \in J,$$

be the components of b. Since I is finite the relation $ba = 0$ implies (or is actually equivalent to) the relations

$$\sum_i b_{ji} a_i = 0, \qquad j \in J.$$

Since

$$f : \mathfrak{A}(N, E) \to \mathfrak{A}(A, E) \text{ is } \mathfrak{A}(E, E)\text{-linear}$$

and since

$$b_{ji} \in \mathfrak{A}(E, E), \qquad a_i \in \mathfrak{A}(N, E),$$

one obtains

$$0 = f\left(\sum_i b_{ji} a_i\right) = \sum_i b_{ji} f(a_i), \qquad j \in J.$$

Let

$$a' : A \to \prod (E_i\ ;\ i \in I)$$

be the morphism with components $f(a_i)$. The preceding relations imply $ba' = 0$, hence the exactness of $(*)$ implies the existence of

$$c : A \to N \qquad \text{with} \qquad ac = a', \quad \text{i.e.}$$
$$a_i c = f(a_i), \qquad \text{all} \qquad i \in I.$$

I finally show that $f = \mathfrak{A}(c, E)$. So let $d : N \to E$ be a morphism. Since E is injective there is a morphism

$$e : \prod (E_i \,;\, i \in I) \to E \quad \text{with} \quad ea = d.$$

If e_i, $i \in I$, are the components of e, then $ea = \sum_i e_i a_i$, hence $\sum_i e_i a_i = d$. Since f is linear this equation implies

$$f(d) = f\Big(\sum_i e_i a_i\Big) = \sum_i e_i f(a_i) = \sum_i e_i a_i c = dc = \mathfrak{A}(c, E)(d),$$

hence $f = \mathfrak{A}(c, E)$.

(2) The ideals $\mathfrak{A}(E/N, E)$, $N \subseteq E$, $N \in \mathfrak{N}$,

form a basis of $\mathfrak{A}(E, E)$. Hence the

$$\mathfrak{A}(N, E) = \mathfrak{A}(E, E)/\mathfrak{A}(E/N, E), \qquad N \subseteq E, \qquad N \in \mathfrak{N},$$

form a family of generators of Dis $\mathfrak{A}(E, E)$ (2.3). More generally, if $N \in \mathfrak{N}$, then $\mathfrak{A}(N, E)$ has the discrete topology since $0 = \mathfrak{A}(N/N, E)$ is open in $\mathfrak{A}(N, E)$. Hence the $\mathfrak{A}(N, E)$, $N \in \mathfrak{N}$ or better $N \in \mathrm{Sk}(\mathfrak{N})$, also form a family of generators of Dis $\mathfrak{A}(E, E)$. For each $N \in \mathfrak{N}$ there is an exact sequence

$$0 \to N \to E^k, \qquad k \in \mathbb{N},$$

which implies the surjection

$$\mathfrak{A}(E, E)^k \to \mathfrak{A}(N, E),$$

and hence that $\mathfrak{A}(N, E)$ is finitely generated. Finally the

$$\mathfrak{A}(N, E), \qquad N \in \mathfrak{N},$$

are additively closed since

$$\mathfrak{A}(N_1, E) \coprod \mathfrak{A}(N_2, E) \cong \mathfrak{A}\Big(N_1 \coprod N_2, E\Big).$$

So the

$$\mathfrak{A}(N, E), \qquad N \in \mathfrak{N},$$

form an additively closed family of finitely generated generators of Dis $\mathfrak{A}(E, E)$. In particular, an $X \in$ Dis $\mathfrak{A}(E, E)$ is of finite type iff there is an epimorphism

$$\mathfrak{A}(N, E) \to X, \qquad N \in \mathfrak{N}.$$

Also $X \in \operatorname{Dis} \mathfrak{A}(E, E)$ is coherent iff X is of finite type and if for each

$$\mathfrak{A}(N, E) \xrightarrow{f} X, \qquad N \in \mathfrak{N},$$

there is a short exact sequence

$$\mathfrak{A}(M, E) \xrightarrow{g} \mathfrak{A}(N, E) \xrightarrow{f} X.$$

This follows from Lemma 2.3.

(3) The modules

$$\mathfrak{A}(N, E), \qquad N \in \mathfrak{N},$$

are coherent in $\operatorname{Dis} \mathfrak{A}(E, E)$. For by (2) they are of finite type. If

$$\mathfrak{A}(N_1, E) \xrightarrow{f} \mathfrak{A}(N, E)$$

is a linear map then there is an $a : N \to N_1$ in $\mathfrak{N}$ with $f = \mathfrak{A}(a, E)$ by (1). There results the exact sequence

$$\mathfrak{A}(\operatorname{coker} a, E) \longrightarrow \mathfrak{A}(N_1, E) \xrightarrow{f} \mathfrak{A}(N, E).$$

(4) Let $X \in \operatorname{Coh}(\operatorname{Dis} \mathfrak{A}(E, E))$. By (2) there is an epimorphism

$$f : \mathfrak{A}(N, E) \to X.$$

Since X is coherent there exists an exact sequence

$$\mathfrak{A}(N_1, E) \xrightarrow{g} \mathfrak{A}(N, E) \longrightarrow X \longrightarrow 0$$

where

$$g = \mathfrak{A}(a, E), \qquad a : N \to N_1.$$

Since $\mathfrak{A}(-, E)$ is exact one obtains

$$X \simeq \mathfrak{A}(\ker a, E).$$

(5) The calculations (1)–(4) establish the equivalence

$$\mathfrak{N}^{op} \to \operatorname{Coh}(\operatorname{Dis} \mathfrak{A}(E, E)) : N \rightsquigarrow \mathfrak{A}(N, E). \quad \|$$

THEOREM 3.4. *Assumptions as above. The functor* $A \rightsquigarrow \mathfrak{A}(A, E)$ *induces an equivalence*

$$\mathfrak{A}^{op} \to \operatorname{STC}(\mathfrak{A}(E, E)).$$

Proof. (1) For $A \in \mathfrak{A}$ the module $\mathfrak{A}(A, E)$ is topologically coherent and complete. The first assertion is clear since the

$$\mathfrak{A}(A/N, E), \qquad N \subseteq A, \qquad N \in \mathfrak{N},$$

are a basis of $\mathfrak{A}(A, E)$, and

$$\mathfrak{A}(A, E)/\mathfrak{A}(A/N, E) = \mathfrak{A}(N, E)$$

is coherent in Dis $\mathfrak{A}(E, E)$ by Proposition 3.3. Since $\mathfrak{N}$ generates $\mathfrak{A}$ and is finitely closed one has

$$A \cong \operatorname{colim}(N;\ N \subseteq A,\ N \in \mathfrak{N}).$$

Hence

$$\mathfrak{A}(A, E) \cong \lim(\mathfrak{A}(N, E);\ N \subseteq A,\ N \in \mathfrak{N})$$

as $\mathfrak{A}(E, E)$-modules. I show that the topology of $\mathfrak{A}(A, E)$ is indeed the lim-topology. Since the $\mathfrak{A}(N, E)$ are discrete the lim-topology on $\mathfrak{A}(A, E)$ has as basis the modules

$$\operatorname{Ker}(\mathfrak{A}(A, E) \to \mathfrak{A}(N, E)) = \mathfrak{A}(A/N, E)$$

where N runs over the subobjects of A in $\mathfrak{N}$. But this is also a basis of the original topology of $\mathfrak{A}(A, E)$. The lim-topology however is complete, so $\mathfrak{A}(A, E)$ is complete.

(2) Let $A \in \mathfrak{A}$ and $N \in \mathfrak{N}$. Then the map

$$\mathfrak{A}(N, A) \to \hom(\mathfrak{A}(A, E), \mathfrak{A}(N, E)) : a \rightsquigarrow \mathfrak{A}(a, E)$$

is a bijection. Here hom denotes the continuous $\mathfrak{A}(E, E)$-linear maps. The map is injective since E is a cogenerator. Now let

$$f : \mathfrak{A}(A, E) \to \mathfrak{A}(N, E)$$

be a continuous linear map. Since f is continuous the kernel of f is open, hence there is an

$$M \subseteq A, \qquad M \in \mathfrak{N},$$

with

$$\mathfrak{A}(A/M, E) \subseteq \ker f.$$

There results the factorization

$$\mathfrak{A}(A, E) \xrightarrow{\text{can}} \mathfrak{A}(M, E) \xrightarrow{f'} \mathfrak{A}(N, E)$$

of f. By Proposition 3.3 there is an $a : N \to M$ with

$$f' = \mathfrak{A}(a, E).$$

Also can $= \mathfrak{A}(\mathrm{inj}, E)$ where inj : $M \to A$ is the canonical injection. Hence

$$f = f' \text{ can} = \mathfrak{A}(a, E)\,\mathfrak{A}(\mathrm{inj}, E) = \mathfrak{A}(\mathrm{inj}\, a, E).$$

Hence

$$\mathfrak{A}(N, A) \to \hom(\mathfrak{A}(A, E), \mathfrak{A}(N, E))$$

is also surjective.

(3) If $A, B \in \mathfrak{A}$, then the map

$$(*) \qquad \mathfrak{A}(B, A) \to \hom(\mathfrak{A}(A, E), \mathfrak{A}(B, E)) : a \rightsquigarrow \mathfrak{A}(a, E)$$

is bijective. Again hom denotes the continuous linear maps, and only the surjectivity is not obvious. But

$$B \cong \operatorname{colim}(N; N \subseteq B, N \in \mathfrak{N}).$$

Hence

$$\begin{aligned} \mathfrak{A}(B, A) &\cong \mathfrak{A}(\operatorname{colim} N, A) \cong \lim \mathfrak{A}(N, A) \cong \lim \hom(\mathfrak{A}(A, E), \mathfrak{A}(N, E)) \\ &\cong \hom(\mathfrak{A}(A, E), \lim \mathfrak{A}(N, E)) \cong \hom(\mathfrak{A}(A, E), \mathfrak{A}(B, E)). \end{aligned}$$

In this sequence of isomorphisms the third one holds by (2), and the fifth one follows from the fact that

$$\mathfrak{A}(B, E) \cong \lim \mathfrak{A}(N, E)$$

is an isomorphism in the category of topological $\mathfrak{A}(E, E)$-modules. The resulting isomorphism

$$\mathfrak{A}(B, A) \cong \hom(\mathfrak{A}(A, E), \mathfrak{A}(B, E))$$

is easily seen to be the one in (*). The preceding considerations show that the functor $A \rightsquigarrow \mathfrak{A}(A, E)$ is a full embedding of $\mathfrak{A}^{op}$ into the category of topological $\mathfrak{A}(E, E)$-modules.

(4) If $B \subseteq A \in \mathfrak{A}$, then $\mathfrak{A}(A/B, E)$ is a special closed submodule of $\mathfrak{A}(A, E)$. For

$$\begin{aligned} &\bigcap_N (\mathfrak{A}(A/B, E) + \mathfrak{A}(A/N, E)) \\ &\quad = \bigcap_N \mathfrak{A}(A/B \cap N, E) = \mathfrak{A}\left(A\Big/\bigcup_N (B \cap N), E\right) \\ &\quad = \mathfrak{A}\left(A\Big/B \cap \left(\bigcup_N N\right), E\right) = \mathfrak{A}(A/B \cap A, E) = \mathfrak{A}(A/B, E). \end{aligned}$$

Here N runs over all subobjects of A in $\mathfrak{N}$, and thus $\mathfrak{A}(A/N, E)$ runs over a basis of $\mathfrak{A}(A, E)$. The equation

$$\bigcap_N (\mathfrak{A}(A/B, E) + \mathfrak{A}(A/N, E)) = \mathfrak{A}(A/B, E)$$

shows that $\mathfrak{A}(A/B, E)$ is closed in $\mathfrak{A}(A, E)$. In the preceding proof the equality

$$\mathfrak{A}(A/B, E) + \mathfrak{A}(A/N, E) = \mathfrak{A}(A/B \cap N, E)$$

follows from the exactness of the functor $\mathfrak{A}(-, E)$. The equality

$$B \cap \left(\bigcup_N N\right) = \bigcup_N (B \cap N)$$

is the (AB5)-condition in a Grothendieck category since

$$\{N;\ N \subseteq A,\ N \in \mathfrak{N}\}$$

is filtered from above.

The module $\mathfrak{A}(A/B, E)$ is also special closed in $\mathfrak{A}(A, E)$ since for

$$N \subseteq A, \qquad N \in \mathfrak{N},$$

one has

$$\begin{aligned} \mathfrak{A}(A/B, E)/\mathfrak{A}(A/B, E) \cap \mathfrak{A}(A/N, E) &= \mathfrak{A}(A/B, E)/\mathfrak{A}(A/B + N, E) \\ &\cong \mathfrak{A}(B + N/B, E) \cong \mathfrak{A}(N/B \cap N, E). \end{aligned}$$

The latter module is in Coh Dis $\mathfrak{A}(E, E)$ since $N/B \cap N \in \mathfrak{N}$.

(5) Each special closed submodule

$$X \quad \text{of} \quad \mathfrak{A}(A, E), \qquad A \in \mathfrak{A},$$

is of the form

$$X = \mathfrak{A}(A/K, E)$$

for a unique $K \subseteq A$. The uniqueness of K is a consequence of the assumption that E is an injective cogenerator. That X is special closed means that X is closed and that for each $N \subseteq A$, $N \in \mathfrak{N}$, the module

$$X/X \cap \mathfrak{A}(A/N, E)$$

is in

$$\mathrm{Coh}(\mathrm{Dis}\ \mathfrak{A}(E, E)).$$

By Proposition 3.3 there are objects $Q_N \in \mathfrak{N}$, unique up to isomorphism, and isomorphisms

$$X/X \cap \mathfrak{A}(A/N, E) \cong \mathfrak{A}(Q_N, E).$$

The injections

$$(*)\qquad \begin{tikzcd} X/X \cap \mathfrak{A}(A/N, E) \arrow[r] \arrow[d, "\wr\|"] & \mathfrak{A}(A, E)/\mathfrak{A}(A/N, E) \arrow[d, "\wr\|"] \\ \mathfrak{A}(Q_N, E) \arrow[r, dashed, "\mathfrak{A}(q_N, E)"] & \mathfrak{A}(N, E) \end{tikzcd}$$

give rise to epimorphisms $N \xrightarrow{q_N} Q_N$ s.t. the dotted arrow in (*) makes (*) commutative. Let $K_N = \ker q_N$. It is easy to see that K_N, $N \subseteq A$, $N \in \mathfrak{N}$, is an increasing sequence of subobjects of A. Let

$$K := \bigcup_N K_N \cong \operatorname*{colim}_N K_N .$$

Then

$$K \subseteq \bigcup_N N = A.$$

Since X is closed and hence complete one has

$$X \cong \lim_N X/X \cap \mathfrak{A}(A/N, E).$$

Altogether one obtains the string of isomorphisms

$$\begin{aligned} X &\cong \lim_N X/X \cap \mathfrak{A}(A/N, E) \cong \lim_N \mathfrak{A}(Q_N, E) \cong \lim_N \mathfrak{A}(N/K_N, E) \\ &\cong \mathfrak{A}(\operatorname*{colim}_N N/K_N, E) \cong \mathfrak{A}(\operatorname*{colim}_N N/\operatorname*{colim}_N K_N, E) \cong \mathfrak{A}(A/K, E) \end{aligned}$$

where all isomorphisms are just considered as $\mathfrak{A}(E, E)$-linear isomorphisms. An easy verification shows that the resulting isomorphism $X \cong \mathfrak{A}(A/K, E)$ is indeed the identity (modulo the identification $\mathfrak{A}(A/K, E) \subseteq \mathfrak{A}(A, E)$). Hence $X = \mathfrak{A}(A/K, E)$ which was to be shown.

(6) For $A \in \mathfrak{A}$ the module $\mathfrak{A}(A, E)$ is strict (see section 2, for the definition). Let namely $X \subseteq \mathfrak{A}(A, E)$ be a special closed submodule of $\mathfrak{A}(A, E)$ such that $\mathfrak{A}(A, E)/X \in \text{Coh Dis } \mathfrak{A}(E, E)$. By (5) one has $X = \mathfrak{A}(A/K, E)$, hence

$$\mathfrak{A}(A, E)/X \cong \mathfrak{A}(K, E) \in \text{Coh Dis } \mathfrak{A}(E, E).$$

Hence there is a linear isomorphism

$$\mathfrak{A}(N, E) \xrightarrow{f} \mathfrak{A}(K, E), \qquad N \in \mathfrak{N}.$$

By the first part of the proof of Proposition 3.3 there is an $a : K \to N$ such that $f = \mathfrak{A}(a, E)$. Since $\mathfrak{A}(-, E)$ is a faithful exact functor (into $\mathfrak{A}(E, E)$-modules) and since $\mathfrak{A}(a, E) = f$ is an isomorphism also a is an isomorphism. This implies $K \in \mathfrak{N}$, and hence $\mathfrak{A}(A/K, E) = X$ is special open. By definition this implies that $\mathfrak{A}(A, E)$ is strict. The preceding six parts show that the functor $A \rightsquigarrow \mathfrak{A}(A, E)$ is a full embedding of $\mathfrak{A}^{op}$ into $\mathrm{STC}(\mathfrak{A}(E, E))$.

(7) I finally show that each $X \in \mathrm{STC}(\mathfrak{A}(E, E))$ is isomorphic to some $\mathfrak{A}(E, E)$. Let X' run over the set of all special open submodules of X. Then the canonical map

$$X \to \lim_{X'} X/X'$$

is an isomorphism in the category of topological $\mathfrak{A}(E, E)$-modules. The inverse system $(X/X'; X' \subseteq X$ special open in $\mathrm{Coh}(\mathrm{Dis}\, \mathfrak{A}(E, E))$ gives rise to the direct system $(N_{X'}; X' \subseteq X$ special open) in $\mathfrak{N}$ by Proposition 3.3. If $X' \subseteq X''$ the morphism $N_{X''} \to N_{X'}$ is a monomorphism since $X/X' \to X/X''$ is surjective. Let

$$A = \operatorname*{colim}_{X'} N_{X'} .$$

Since $\mathfrak{A}$ is $(AB5)$ I assume w.l.o.g. that $N_{X'} \subseteq A$, $X' \subseteq X$ special open. One obtains the linear isomorphism

$$\mathfrak{A}(A, E) \cong X \quad \text{via}$$
$$\mathfrak{A}(A, E) \cong \mathfrak{A}(\operatorname*{colim}_{X'} N_{X'}, E) \cong \lim_{X'} \mathfrak{A}(N_{X'}, E) \cong \lim_{X'} X/X' \cong X.$$

Hence one may identify $X = \mathfrak{A}(A, E)$ and $X/X' = \mathfrak{A}(N_{X'}, E)$ as $\mathfrak{A}(E, E)$-modules. It remains to be shown that the given topology on $\mathfrak{A}(A, E)$ is the $(\mathfrak{N}, E)$-topology, i.e. that every $\mathfrak{A}(A/N, E)$, $N \subseteq A$, $N \in \mathfrak{N}$, is open w.r.t. the given topology on $\mathfrak{A}(A, E)$ which has the basis $\mathfrak{A}(A/N_{X'}, E)$, all X'. As in (4) one shows that $\mathfrak{A}(A/N, E)$ is special closed w.r.t. the given topology on $\mathfrak{A}(A, E) = X$. Also

$$\mathfrak{A}(A, E)/\mathfrak{A}(A/N, E) = \mathfrak{A}(N, E) \in \mathrm{CohDis}\, \mathfrak{A}(E, E).$$

Since $X = \mathfrak{A}(A, E)$ with the given topology is strict, the preceding properties imply that $\mathfrak{A}(A/N, E)$ is open. Hence $\mathfrak{A}(A, E) \cong X$ where the isomorphism is also topological, i.e., an isomorphism in $\mathrm{STC}(\mathfrak{A}(E, E))$. This completes the proof of the theorem. ||

Remark 3.5. The preceding theorem can be generalized in the following way which, however, does not seem to have valuable applications. Again one starts with a Grothendieck category $\mathfrak{A}$, a full, skeletal-small, finitely closed and generating subcategory $\mathfrak{N}$ of $\mathfrak{A}$, and an injective cogenerator E of $\mathfrak{A}$ such that for $N \in \mathfrak{N}$ there is an exact sequence $0 \to N \to E^k$, some $k \in \mathbb{N}$. One defines a new category $\mathfrak{A}'$: The objects of $\mathfrak{A}'$ are pairs $(A, \mathfrak{U})$ of an object $A \in \mathfrak{A}$ and an $\mathfrak{N}$-*covering* $\mathfrak{U}$ of A. The latter means that $\mathfrak{U}$ is a directed set of subobjects of A which lie in $\mathfrak{N}$ and that the supremum of $\mathfrak{U}$ is A. A morphism $f : (A, \mathfrak{U}) \to (B, \mathfrak{B})$ is a morphism $f : A \to B$ in $\mathfrak{A}$ such that for each $M \in \mathfrak{U}$ there is an $N \in \mathfrak{B}$ with $f(M) \subseteq N$. For each $(A, \mathfrak{U}) \in \mathfrak{A}'$ there is a unique $\mathfrak{A}(E, E)$-linear topology on $\mathfrak{A}(A, E)$ which has the submodules $\mathfrak{A}(A/M, E)$, $M \in \mathfrak{U}$, as basis. Write $\mathfrak{A}(A, E)_{\mathfrak{U}}$ for $\mathfrak{A}(A, E)$ with this topology. If $f : (A, \mathfrak{U}) \to (B, \mathfrak{B})$ is a morphism in $\mathfrak{A}'$ then

$$\mathfrak{A}(f, E) : \mathfrak{A}(B, E)_{\mathfrak{B}} \to \mathfrak{A}(A, E)_{\mathfrak{U}}$$

is continuous.

THEOREM 3.6. *The functor* $(A, \mathfrak{U}) \rightsquigarrow \mathfrak{A}(A, E)_{\mathfrak{U}}$ *defines an equivalence*

$$\mathfrak{A}'^{op} \to \mathrm{CTC}(\mathfrak{A}(E, E)).$$

Here the topology on $\mathfrak{A}(E, E)$ *is the* $(\mathfrak{N}, E)$-*topology.*

The proof of this result is almost the same as that of Theorem 3.4 and omitted. ||

The preceding two theorems furnish a commutative diagram of categories

$$\begin{array}{ccc} \mathfrak{A}^{op} & \longrightarrow & \mathrm{STC}(\mathfrak{A}(E, E)) \\ \cap & & \cap \\ \mathfrak{A}'^{op} & \longrightarrow & \mathrm{CTC}(\mathfrak{A}(E, E)) \end{array}$$

with vertical full embeddings and horizontal equivalences.

The following proposition gives the connection between the work of Jan-Erik Roos ([11], Section 5, th. 6) and the preceding theorems.

PROPOSITION. 3.7. $\mathfrak{A}, \mathfrak{N}, E$ *as in Theorem* 3.4. *The following assertions are equivalent*:

(i) $\mathfrak{N}$ *is noetherian*, i.e. *each object in* $\mathfrak{N}$ *satisfies the ascending chain condition.*

(ii) *Each complete topologically coherent* $\mathfrak{A}(E, E)$-*module is strict. If* (i) *and* (ii) *are satisfied, then* $\mathfrak{N}$ *is the class of all noetherian objects of* $\mathfrak{A}$.

Proof. (1) Using the results 3.4, 3.5, 3.6 one sees that the following assertions are equivalent:

Each complete topologically coherent $\mathfrak{A}(E, E)$-module is strict. $\Leftrightarrow$

The full embedding $\mathfrak{A} \to \mathfrak{A}'$ is an equivalence. $\Leftrightarrow$
For each $(A, \mathfrak{U}) \in \mathfrak{A}'$ the morphism

$$\mathrm{id}_A : (A, \mathfrak{U}) \to (A, \{N; N \subseteq A, N \in \mathfrak{N}\})$$

is an isomorphism. $\Leftrightarrow$
For each $(A, \mathfrak{U}) \in \mathfrak{A}'$ the morphism id_A is also a morphism in $\mathfrak{A}'$ from $(A, \{N; N \subseteq A, N \in \mathfrak{N}\})$ to $(A, \mathfrak{U})$. $\Leftrightarrow$
For each $(A, \mathfrak{U}) \in \mathfrak{A}'$ and each $N \subseteq A$, $N \in \mathfrak{N}$, there is an $M \in \mathfrak{U}$ with $N \subseteq M$.

(2) Now assume first that (i) is satisfied, i.e., that $\mathfrak{N}$ is noetherian. Let $(A, \mathfrak{U}) \in \mathfrak{A}'$ and $N \subseteq A$, $N \in \mathfrak{N}$. Since $A = \bigcup(M; M \in \mathfrak{U})$ and since $\mathfrak{U}$ is filtered from above one gets $N = \bigcup(M \cap N; M \in \mathfrak{U})$. Since N is noetherian there is a $M \in \mathfrak{U}$ with $N = M \cap N$, i.e., $N \subseteq M$. By the above string of equivalences one sees that (ii) is true.

(3) On the other hand assume that (ii) is satisfied. In order to show that $\mathfrak{N}$ is noetherian it is enough to show that each $N \in \mathfrak{N}$ is of finite type in $\mathfrak{A}$. Thus let $\mathfrak{U}$ be a directed set of subobjects of $N \in \mathfrak{N}$ with $\bigcup(M; M \in \mathfrak{U}) = N$. The properties of $\mathfrak{N}$ and the definition of $\mathfrak{A}'$ imply that $(N, \mathfrak{U}) \in \mathfrak{A}'$. By (ii) and the above string of equivalences and since N itself is in $\mathfrak{N}$ there is an $M \in \mathfrak{U}$ with $N \subseteq M$, i.e. $N = M$ hence $N \in \mathfrak{U}$ and N is of finite type.

(4) Assume that (i) and (ii) are satisfied. Since $\mathfrak{N}$ is noetherian and finitely closed in $\mathfrak{A}$ the objects of $\mathfrak{N}$ are noetherian in $\mathfrak{A}$ too. Let, on the other hand, A be noetherian in $\mathfrak{A}$. Since $A = \bigcup(N; N \subseteq A, N \in \mathfrak{N})$ there is an $N \subseteq A$, $N \in \mathfrak{N}$, with $N = A$. Hence $A \in \mathfrak{N}$. Thus $\mathfrak{N}$ consists exactly of the noetherian objects of $\mathfrak{A}$. ||

IV. Topologically Linearly Compact Rings

Proposition. 4.1. *$\mathfrak{A}$, $\mathfrak{N}$, E as in Theorem* 3.4. *Each module*

$$X \in \mathrm{Coh}(\mathrm{Dis}\, \mathfrak{A}(E, E))$$

has the following property: If $(X_i\,; i \in I)$, I directed, is a decreasing family of submodules of X in Coh(Dis $\mathfrak{A}(E, E)$), *then the submodule $\bigcap_{i \in I} X_i$ of X is again coherent, and the canonical linear map*

$$X \to \varprojlim_i X/X_i$$

is surjective. The limit is taken in mod R.

Proof. By 3.3 $X \cong \mathfrak{A}(N, E)$, $N \in \mathfrak{N}$. The increasing family of surjections $X \to X/X_i$ in Coh(Dis $\mathfrak{A}(E, E)$) gives rise to the increasing family $N_i \subseteq N$,

$i \in I$, of subobjects of N. Since $\mathfrak{A}$ is a Grothendieck category $\operatorname{colim}_{i\in I} N_i$ is again a subobject of N, indeed $\operatorname{colim}_{i\in I} N_i \cong \bigcup_{i\in I} N_i$. The monomorphism $\operatorname{colim}_i N_i \to N$ gives rise to the epimorphism

$$X \cong \mathfrak{A}(N, E) \to \mathfrak{A}(\operatorname{colim}_i N_i, E) \cong \lim_i \mathfrak{A}(N_i, E) \cong \lim_i X/X_i .$$

This shows that $\lim_i X/X_i \in \operatorname{Coh}(\operatorname{Dis} \mathfrak{A}(E, E))$, and that $X \xrightarrow{\text{can}} \lim_i X/X_i$ is surjective. Moreover the kernel $\bigcap_i X_i$ of the map can : $X \to \lim_i X/X_i$ is also coherent, hence the proposition. ||

I am going to show now that the preceding proposition is a statement on linear compactness.

DEFINITION. 4.2. An ordered set I is called compact if it is complete and satisfies the following compactness condition: If $(i_k;\ k \in K)$ is a family of elements of I with $\bigcap_{k\in K} i_k = 0$, then there is a finite subset K' of K with $\bigcap_{k\in K'} i_k = 0$.

It is clear that the compactness condition is equivalent to the following condition: If I' is a subset of I which is filtered from below and satisfies $\bigcap I' = 0$, then $i' = 0$ for some $i' \in I'$. Here 0 denotes the smallest element of I.

Now let R be a ring and $\mathfrak{B}$ a full, skeletal-small subcategory of mod R closed under finite limits and colimits. For each $B \in \mathfrak{B}$ let $sub_{\mathfrak{B}}\, B$ denote the ordered set of submodules of B in $\mathfrak{B}$. Since $\mathfrak{B}$ is closed under finite limits and colimits $sub_{\mathfrak{B}}\, B$ is a lattice. For $B \in \mathfrak{B}$ let

$$\operatorname*{res}_{\mathfrak{B}} B = \{\emptyset\} \cup \{x + B';\ x \in B,\ B' \subseteq B,\ B' \in \mathfrak{B}\}$$

be the set of all residue classes of B w.r.t. submodules of B in $\mathfrak{B}$. Then $\operatorname{res}_{\mathfrak{B}} B$ is ordered by inclusion, and $\operatorname{sub}_{\mathfrak{B}} B$ is a subset of $\operatorname{res}_{\mathfrak{B}} B$. Moreover $\operatorname{res}_{\mathfrak{B}} B$ is closed under finite intersections.

DEFINITION. 4.3. Let $\mathfrak{B}$ be as above, and $B \in \mathfrak{B}$. The module B is called $\mathfrak{B}$-linearly compact if:

(1) $\operatorname{sub}_{\mathfrak{B}} B$ is closed under arbitrary intersections.

(2) $\operatorname{res}_{\mathfrak{B}} B$ is compact.

Remark here that (1) means that $\operatorname{sub}_{\mathfrak{B}} B$ is a complete lattice with intersection as infinum. The condition (1) implies that $\operatorname{res}_{\mathfrak{B}} B$ is a complete lattice with intersection as infinum. The condition (2) means that if $(x_i + B_i;\ i \in I)$, I directed, is a decreasing family of nonempty residue classes of B in $\operatorname{res}_{\mathfrak{B}} B$, then $\bigcap_{i\in I} (x_i + B_i) \neq \emptyset$, i.e. there is an $x \in B$ with $x + B_i = x_i + B_i$, all $i \in I$.

LEMMA 4.4. $\mathfrak{B}$ *as above.* $B \in \mathfrak{B}$. *Assume that* $\mathrm{sub}_{\mathfrak{B}} B$ *is closed under arbitrary intersections. Then* B *is* $\mathfrak{B}$*-linearly compact iff for each decreasing family* $(B_i\,;\, i \in I)$, I *directed, of submodules of* B *in* $\mathfrak{B}$ *the canonical map* $B \to \lim_{i \in I} B/B_i$ *is surjective. The limit is taken in* mod R.

Proof. The elements of $\lim_{i \in I} B/B_i$ are families $(x_i + B_i\,;\, i \in I)$ such that $x_i + B_j = x_j + B_j$ whenever $i \geqslant j$. This is the same as

$$x_i + B_i \subseteq x_j + B_j \qquad \text{for} \qquad i \geqslant j,$$

i.e., that $(x_i + B_i\,;\, i \in I)$ is a decreasing family of nonempty residue classes. That $B \to \lim_i B/B_i$ is surjective means that for each such decreasing family $(x_i + B_i\,;\, i \in I)$ there is an $x \in B$ with $x + B_i = x_i + B_i$, all i. By the above remark this means that B is $\mathfrak{B}$-linearly compact. ||

DEFINITION. 4.5. A full, skeletal-small subcategory $\mathfrak{B}$ of mod R, R a ring, is called linearly compact if it is closed under finite limits and colimits, if all $\mathrm{sub}_{\mathfrak{B}} B$, $B \in \mathfrak{B}$, are closed under arbituary intersections and if all $\mathrm{res}_{\mathfrak{B}} B$, $B \in \mathfrak{B}$, are compact.

From proposition 4.1 one obtains the

COROLLARY 4.6. $\mathfrak{A}$, $\mathfrak{N}$, E *as in Theorem* 3.4. *Let* $R = \mathfrak{A}(E, E)$ *with the topology defined in section* 3. *Then* Coh(Dis R) *is a linearly compact subcategory of* mod R. ||

Remark 4.7. My definition of linear compactness differs from and is weaker than all previous definitions of this term in the literature. (Compare [5] p. 390 ff; [11], Section 4, def. 3). Until now the definition always included that $\mathfrak{B}$ was artinian which is obviously a stronger requirement than linear compactness in the sense defined here. Nevertheless I consider my terminology justified because it is the precise counterpart of topological compactness for linear topologies.

Corollary 4.6 shows how one can obtain linearly compact categories of modules from Grothendieck categories. The following considerations explain another possibility.

DEFINITION. 4.8. Let R be a ring. An R-module M is called algebraically linearly compact if for each decreasing family $(M_i\,;\, i \in I)$, I directed, of finitely generated submodules M_i of M the module $\bigcap_{i \in I} M_i$ is also finitely generated and the canonical map $M \to \lim_i M/M_i$ is surjective. The ring R is called algebraically left linearly compact if the module ${}_R R$ is algebraically linearly compact.

In the topological case the following definition is appropriate.

DEFINITION 4.9. A left linear topological ring R is called topologically left linearly compact if it admits a basis of left ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ is algebrically linearly compact.

The two notions of linear compactness defined in 4.3 resp. 4.8 are connected by

LEMMA 4.10. *Let R be a left linear topological ring. Let* $\mathfrak{B} = \operatorname{Coh}(\operatorname{Dis} R)$. *A module* $X \in \operatorname{Coh}(\operatorname{Dis} R)$ *is $\mathfrak{B}$-linearly compact iff it is algebraically linearly compact.*

Proof. One has just to notice that the finitely generated submodules of X are exactly the subobjects of X in $\operatorname{Coh}(\operatorname{Dis} R)$. ||

PROPOSITION 4.11. $\mathfrak{A}$, $\mathfrak{N}$, *E as in Theorem* 3.4. *The ring* $\mathfrak{A}(E, E)$ *is topologically left linearly compact.*

Proof. The ring $\mathfrak{A}(E, E)$ has the basis $\mathfrak{A}(E/N, E)$, $N \subseteq E$, $N \in \mathfrak{N}$. By Proposition 4.1 and Lemma 4.10 the modules $\mathfrak{A}(E, E)/\mathfrak{A}(E/N, E) = \mathfrak{A}(N, E)$ are $\operatorname{Coh}(\operatorname{Dis} \mathfrak{A}(E, E))$-linearly compact. Hence $\mathfrak{A}(E, E)$ is topologically left linearly compact. ||

PROPOSITION 4.12. *If R is a topologically left linearly compact ring then* $\operatorname{Coh}(\operatorname{Dis} R)$ *is linearly compact.*

Proof. (1) Let R be any ring, M an R-module and M' a finitely generated submodule of M. If M is algebraically linearly compact, then so is M/M'. For let $g : M \to M/M' =: M''$ be the canonical map, and let $(M_i'' ; i \in I)$, I directed, be a decreasing family of finitely generated submodules of M''. Since $M' = \ker g$ is finitely generated so are all $g^{-1}(M_i'')$, hence $(g^{-1}(M_i''); i \in I)$ is a decreasing family of finitely generated submodules of the algebraically linearly compact module M. Thus $\bigcap_i g^{-1}(M_i'')$ is finitely generated, and $M \to \lim_i M/g^{-1}(M_i'')$ is surjective. Since

$$g\left(\bigcap_i g^{-1}(M_i'')\right) = \bigcap_i M_i''$$

this implies the finite generation of $\bigcap_i M_i''$. Also $M/g^{-1}(M_i'') \cong M''/M_i''$, and the canonical map

$$M \to \lim_i M/g^{-1}(M_i'')$$

factorizes as

$$M \to M'' \to \lim_i M''/M_i'' .$$

So $M'' \to \lim_i M''/M_i''$ is surjective too.

(2) Let $\mathfrak{B} = \text{Coh Dis } R$ where R is the ring given in the proposition. Then $\mathfrak{B}$ is a full, skeletal-small subcategory of mod R closed under finite limits and colimits. If $B \in \mathfrak{B}$, then B is algebraically linearly compact iff it is $\mathfrak{B}$-linearly compact by Lemma 4.10.

I show now that for each special open left ideal $\mathfrak{a}$ of R the module $R/\mathfrak{a}(\in \mathfrak{B})$ is algebraically linearly compact, hence $\mathfrak{B}$-linearly compact. By assumption there is an open left ideal $\mathfrak{b}$ of R such that $\mathfrak{b} \subseteq \mathfrak{a}$ and such that $R/\mathfrak{b}$ is algebraically linearly compact. Moreover, $R/\mathfrak{b}$ is of finite type in Dis R. So

$$0 \to \mathfrak{a}/\mathfrak{b} \to R/\mathfrak{b} \to R/\mathfrak{a} \to 0$$

is exact in Dis R, $R/\mathfrak{a}$ is coherent and $R/\mathfrak{b}$ is of finite type in Dis R. By definition of coherence $\mathfrak{a}/\mathfrak{b}$ is of finite type in Dis R, hence finitely generated. Since $R/\mathfrak{b}$ is algebraically linearly compact and $\mathfrak{a}/\mathfrak{b}$ finitely generated also $R/\mathfrak{a}$ is algebraically linearly compact by (1). In particular $R/\mathfrak{a}$ is $\mathfrak{B}$-linearly compact.

(3) If $B \in \mathfrak{B}$ is $\mathfrak{B}$-linearly compact, then so is any factor module of B in $\mathfrak{B}$. This follows directly from (1) since $\mathfrak{B}$-linear compactness and algebraic linear compactness coincide for modules in $\mathfrak{B}$.

(4) I show that if $0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$ is a short exact sequence in $\mathfrak{B}$ and if B' and B'' are $\mathfrak{B}$-linearly compact, then so is B. Let $(B_i\,;\, i \in I)$ I directed, be a decreasing family of submodules of B in $\mathfrak{B}$. Since $\mathfrak{B}$ is closed under finite limits and colimits this family gives rise to the decreasing families $(f^{-1}(B_i); i \in I)$ resp. $(g(B_i); i \in I)$ of submodules of B' resp. B'' in $\mathfrak{B}$. The sequences

$$0 \to B'/f^{-1}(B_i) \to B/B_i \to B''/g(B_i) \to 0$$

are exact. There results the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \bigcap_i f^{-1}(B_i) \arrow[r] \arrow[d, "\text{inj}"] & \bigcap_i B_i \arrow[r] \arrow[d, "\text{inj}"] & \bigcap_i g(B_i) \arrow[r] \arrow[d, "\text{inj}"] & 0 \\
0 \arrow[r] & B' \arrow[r, "f"] \arrow[d, "\text{can}"] & B \arrow[r, "g"] \arrow[d, "\text{can}"] & B'' \arrow[r] \arrow[d, "\text{can}"] & 0 \\
0 \arrow[r] & \varprojlim_i B'/f^{-1}(B_i) \arrow[r] \arrow[d] & \varprojlim_i B/B_i \arrow[r] & \varprojlim_i B''/g(B_i) \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}$$

The two outer columns of this diagram are exact and lie in $\mathfrak{B}$ since B' and B'' are $\mathfrak{B}$-linearly compact. The lower row is exact since $\lim_i$ is left-exact. The Snake lemma ([4], ch. 3, Lemma 3.2) implies the exactness of the first row and the surjectivity of can : $B \to \lim_i B/B_i$. As a submodule of $B \in \text{Dis } R$ the module $\bigcap_i B_i$ also lies in Dis R. Hence the sequence

$$0 \to \bigcap_i f^{-1}(B_i) \to \bigcap_i B_i \to \bigcap_i g(B_i) \to 0$$

is exact in Dis R and the two outer objects lie in $\mathfrak{B} = \text{Coh}(\text{Dis } R)$. This implies that $\bigcap_i B_i \in \mathfrak{B}$, and hence B is $\mathfrak{B}$-linearly compact.

(5) The parts (1)–(4) imply the assertion of the proposition. For by (2) all $R/\mathfrak{a}$ where $\mathfrak{a}$ is a special open ideal are $\mathfrak{B}$-linearly compact. By (4) this implies that all $R/\mathfrak{a}_1 \coprod \cdots \coprod R/\mathfrak{a}_k$, $\mathfrak{a}_i$ special open, are $\mathfrak{B}$-linearly compact. For each $B \in \mathfrak{B}$ there is an exact sequence

$$R/\mathfrak{a}_1 \coprod \cdots \coprod R/\mathfrak{a}_k \to B \to 0$$

in $\mathfrak{B}$ with special open $\mathfrak{a}_i \subseteq R$. By (3) B is also $\mathfrak{B}$-linearly compact. Hence $\mathfrak{B}$ is linearly compact. ∎

The distinguishing property of linearly compact categories of modules is that they give rise to co-Grothendieck categories of topological modules.

V. Construction of Co-Grothendieck Categories

In this section R is a ring and $\mathfrak{B}$ a full, skeletal-small subcategory of mod R closed under finite limits and colimits. In particular, $\mathfrak{B}$ is an Abelian category, and $\mathfrak{B} \xrightarrow{\text{inj}} \text{mod } R$ is exact. Let $\hat{\mathfrak{B}}$ be the following category: The objects of $\hat{\mathfrak{B}}$ are R-modules X together with a topology which makes X into a complete (and Hausdorff) topological group and which admits a basis (of neighborhoods of 0) consisting of submodules X' with $X/X' \in \mathfrak{B}$. The morphisms of $\hat{\mathfrak{B}}$ are the continuous R-linear maps. I consider $\mathfrak{B}$ as a full subcategory of $\hat{\mathfrak{B}}$ by equipping the modules in $\mathfrak{B}$ with the discrete topology. A submodule X' of $X \in \hat{\mathfrak{B}}$ is called *special open* if it is open and $X/X' \in \mathfrak{B}$. A submodule $Y \subseteq X \in \hat{\mathfrak{B}}$ is called *special closed* if it is closed in X and if $Y \in \hat{\mathfrak{B}}$ with the induced topology. Notice here that $\{X' \cap Y;\ X' \subseteq X \text{ special open}\}$ is a basis of Y and that Y is complete w.r.t. the induced topology if it is closed. E.g., if $B \in \mathfrak{B}$, then a submodule B' of B (with the discrete topology) is special open iff it is special closed iff it lies itself in $\mathfrak{B}$. Finally a module $X \in \hat{\mathfrak{B}}$ is called *strict* if each special closed submodule Y of X with $X/Y \in \mathfrak{B}$ is open. Here no topology is considered on X/Y. E.g., the modules $B \in \mathfrak{B}$ are strict w.r.t. the discrete topology according to the above remark. The strict modules of $\hat{\mathfrak{B}}$ form the full subcategory $\tilde{\mathfrak{B}}$ of $\hat{\mathfrak{B}}$.

EXAMPLE 5.1. If R is a left linear topological ring and if $\mathfrak{B} = \mathrm{Coh}(\mathrm{Dis}\ R)$, then $\hat{\mathfrak{B}} = \mathrm{CTC}(R)$ and $\tilde{\mathfrak{B}} = \mathrm{STC}(R)$.

LEMMA 5.2. *Let X_1 and X_2 be special open submodules of $X \in \hat{\mathfrak{B}}$. Then $X_1 \cap X_2$ and $X_1 + X_2$ are special open in X.*

Proof. Since $X_1 \cap X_2$ is obviously open in X and since $X \in \hat{\mathfrak{B}}$ there is a special open $X' \subseteq X$ with $X' \subseteq X_1 \cap X_2$. In mod R one obtains the diagram

$$X/X' \xrightarrow{e} X/X_1 \cap X_2 \xrightarrow{i} X/X_1 \times X/X_2$$

where e and i are the canonical surjection resp. injection. But X/X', X/X_1, X/X_2, and hence $X/X_1 \times X/X_2 \in \mathfrak{B}$. Since $\mathfrak{B}$ is closed under image this implies $X/X_1 \cap X_2 \in \mathfrak{B}$.

Moreover $X_1 + X_2$ contains X', and is hence open in X. The canonical sequence of R-modules

$$0 \to X/X_1 \cap X_2 \to X/X_1 \times X/X_2 \to X/X_1 + X_2 \to 0$$

is exact with $X/X_1 \cap X_2$, $X/X_1 \times X/X_2 \in \mathfrak{B}$. Hence $X/X_1 + X_2 \in \mathfrak{B}$, and $X_1 + X_2$ is also special open. ||

LEMMA 5.3. *Let $X \in \hat{\mathfrak{B}}$ and Y be a closed submodule of X. Then Y is special closed iff for each special open submodule $X' \subseteq X$ the module $Y/X' \cap Y$ is in $\mathfrak{B}$.*

Proof. The condition is obviously sufficient. Assume that Y is special closed in X and that X' is special open in X. Since $X' \cap Y$ is an open submodule of Y w.r.t. the induced topology and since Y is special closed there is a special open $Y' \subseteq Y$ such that $Y' \subseteq X' \cap Y$. There results the diagram

$$Y/Y' \xrightarrow{e} Y/X' \cap Y \xrightarrow{i} X/X'$$

of R-modules where e resp. i are surjective resp. injective and where Y/Y' and X/X' are in $\mathfrak{B}$. Since $\mathfrak{B}$ is closed under finite limits and colimits it is closed under images, hence $Y/X' \cap Y \in \mathfrak{B}$. ||

LEMMA 5.4. *Let $X \in \hat{\mathfrak{B}}$. A special open submodule of X is special closed in X. An open and special closed submodule of X is special open.*

Proof. Let Y be special open in X and let X' be any other special open submodule of X. Then $X' \cap Y$ is special open in X by Lemma 5.2. The short exact sequence

$$0 \to Y/X' \cap Y \to X/X' \cap Y \to X/Y \to 0$$

lies fully in $\mathfrak{B}$ since both $X/X' \cap Y$ and X/Y lie in $\mathfrak{B}$. Since all $Y/X' \cap Y$ lie in $\mathfrak{B}$ and since an open submodule is always also closed the submodule Y is special closed in X.

Let, on the other side, Y be an open and special closed submodule of X. There is a special open $X' \subseteq Y$ which gives rise to the exact sequence

$$0 \to Y/X' = Y/X' \cap Y \to X/X' \to X/Y \to 0$$

Since Y is special closed the module $Y/X' \cap Y$ lies in $\mathfrak{B}$ (Lemma 5.3). Also X/X' and hence X/Y lie in $\mathfrak{B}$. This means that Y is special open. ||

Additional assumption: In the remaining part of this paragraph I assume that $\mathfrak{B}$ is linearly compact in the sense of 4.5. I am going to show by a series of lemmas that $\tilde{\mathfrak{B}}$ is a co-Grothendieck category.

Most of the results of this paragraph are based on the following valuable result of N. Bourbaki ([1], p. 85) on inverse limits. I state this in detail. Assume that the following data are given:

(a) A directed set I and an inverse system $(E_i, f_{ij} : E_j \to E_i,\ i \leqslant j$ in $I)$ of sets and functions.

(b) For each $i \in I$ a subset $\mathfrak{S}_i$ of the power set of E_i.

These data are supposed to satisfy the following conditions:

(1) For each $i \in I$ the set $\mathfrak{S}_i$ is closed under arbitrary intersections and compact.

(2) For each $i \leqslant j$ in I and each $x \in E_i$ the fiber $f_{ij}^{-1}(x)$ of x in E_j lies in $\mathfrak{S}_j$.

(3) For each $i \leqslant j$ in I and each $S \in \mathfrak{S}_j$ the image $f_{ij}(S)$ lies in $\mathfrak{S}_i$.

THEOREM 5.5. ([1], p. 85). *Data and conditions as above.* Let $E = \lim_i E_i$ *with the canonical projections* $f_i : E \to E_i$. *Then*

$$(*) \qquad f_i(E) = \bigcap_{i \leqslant j} f_{ij}(E_j), \qquad \text{all } i \in I. \quad ||$$

Remark also that the conditions (1), (3), and (*) imply that $f_i(E) \in \mathfrak{S}_i$.

If $(B_i, f_{ij} : B_j \to B_i,\ i \leqslant j$ in (I), I directed, is an inverse system in the linearly compact category $\mathfrak{B} \subset \operatorname{mod} R$ then this inverse system together with $\mathfrak{S}_i = \operatorname{res}_{\mathfrak{B}} B_i$, $i \in I$, satisfies the conditions of Bourbaki's theorem. Hence $f_i(B) = \bigcap_{i \leqslant j} f_{ij}(B_j)$, all $i \in I$, and $f_i(B) \in \mathfrak{B}$.

LEMMA 5.6. *Let*

$$(f_i\,;\, i \in I) : (B_i\,;\, i \in I) \to (C_i\,;\, i \in I),$$

I directed, be a morphism between inverse systems in $\mathfrak{B}$. *Assume that the* f_i *are monomorphisms and that* $C = \lim_i C_i \in \mathfrak{B}$ *where the latter limit is taken in* mod R. *Then* $\lim_i B_i \in \mathfrak{B}$.

Proof. I assume $B_i \subseteq C_i$ w.l.o.g. The sequence

$$0 \longrightarrow \lim_i B_i \longrightarrow \lim_i C_i = C \xrightarrow{\text{can}} \lim_i C_i/B_i$$

is exact. Hence

$$\lim_i B_i = \bigcap_i \ker(C \xrightarrow{\text{proj}} C_i \xrightarrow{\text{can}} C_i/B_i).$$

Since C is $\mathfrak{B}$-linearly compact this intersection lies in $\mathfrak{B}$. ||

LEMMA 5.7. *Let I be directed and let*

$$0 \longrightarrow (B_i'\,;\, i \in I) \xrightarrow{(f_i)} (B_i\,;\, i \in I) \xrightarrow{(g_i)} (B_i''\,;\, i \in I) \longrightarrow 0$$

be an exact sequence of inverse systems in $\mathfrak{B}$. *Then*

$$0 \longrightarrow \lim_i B_i' \xrightarrow{f=\lim_i f_i} \lim_i B_i \xrightarrow{g=\lim_i g_i} \lim_i B_i'' \longrightarrow 0$$

is exact in mod R.

Proof. There is only to show that g is surjective. Let $p_{ij} : B_j \to B_i$ be the maps of the inverse system. Let $(x_i''\,;\, i \in I)$ be an element of $\lim_i B_i''$. Then $(g_i^{-1}(x_i''); p_{ij})$ is an inverse system of nonempty sets. It is enough to show that $\lim_i g_i^{-1}(x_i'') \neq \emptyset$. For if $(x_i\,;\, i \in I)$ is an element of this limit, then $(x_i\,;\, i \in I) \in \lim_i B_i$ and $g_i(x_i) = x_i''$, all $i \in I$, i.e., $g(x_i\,;\, i \in I) = (x_i''\,;\, i \in I)$. Let

$$\mathfrak{S}_i = \{K;\, K \in \operatorname{res}_{\mathfrak{B}} B_i\,,\, K \subseteq g_i^{-1}(x_i'')\}.$$

Since $\mathfrak{S}_i$ is a section of $\operatorname{res}_{\mathfrak{B}} B_i$ it is easily verified that the inverse system $(g_i^{-1}(x_i''); i \in I)$ with these $\mathfrak{S}_i$ satisfies the assumptions of Bourbaki's theorem. In particular $\lim_i g^{-1}(x_i'') \neq \emptyset$. ||

LEMMA 5.8. *Let* $(B_i\,;\, i \in I)$, *I directed, be an inverse system in* $\mathfrak{B}$. *Then the R-module* $X = \lim_i B_i$ *with the inverse limit topology lies in* $\hat{\mathfrak{B}}$.

Proof. Let $p_i : X \to B_i$ be the canonical projections. Obviously X is an R-module with a topology which has the basis $\ker p_i$, $i \in I$, and makes X into a complete Abelian group. In order to show $X \in \hat{\mathfrak{B}}$ it is enough to show that $X/\ker p_i \in \mathfrak{B}$, $i \in I$. But

$$X/\ker p_i \cong p_i(X) = \bigcap_{i \leqslant j} p_{ij}(B_j)$$

lies in $\mathfrak{B}$ by Bourbaki's theorem applied to the inverse system $(B_i\,;\, i \in I)$. ||

LEMMA 5.9. *If $(X_i\,;\, i \in I)$, I any set, is a family of $X_i \in \hat{\mathfrak{B}}$, then $X = \prod_i X_i$ with the product topology is again in $\hat{\mathfrak{B}}$.*

Proof. Obviously X is an R-module and complete w.r.t. the product topology. The product topology has a basis consisting of the modules $\prod_i X_i'$ where $X_i' \subseteq X_i$ is special open and $X_i = X_i'$, almost all $i \in I$. The factor modules

$$\prod_i X_i \Big/ \prod_i X_i' \cong \prod_{i \in I'} X_i/X_i'$$

are in $\mathfrak{B}$ where I' is the finite set of those i with $X_i' \neq X_i$. Hence $\prod X_i \in \hat{\mathfrak{B}}$. ||

LEMMA 5.10. *Let $X \in \hat{\mathfrak{B}}$ and Y be a special closed submodule of X. Then X/Y with the coinduced topology lies in $\hat{\mathfrak{B}}$.*

Proof. The coinduced topology on X/Y has the basis $X' + Y/Y$ where X' runs over the special open submodules of X. Moreover, $(X/Y)/(X' + Y/Y) \cong X/X' + Y$, and the sequence

$$0 \to X' + Y/X' \cong Y/X' \cap Y \to X/X' \to X/X' + Y \to 0$$

is exact. Since Y is special closed one obtains $Y/X' \cap Y \in \mathfrak{B}$ (Lemma 5.3). Since also $X/X' \in \mathfrak{B}$ this implies $X/X' + Y \in \mathfrak{B}$. There remains to be shown that the canonical map

$$X/Y \to \varprojlim_{X'} X/X' + Y$$

is a bijection. Here X' runs over all special open submodules of X. This follows from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} & & \\
0 & \longrightarrow & \varprojlim_{X'} Y/X' \cap Y & \longrightarrow & \varprojlim_{X'} X/X' & \longrightarrow & \varprojlim_{X'} X/X' + Y & \longrightarrow & 0
\end{array}$$

where the lower sequence is exact by Lemma 5.7. Since Y and X are in $\hat{\mathfrak{B}}$ the two left vertical arrows are bijections, hence

$$X/Y \xrightarrow{\text{can}} \lim_{X'} X/X' + Y$$

is a bijection too. ||

Lemma 5.11. *Let $(X_i\,;\, i \in I)$, I directed, be a decreasing family of special closed submodules of $X \in \hat{\mathfrak{B}}$. Then the R-module $Y := \lim_{i \in I} X/X_i$ with the limit topology lies in $\hat{\mathfrak{B}}$, and the canonical map*

$$X \xrightarrow{\text{can}} Y = \lim_{i} X/X_i$$

is surjective.

Proof. (1) By 5.10 $X/X_i \in \hat{\mathfrak{B}}$. The module X/X_i has the basis of open submodules $X_i + X'/X_i$ where X' runs over all special open submodules of X. There results the isomorphism

$$X/X_i \to \lim_{X'} X/X' + X_i$$

in $\hat{\mathfrak{B}}$. Here $X/X' + X_i \in \mathfrak{B}$ and $\lim_{X'}$ has the limit topology. There are the topological isomorphisms

$$Y = \lim_{i} X/X_i \cong \lim_{i} \lim_{X'} X/X' + X_i \cong \lim_{i,X'} X/X' + X_i\,.$$

Since the $X/X' + X_i$ are discrete and in $\mathfrak{B}$, Lemma 5.8 shows $Y \in \hat{\mathfrak{B}}$. The canonical surjections

$$X/X' \to X/X' + X_i$$

induce the surjection

$$X \cong \lim_{X'} X/X' \cong \lim_{i,X'} X/X' \xrightarrow{\text{can}} \lim_{i,X'} X/X_i + X' \cong Y$$

by Lemma 5.7. ||

Lemma 5.12. *If $X \xrightarrow{f} Y$ is a morphism in $\hat{\mathfrak{B}}$ and if $Y' \subseteq Y$ is special open, then $f^{-1}(Y')$ is special open.*

Proof. Obviously $f^{-1}(Y')$ is open. Hence there is a special open $X' \subseteq X$ with $X' \subseteq f^{-1}(Y')$. One obtains the diagram

$$X/X' \xrightarrow{e} X/f^{-1}(Y') \xrightarrow{i} Y/Y'$$

where e is surjective, i is injective, and X/X' and Y/Y' are in $\mathfrak{B}$. So $X/f^{-1}(Y')$ is in $\mathfrak{B}$, i.e., $f^{-1}(Y')$ is special open. ||

The following lemma is crucial.

LEMMA 5.13. *If $f: X \to Y$ is a morphism in $\hat{\mathfrak{B}}$, then $f(X)$ is special closed.*

Proof. (1) Let I be the set of all pairs (X', Y') where X' resp. Y' are special open in X resp. Y and where $f(X') \leqslant Y'$. The set I is ordered by

$$i' = (X', Y') \leqslant i'' = (X'', Y'') \quad \text{iff} \quad X' \supseteq X'', \quad Y' \supseteq Y''.$$

It is obvious that I is a directed set. For $i = (X', Y') \in I$ define

$$X_i = X', \quad Y_i = Y', \quad Z_i = f^{-1}(Y').$$

If Y' is any special open submodule of Y, then $i = (f^{-1}(Y'), Y') \in I$ by the preceding lemma, hence $Y' = Y_i$. If X' is any special open submodule of X, then $i = (X', Y) \in I$, hence $X' = X_i$. Since the special open submodules of X resp. Y form a basis of X resp. Y one obtains that

$$\{X_i\,;\, i \in I\} \quad \text{resp.} \quad \{Y_i\,;\, i \in I\}$$

is a basis of X resp. Y. In particular there result the canonical isomorphisms in $\hat{\mathfrak{B}}$

$$X \to \varprojlim_i X/X_i\,,$$

and

$$Y \to \varprojlim_i Y/Y_i\,.$$

For $i = (X', Y') \in I$ one has the sequences

$$X/X_i = X/X' \xrightarrow{g_i} X/Z_i = X/f^{-1}(Y') \xrightarrow{h_i} Y/Y_i = Y/Y'$$

in $\mathfrak{B}$ where g_i is surjective, h_i is injective and $h_i g_i$ is the function induced from f. There results the sequence in mod R

$$\varprojlim_i X/X_i \xrightarrow{g=\lim_i g_i} \varprojlim_i X/Z_i \xrightarrow{h=\lim_i h_i} \varprojlim_i Y/Y_i$$

where, by Lemma 5.7, g is again surjective and h is injective. Identifying the two outer limits with X resp. Y one gets $hg = f$, and w.l.o.g.

$$f(X) = \varprojlim_i X/Z_i \quad \text{and} \quad h = \text{inj}.$$

The identification $f(X) = \lim_i X/Z_i$ is first meant algebraically only. The module $f(X)$ with the limit topology is in $\hat{\mathfrak{B}}$ by Lemma 5.8. So this lemma will be proved once one has shown that the limit topology is the same as the induced topology. But the limit topology on $f(X)$ has the basis

$$\{\ker(f(X) \xrightarrow{\text{can}} X/f^{-1}(Y')); (X', Y') \in I\}.$$

Since $\ker(f(X) \xrightarrow{\text{can}} X/f^{-1}(Y')) = f(X) \cap Y'$, the limit topology on $f(X)$ has the basis $\{f(X) \cap Y'; Y' \text{ special open in } Y\}$.

But this is obviously also a basis of the induced topology. Hence $f(X)$ with the induced topology lies in $\hat{\mathfrak{B}}$, i.e., $f(X)$ is special closed. ||

LEMMA 5.14. *Let $f: B \to X$ be a continuous linear map from $B \in \mathfrak{B}$ to $X \in \hat{\mathfrak{B}}$. Then* $\ker f \in \mathfrak{B}$.

Proof. For special open $X' \subseteq X$

$$f^{-1}(X') = \ker(B \xrightarrow{f} X \xrightarrow{\text{can}} X/X') \in \mathfrak{B}$$

since $\mathfrak{B}$ is closed under taking kernels. The $(f^{-1}(X'); X' \subseteq X, X' \text{ special open})$ are a decreasing family of subobjects of B in $\mathfrak{B}$, hence

$$\ker f = f^{-1}(0) = f^{-1}\Big(\bigcap_{X'} X'\Big) = \bigcap_{X'} f^{-1}(X') \in \mathfrak{B}$$

since $\mathfrak{B}$ is linearly compact. ||

LEMMA 5.15. *If $f: X \to Y$ is a morphism in $\hat{\mathfrak{B}}$ and if X'* resp. *Y' are special closed submodules of X* resp. *Y, then $f^{-1}(Y')$* resp. *$f(X')$ are special closed in X* resp. *Y.*

Proof. (1) Since X' is special closed in X it lies in $\hat{\mathfrak{B}}$. Then $f(X')$ is the image of the morphism $X' \xrightarrow{\text{inj}} X \xrightarrow{f} Y$ in $\hat{\mathfrak{B}}$, so $f(X')$ is special closed in Y by Lemma 5.13.

(2) I show first that the kernel $\ker f$ of f is special closed in X. It is obviously closed in X. So let X' be special open in X. Then X' is also special closed in X, hence $f(X')$ is special closed in Y by 1). There results the commutative diagram in $\hat{\mathfrak{B}}$

$$\begin{array}{ccccccc}
 & & & & X & \xrightarrow{f} & Y \\
 & & & & \downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle \text{can}} \\
0 & \longrightarrow & X' + \ker f/X' = f^{-1}(f(X'))/X' & \longrightarrow & X/X' & \xrightarrow{f'} & Y/f(X')
\end{array}$$

where the lower sequence is an exact sequence of R-modules with continuous

f'. Remark here that $Y/f(X') \in \hat{\mathfrak{B}}$ by Lemma 5.10, and $X/X' \in \mathfrak{B}$. By Lemma 5.14

$$f^{-1}(f(X'))\, X' = X' + \ker f/X' \cong \ker f/\ker f \cap X'$$

is in $\mathfrak{B}$ which shows that $\ker f$ is special closed.

(3) Let now Y' be any special closed submodule of Y. By Lemma 5.10 $Y \to Y/Y'$ is a morphism in $\hat{\mathfrak{B}}$. Then $f^{-1}(Y')$ is the kernel of the morphism $X \xrightarrow{f} Y \xrightarrow{\text{can}} Y/Y'$ in $\hat{\mathfrak{B}}$. By (2) $f^{-1}(Y')$ is special closed in X. ||

Corollary 5.16. *If $f : X \to B$ is a morphism in $\hat{\mathfrak{B}}$ with $B \in \mathfrak{B}$, then $f(X)$ lies in $\mathfrak{B}$,* i.e. $\ker f$ *is special open.*

Proof. Since B is discrete $\ker f$ is open. By Lemma 5.15 $\ker f$ is special closed. These properties together mean that $\ker f$ is special open (Lemma 5.4), i.e., that $X/\ker f \cong f(X) \in \mathfrak{B}$. ||

Corollary 5.17. *The category $\mathfrak{B} \subset \hat{\mathfrak{B}}$ is dense in $\hat{\mathfrak{B}}$.*

Proof. For $X \in \mathfrak{B}$ let $(X/\mathfrak{B})'$ be the full subcategory of $X/\mathfrak{B}$ consisting of all surjections

$$f : X \to B \quad \text{in} \quad \hat{\mathfrak{B}}, \qquad B \in \mathfrak{B}.$$

This category $(X/\mathfrak{B})'$ is equivalent to the full subcategory of $X/\mathfrak{B}$ consisting of all $X \to X/X'$ where X' is special open in X. Hence $X \cong \lim_{(X/\mathfrak{B})'} B$. In order to show that $\mathfrak{B}$ is dense in $\hat{\mathfrak{B}}$ it is thus enough to show that for all $X \in \hat{\mathfrak{B}}$ the category $X/\mathfrak{B}$ is filtered from below and that for each $X \xrightarrow{f} B$ in $X/\mathfrak{B}$ there is an object $X \xrightarrow{f'} B'$ in $(X/\mathfrak{B})'$ and a morphism $(X \xrightarrow{f'} B') \to (X \xrightarrow{f} B)$ in $X/\mathfrak{B}$. (Lemma 1.1). The latter assertion is trivial since f factorizes as $X \xrightarrow{f'} f(X) \xrightarrow{\text{inj}} B$ where f' is surjective and $f(X) \in \mathfrak{B}$ by Lemma 5.15. I now check the two conditions for "filtered from below." If $f_i : X \to B_i$, $i = 1, 2$, are two objects in $X/\mathfrak{B}$, then $\ker f_i$, $i = 1, 2$, is special open by 5.15, hence $\ker f_1 \cap \ker f_2$ is special open. There are the canonical morphisms

$$(X \longrightarrow X/\ker f_1 \cap \ker f_2) \longrightarrow (X \xrightarrow{f_i} B_i)$$

in $X/\mathfrak{B}$.

If

$$(X \xrightarrow{f} B) \underset{g}{\overset{h}{\rightrightarrows}} (X \xrightarrow{f'} B')$$

are two morphisms in $X/\mathfrak{B}$, then they are equalized by

$$(X \xrightarrow{f_{\text{ind}}} \ker(h, g)) \xrightarrow{\text{inj}} (X \xrightarrow{f} B).$$

Hence $X/\mathfrak{B}$ is filtered from below. ||

COROLLARY 5.18. *If X_1 and X_2 are special closed in $X \in \hat{\mathfrak{B}}$, then $X_1 \cap X_2$ is special closed in X.*

Proof. $X_1 \cap X_2$ is the kernel of the diagonal $X \to X/X_1 \times X/X_2$ where both X/X_1 and X/X_2 are in $\hat{\mathfrak{B}}$ by 5.10. The assertion follows from 5.15.

I now consider the problem of infinite limits in $\mathfrak{B}$. For $X \in \hat{\mathfrak{B}}$ let

$$\operatorname*{res}_{\hat{\mathfrak{B}}} X = \{\phi\} \cup \{x + Y;\ x \in X,\ Y \subseteq X \text{ special closed}\}.$$

I show that $\operatorname{res}_{\hat{\mathfrak{B}}} X$ satisfies the properties needed for the application of Theorem 5.5.

LEMMA 5.19. (1) *If $X \in \hat{\mathfrak{B}}$ then $\operatorname{res}_{\hat{\mathfrak{B}}} X$ is closed under intersection and a compact lattice.*

(2) *A morphism $f : X \to Y$ in $\hat{\mathfrak{B}}$ induces the order preserving maps*

$$f_* : \operatorname*{res}_{\hat{\mathfrak{B}}} X \longrightarrow \operatorname*{res}_{\hat{\mathfrak{B}}} Y : X' \rightsquigarrow f(X')$$

and

$$f^* : \operatorname*{res}_{\hat{\mathfrak{B}}} X \to \operatorname*{res}_{\hat{\mathfrak{B}}} Y : f^{-1}(Y') \rightsquigarrow Y'.$$

In particular if $y \in Y$, then $f^{-1}(y) \in \operatorname{res}_{\hat{\mathfrak{B}}} X$.

Proof. (2) and (3) follow directly from 5.15. By 5.18 $\operatorname{res}_{\hat{\mathfrak{B}}} X$ is closed under finite intersections. So show only that if $(X_i\,;\, i \in I)$, I directed, is a decreasing family of special closed submodules of X, then $\bigcap_i X_i$ is again special closed and the canonical map $X \to \lim_i X/X_i$ is surjective. But this canonical map is a surjection in $\hat{\mathfrak{B}}$ by Lemma 5.11, and its kernel is $\bigcap_i X_i$. Lemma 5.15 implies that $\bigcap_i X_i$ is special closed in X, hence the assertion. ||

LEMMA 5.20. *Let*

$$(f_i\,;\, i \in I) : (X_i\,;\, i \in I) \to (Y_i\,;\, i \in I)$$

be a morphism of inverse systems in $\hat{\mathfrak{B}}$ where I is directed and all f_i are surjective. Then the function

$$\lim_i f_i : \lim_i X_i \to \lim_i Y_i$$

is surjective.

Proof. By use of the preceding lemmas, in particular Lemmas 5.15 and 5.19, the proof is totally the same as that of 5.7. ||

I take all the preceding lemmas together, and obtain the following

THEOREM 5.21. *Assumptions on $\mathfrak{B}$ as above. Then the category $\hat{\mathfrak{B}}$ admits products, limits, kernels, and cokernels. Indeed, these are the algebraic ones with the product, limit, induced* resp. *coinduced topologies. In particular a morphism f is a monomorphism* resp. *epimorphism iff it is injective* resp. *surjective. If*

$$(f_i\,;\, i \in I) : (X_i\,;\, i \in I) \to (Y_i\,;\, i \in I)$$

is an epimorphism of inverse systems in $\hat{\mathfrak{B}}$ with directed I, then $\lim_i f_i$ *is an epimorphism. The full subcategory $\mathfrak{B}$ of $\hat{\mathfrak{B}}$ is dense in $\hat{\mathfrak{B}}$.* ||

In general the category $\hat{\mathfrak{B}}$ will not be Abelian (see Proposition 5.24 for details). However $\tilde{\mathfrak{B}}$ is a co-Grothendieck category.

PROPOSITION 5.22. *$\mathfrak{B}$ as above. The category $\tilde{\mathfrak{B}}$ is coreflective in $\hat{\mathfrak{B}}$,* i.e., *the injection functor* $\mathrm{inj} : \tilde{\mathfrak{B}} \to \hat{\mathfrak{B}}$ *has a right adjoint* $s : \hat{\mathfrak{B}} \to \tilde{\mathfrak{B}}$ *such that* $s \cdot \mathrm{inj} \simeq \mathrm{id}_{\tilde{\mathfrak{B}}}$. *If $X \in \hat{\mathfrak{B}}$, then $s(X)$ has the same underlying module as X, and $s(X)$ has the basis*

$$\{Y;\ Y \subseteq X \text{ special closed, } X/Y \in \mathfrak{B}\}.$$

Here no topology is considered on X/Y.

Proof. (1) The topology on X will be distinguished by an upper index 1, if necessary. The upper index 2 is used for the new topology on X which defines $s(X)$.

I show first that $\{Y;\ Y \subseteq X$ special closed, $X/Y \in \mathfrak{B}\}$ is closed under finite intersections. Let Y_i, $i = 1, 2$, be special closed in X with $X/Y_i \in \mathfrak{B}$. By 5.18 $Y_1 \cap Y_2$ is special closed. With the coinduced topology X/Y_i lies in $\hat{\mathfrak{B}}$, and has the basis X_i/Y_i where X_i runs over all special open submodules X_i of X containing Y_i. In particular $X/Y_i \simeq \lim_{X_i} X/X_i$ in $\hat{\mathfrak{B}}$, hence

$$X/Y_1 \times X/Y_2 \cong \lim_{X_1, X_2} X/X_1 \times X/X_2$$

in $\hat{\mathfrak{B}}$.

The commutative diagrams

$$\begin{tikzcd} X/Y_1 \cap Y_2 \arrow[r, "\Delta"] \arrow[d, "\mathrm{can}"] & X/Y_1 \times X/Y_2 \arrow[d, "\mathrm{can}"] \\ X/X_1 \cap X_2 \arrow[r, "\Delta"] & X/X_1 \times X/X_2 \end{tikzcd}$$

with horizontal injections give rise to the commutative diagram

$$\begin{tikzcd} X/Y_1 \cap Y_2 \arrow[r] \arrow[d, "f"] & X/Y_1 \times X/Y_2 \arrow[d, "\cong"] \\ Z := \lim_{X_1, X_2} X/X_1 \cap X_2 \arrow[r, "g"] & \lim_{X_1, X_2} X/X_1 \times X/X_2 . \end{tikzcd}$$

By Lemma 5.6 Z is again in $\mathfrak{B}$ (no topology is considered) because

$$X/Y_1 \times X/Y_2 \cong \varprojlim_{X_1, X_2} X/X_1 \times X/X_2 \quad \text{is.}$$

But f is surjective by 5.11. Hence f is a linear isomorphism, and $X/Y_1 \cap Y_2$ is in $\mathfrak{B}$ (disregard the topology).

(2) By (1) there is a unique topology on X having

$$\{Y;\ Y \subseteq X \text{ special closed, } X/Y \in \mathfrak{B}\}$$

as basis and making X into a topological group. I show that X^2 (X with this new topology) is complete by showing that $X \xrightarrow{\text{can}} \lim_Y X/Y$ is bijective. The canonical isomorphism $X \xrightarrow{\text{can}} \lim_Z X/Z$ where Z runs over all special open submodules of X^1 (X with its original topology) factorizes as

$$X \xrightarrow{\text{can}} \varprojlim_Y X/Y \xrightarrow{p} \varprojlim_Z X/Z$$

where p is the canonical projection. But p is injective. For let

$$(x_Y + Y)_Y \in \varprojlim_Y X/Y,$$

and assume that $p((x_Y + Y)_Y) = (x_Z + Z)_Z = 0$. For fixed special closed Y one has $Y = \bigcap Z$ where Z runs over all special open submodules of X containing Y. In particular for each such Z the relation $x_Y + Z = x_Z + Z = Z$ holds, hence $x_Y \in \bigcap Z = Y$, i.e., $x_Y + Y = Y$. This implis $(x_Y + Y)_Y = 0$. The preceding calculation shows that $X \to \lim_Y X/Y$ is bijective, i.e., that X^2 ($= X$ with the new topology) is complete. But then obviously $X^2 \in \hat{\mathfrak{B}}$, and the identity id : $X^2 \to X^1$ is a morphism in $\hat{\mathfrak{B}}$.

(3) If $X \in \hat{\mathfrak{B}}$, then $X^2 \in \tilde{\mathfrak{B}}$. For let $Y \subseteq X^2$ be a special closed submodule of X^2 with $X/Y \in \mathfrak{B}$. Since id : $X^2 \to X^1$ is in $\hat{\mathfrak{B}}$ and by Lemma 5.15 Y is a special closed submodule of X^1 with $X/Y \in \mathfrak{B}$. By definition of the topology on X^2 this implies that Y is special open in X^2. Hence X^2 is strict.

(4) I show finally for each $X \in \hat{\mathfrak{B}}$ and each $Y \in \tilde{\mathfrak{B}}$ that

$$\hat{\mathfrak{B}}(Y, X^2) = \hat{\mathfrak{B}}(Y, X^1).$$

This relation obviously implies that the assignment

$$s : \hat{\mathfrak{B}} \to \tilde{\mathfrak{B}} : X^1 \rightsquigarrow X^2$$

is a functor, and indeed the right adjoint of the injection. Moreover, if $X \in \tilde{\mathfrak{B}}$, then $X^1 = X^2$ by the definiton of strict modules. Now obviously

$$\hat{\mathfrak{B}}(Y, X^2) \subseteq \hat{\mathfrak{B}}(Y, X^1).$$

Let on the other side $f: Y \to X^1$ be continuous and let $Z \subseteq X^2$ be a module of the basis constructed above, i.e., let Z be a special closed submodule of X^1 with $X/Z \in \mathfrak{B}$. I show that $f^{-1}(Z)$ is open. By lemma 5.15 and since $f: Y \to X^1$ is continuous, $f^{-1}(Z)$ is special closed. Also Z is the intersection of all special open X' in X^1 containing Z, and the $f^{-1}(X')$ are special open in Y. The commutative diagrams

$$\begin{array}{ccc} Y/f^{-1}(Z) & \xrightarrow{\text{inj}} & X/Z \\ \downarrow{\scriptstyle\text{can}} & & \downarrow{\scriptstyle\text{can}} \\ Y/f^{-1}(X') & \xrightarrow{\text{inj}} & X/X' \end{array}$$

give rise to the commutative diagram

$$\begin{array}{ccc} Y/f^{-1}(Z) & \xrightarrow{\text{inj}} & X/Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \wr\wr} \\ \varprojlim_{X'} Y/f^{-1}(X') & \xrightarrow{h} & \varprojlim_{X'} X/X'. \end{array}$$

By Lemma 5.6 and since $\lim_{X'} X/X' \cong X/Z$ is in $\mathfrak{B}$ (disregard the topology) the module $\lim_{X'} Y/f^{-1}(X')$ lies in $\mathfrak{B}$. By Lemma 5.20 g is surjective. Hence g is bijective, and thus $Y/f^{-1}(Z) \in \mathfrak{B}$ (disregard the topology). Since Y is strict and $f^{-1}(Z)$ is special closed in Y this implies that $f^{-1}(Z)$ is open. Hence $f \in \hat{\mathfrak{B}}(Y, X^2)$. ||

THEOREM 5.23. *Let $\mathfrak{B}$ be a linearly compact subcategory of the category* mod R, R *a ring. Then the category $\tilde{\mathfrak{B}}$ is a co-Grothendieck category. The cokernel in $\tilde{\mathfrak{B}}$ is the algebraic one with the coinduced topology. If $(X_i\,;\, i \in I)$ is an inverse system in $\tilde{\mathfrak{B}}$, I any category, then*

$$\lim{}^{\tilde{\mathfrak{B}}} X_i \cong s(\lim{}^{\hat{\mathfrak{B}}} X_i).$$

The category $\mathfrak{B}$ is a full, skeletal-small, dense and finitely closed subcategory of $\tilde{\mathfrak{B}}$.

Proof. (1) Since $\tilde{\mathfrak{B}}$ is a coreflective subcategory of $\hat{\mathfrak{B}}$ it is obvious that $\tilde{\mathfrak{B}}$ admits arbitrary limits and that

$$\lim{}^{\tilde{\mathfrak{B}}} X_i \cong s(\lim{}^{\hat{\mathfrak{B}}} X_i).$$

(2) I show that for $X \in \tilde{\mathfrak{B}}$ and a special closed $Y \subseteq X$ the module X/Y with the coinduced topology is also strict. From 5.10 one has $X/Y \in \hat{\mathfrak{B}}$. Let

$f : X \to X/Y$ be the canonical map, and let Z be a special closed submodule of X/Y with $(X/Y)/Z \in \mathfrak{B}$ (disregard the topology). By 5.15 $f^{-1}(Z)$ is special closed in X, and obviously

$$X/f^{-1}(Z) \cong (X/Y)/Z \in \mathfrak{B}.$$

Since X is strict this implies that $f^{-1}(Z)$ is special open. But then $Z = f(f^{-1}(Z))$ is open, and hence X/Y is also strict. This proves the assertion on cokernels in $\tilde{\mathfrak{B}}$.

(3) I show that every bijection in $\tilde{\mathfrak{B}}$ is an isomorphism which implies that $\tilde{\mathfrak{B}}$ is Abelian. So let $f : X \to Y$ be a bijection in $\tilde{\mathfrak{B}}$. Show that f^{-1} is continuous. Let $X' \subseteq X$ be special open. Then $(f^{-1})^{-1}(X') = f(X')$ is special closed in Y by 5.15. Also $X/X' \cong Y/f(X') \in \mathfrak{B}$ (disregard the topology). Since Y is strict this implies that $f(X') = (f^{-1})^{-1}(X')$ is open, so f^{-1} is continuous.

(4) By Lemma 5.20 $\lim_I$, where I is directed, preserves surjections in $\hat{\mathfrak{B}}$. Since for $X \in \hat{\mathfrak{B}}$ the modules X and $s(X)$ have the same underlying R-module it follows that $\lim_I$ preserves surjections in $\tilde{\mathfrak{B}}$ too, i.e., $\lim_I$ is exact. In other words: The category $\tilde{\mathfrak{B}}$ satisfies $(\mathrm{AB5})^{op}$.

(5) Since $\mathfrak{B}$ is dense in $\hat{\mathfrak{B}}$ (5.17) and contained in $\tilde{\mathfrak{B}}$ it is also dense in $\tilde{\mathfrak{B}}$. Of course $\mathfrak{B}$ is a skeletal-small, full subcategory of $\tilde{\mathfrak{B}}$ closed under finite limits and colimits. In order to show that $\mathfrak{B}$ is finitely closed in $\tilde{\mathfrak{B}}$ it is hence enough to show that it is closed under subobjects. But if $X \xrightarrow{f} B$ is a monomorphism in $\tilde{\mathfrak{B}}$ with $B \in \mathfrak{B}$ then $\ker f$ is special open in X by 5.15, hence X has the discrete topology and $X = X/0$ is in $\mathfrak{B}$ as module. But this means that $X \in \mathfrak{B}$ (including the topology).

(6) Since $\mathfrak{B}$ is dense in $\tilde{\mathfrak{B}}$ and since $\mathfrak{B}$ is skeletal-small the skeleton of $\mathfrak{B}$ is a family of cogenerators of $\tilde{\mathfrak{B}}$. This and the preceding considerations show that $\tilde{\mathfrak{B}}$ is a co-Grothendieck category. ||

The case which is covered in J.-E. Roos' paper ([11], Section 5, th. 5) is contained in the following

Proposition 5.24. *Let $\mathfrak{B}$ be a linearly compact subcategory of* mod R, R *any ring. The following assertions are equivalent.*

(1) $\hat{\mathfrak{B}}$ *is Abelian.*

(2) $\tilde{\mathfrak{B}} = \hat{\mathfrak{B}}$, i.e., *each module $X \in \hat{\mathfrak{B}}$ is strict.*

(3) $\mathfrak{B}$ *is artinian, i.e., for each $B \in \mathfrak{B}$ the ordered set* $\mathrm{sub}_{\mathfrak{B}}\, B$ *of all subobjects of B in $\mathfrak{B}$ satisfies the descending chain condition.*

Proof. (2) $\Rightarrow$ (1): trivial (1) $\Rightarrow$ (3): Let $B \in \mathfrak{B}$ and let $(B_i\,; i \in I)$, I

directed, be a decreasing family of submodules of B. By 5.8 $\lim_i B/B_i$ is in $\hat{\mathfrak{B}}$ with the limit topology, and by 5.7

$$B\Big/\bigcap_i B_i \xrightarrow{\text{can}} \lim_i B/B_i$$

is a bijection. With the discrete topology on $B/\bigcap_i B_i$ this canonical map is a bijective morphism in $\hat{\mathfrak{B}}$, hence an isomorphism, since $\hat{\mathfrak{B}}$ is assumed Abelian. Therefore the limit topology is discrete, which implies the existence of an index j with $B_j/\bigcap_i B_i = 0$, i.e., $B_j = \bigcap_i B_i$. But this means that $\text{sub}_{\mathfrak{B}} B$ is artinian.

(3) $\Rightarrow$ (2). Let $X \in \hat{\mathfrak{B}}$ and let Y be a special closed submodule of X with $X/Y \in \mathfrak{B}$ (disregard the topology). Then

$$0_{X/Y} = \bigcap_{X'} X'/Y$$

where X' runs over all special open submodules of X containing Y. In particular $X'/Y \in \text{sub}_{\mathfrak{B}} X/Y$. Since the latter set is artinian there is an X' with $X'/Y = 0$, i.e. $X' = Y$. This means that Y itself is special open.

VI. Co-Grothendieck Categories over Topological Rings

The Theorems 5.21, 5.23, and 4.12 applied to Coh(Dis R) where R is a topologically left linearly compact ring imply

THEOREM 6.1. *Let R be a topologically left linearly compact ring.*

(1) *The category* CTC(R) *of all complete, topologically coherent R-left modules admits all limits and cokernels. Indeed, these are the algebraic ones with the limit* resp. *coinduced topology. The epimorphisms* resp. *monomorphisms in* CTC(R) *are the surjections* resp. *injections. The category* Coh(Dis R) *is a skeletal-small, full, and dense subcategory of* CTC(R) *closed under finite limits and colimits.*

(2) *The category* STC(R) *of all strict modules in* CTC(R) *is a coreflective subcategory. The right adjoint s of the injection* STC(R) $\to$ CTC(R) *is the identity on the underlying modules.*

(3) *The category* STC(R) *is a co-Grothendieck category which contains* Coh(Dis R) *as a skeletal-small, full, dense, and finitely closed subcategory.*

Taking 3.4 and 6.1 and together one obtains the

THEOREM 6.2. *A category $\mathfrak{A}$ is a Grothendieck category iff it is dual to a category* STC(R) *where R is a strict complete topologically left coherent and linearly compact ring.*

Again, the results of J.-E. Roos ([11], Section 5, th. 6) are given by the following

PROPOSITION 6.3. *Let R be a strict complete topologically left coherent and linearly compact ring. The following assertions are equivalent:*

(1) *R is topologically left coperfect.*

(2) Coh(Dis $_R R$) *is artinian.*

(3) $\mathrm{STC}({}_R R) = \mathrm{CTC}({}_R R)$.

The proof follows from 5.24. ||

It is clear that in the situation of Theorem 6.1 the category STC(R) admits a projective generator. I am going to show that indeed $s\ (\hat{R})$ is one where $\hat{R}$ is the coherent completion of R. For the proof I need the following definitions: Let R be a left linear topological ring. A left linear topological R-module is called *topologically of finite type* if it has a basis of open submodules X' such that X/X' is finitely generated. The modules which are topologically of finite type and the continuous linear maps form the category TF(R). The categories CTC(R) and STC(R) are full subcategories of TF(R).

LEMMA 6.4. *If R is a topologically left linearly compact ring then the category* CTC(R) *is reflective in* TF(R), *i.e., the injection functor* $\mathrm{inj} : \mathrm{CTC}(R) \to \mathrm{TF}(R)$ *admits a left adjoint* $\hat{\ } : \mathrm{TF}(R) \to \mathrm{CTC}(R)$ with $\hat{\ } \cdot \mathrm{inj} \cong \mathrm{id}$. *If $X \in \mathrm{TF}(R)$, then $X \cong \lim_{X'} X/X'$ where X' runs over all open submodules X' of X such that* $X/X' \in \mathrm{Coh}(\mathrm{Dis}\ R)$.

Proof. (1) I show first that for $X \in \mathrm{TF}(R)$ the set $\{X';\ X' \subseteq X$ open submodule, $X/X' \in \mathrm{Coh}(\mathrm{Dis}\ R)\}$ is closed under finite intersections, and hence filtered from below in particular. Let X_i', $i = 1, 2$, be open submodules of X with $X/X_i' \in \mathrm{Coh}(\mathrm{Dis}\ R)$. Then $X_1' \cap X_2'$ is obviously open, hence there is an open submodule X' of X with $X' \subseteq X_1' \cap X_2'$ and X/X' of finite type. Since $X/X' \xrightarrow{\mathrm{can}} X/X_1' \cap X_2'$ is surjective this shows that $X/X_1' \cap X_2'$ is of finite type, too. On the other hand, there is the injection $X/X_1' \cap X_2' \to X/X_1' \times X/X_2'$. Hence $X/X_1' \cap X_2'$ is a submodule of finite type of the coherent module $X/X_1' \times X/X_2'$ which implies that $X/X_1' \cap X_2'$ is itself coherent.

(2) By (1) and Lemma 5.8 one sees that for $X' \in \mathrm{TF}(R)$ the module $\hat{X} := \lim_{X'} X/X'$ is complete and topologically coherent. Here X' runs over all open submodules of X with $X/X' \in \mathrm{Coh}(\mathrm{Dis}\ R)$. The limit is the algebraic

one with the limit topology. The module $\hat{X}$ is the completion of X w.r.t. the unique linear topology having $\{X'; X' \subseteq X$ open submodule, $X/X' \in \mathrm{Coh}(\mathrm{Dis}\, R)\}$ as a basis, and hence is called the coherent completion of X. If X is already complete and topologically coherent, then $X \simeq \hat{X}$. In general one has the canonical continuous linear map can : $X \to \hat{X}$.

(3) If $X \in \mathrm{TF}(R)$ and $Y \in \mathrm{CTC}(R)$ the function

$$(*) \qquad \mathrm{CTC}(\hat{X}, Y) \to \mathrm{TF}(X, Y) : f \rightsquigarrow f\,\mathrm{can}$$

is a bijection.

Let $g : X \to Y$ be continuous and linear. If Y' is a special open submodule of Y, then $g^{-1}(Y')$ is obviously open. Hence there is an open submodule X' of X s.t. $X' \subseteq g^{-1}(Y')$ and X/X' of finite type, thus $X/g^{-1}(Y')$ is a submodule of finite type of the coherent module $Y/Y' \in \mathrm{Coh}(\mathrm{Dis}\, R)$ which implies $X/g^{-1}(Y') \in \mathrm{Coh}(\mathrm{Dis}\, R)$. Hence $g : X \to Y$ is also continuous w.r.t. the topology on X having $\{X'; X' \subseteq X$ open, X/X' coherent$\}$ as basis. But then g factorizes uniquely as $X \xrightarrow{\mathrm{can}} \hat{X} \xrightarrow{f} Y$ because $\hat{X}$ is the completion of X w.r.t. this weaker topology. This implies that $(*)$ is bijective.

(4) The considerations in (3) imply that $X \rightsquigarrow \hat{X}$ is a functor, and indeed the left adjoint of the injection. Since $X \simeq \hat{X}$ for $X \in \mathrm{CTC}(R)$ the lemma is shown. ||

In particular, if R is a topologically left linearly compact ring, then ${}_R R$ is an object of $\mathrm{TF}(R)$. For every linear topological R-module X the function

$$\mathrm{hom}(R, X) \to X : f \rightsquigarrow f(1)$$

is a bijection. Here hom denotes the continuous linear maps. Hence R is a projective generator of $\mathrm{TF}(R)$. Using the left adjoint ^ one obtains the

COROLLARY 6.5. *If R is a topologically left linearly compact ring, then $\hat{R}$ is a projective generator of* $\mathrm{CTC}(R)$. *Indeed, if $X \in \mathrm{CTC}(R)$, then the function*

$$\mathrm{CTC}(\hat{R}, X) \to X : f \rightsquigarrow f(\hat{1})$$

is bijective. Here $\hat{1}$ denotes the image of 1 *under the canonical map $R \to \hat{R}$.* ||

THEOREM 6.6. *Let R be a topologically left linearly compact ring. Then $s(\hat{R})$ is a projective generator of* $\mathrm{STC}(R)$, *and for each $X \in \mathrm{Coh}(\mathrm{Dis}\, R) \subset \mathrm{STC}(R)$ there is a short exact sequence*

$$s(R)^k \to X \to 0, \qquad k \in \mathbb{N},$$

in $\mathrm{STC}(R)$.

Proof. (1) Remark that the underlying modules of $s(\hat{R})$ and $\hat{R}$ are the same, so that $\hat{1} \in s(\hat{R})$. Let $i : s(\hat{R}) \to \hat{R}$ be the identity function which is a bijection in $\mathrm{CTC}(R)$. For each $X \in \mathrm{STC}(R)$ the bijection

$$\mathrm{CTC}(\hat{R}, X) \to X : f \rightsquigarrow f(\hat{1})$$

factorizes as

$$\mathrm{CTC}(\hat{R}, X) \xrightarrow{\mathrm{inj}} \mathrm{CTC}(s(\hat{R}), X) = \mathrm{STC}(s(\hat{R}), X) \longrightarrow X,$$

hence the function

$$\mathrm{STC}(s(\hat{R}), X) \to X : f \rightsquigarrow f(\hat{1})$$

is surjective. This implies that $s(\hat{R})$ is a generator of $\mathrm{STC}(R)$.

(2) In order to show that $s(\hat{R})$ is projective it is enough to prove that each surjection $f : X \to s(\hat{R})$ in $\mathrm{STC}(R)$ has a right inverse. But $(if) : X \to \hat{R}$ is a surjection in $\mathrm{CTC}(R)$ and $\hat{R}$ is projective in $\mathrm{CTC}(R)$ by 6.4, hence there is a continuous linear $g : \hat{R} \to X$ with $ifg = \mathrm{id}_{\hat{R}}$. This implies $ifgi = i$ and then $f(gi) = \mathrm{id}_{s(\hat{R})}$, so f has a right inverse.

(3) Let $X \in \mathrm{Coh}(\mathrm{Dis}\, R)$. In particular X is finitely generated, hence there is a surjection

$$R^k \to X \to 0 \quad \text{in} \quad \mathrm{TF}(R).$$

The two functors $\hat{}$ and s preserve surjections, hence

$$s(\hat{R})^k \to s(\hat{X}) = X$$

is surjective. This shows the last assertion. ||

The Theorems 6.1 and 6.6 show that the triple $\mathrm{STC}({}_RR)^{op}$, $\mathrm{Coh}(\mathrm{Dis}\, {}_RR)^{op}$, and $s(\hat{R})$ is one of the triples $\mathfrak{A}$, $\mathfrak{N}$, E considered in Theorem 3.4. Hence the ring $\mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))$ is a strict complete topologically right coherent and linearly compact ring. Let

$$\bar{R} := \mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))^{op},$$

the opposite ring of $\mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))$ with the same topology. Then $\bar{R}$ is a strict complete topologically left coherent and linearly compact ring. More generally, if $X \in \mathrm{STC}({}_RR)$, then $\mathrm{STC}({}_RR)(s(\hat{R}), X)$ with the topology defined in section 3 in $\mathrm{STC}({}_{\bar{R}}\bar{R})$. Remark here that $\mathrm{STC}({}_RR)(s(\hat{R}), X)$ is a right $\mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))$-module, but a left $\bar{R}$-module. If X' runs over the open submodules of X with $X/X' \in \mathrm{Coh}(\mathrm{Dis}\, {}_RR)$, then $\mathrm{STC}({}_RR)(s(\hat{R}), X')$ runs over a basis of neighborhoods of 0 in $\mathrm{STC}({}_RR)(s(\hat{R}), X)$. Hence Theorem 3.4 implies the

Theorem 6.7. *Let R be a topologically left linearly compact ring, and let $\hat{R}$, $s(\hat{R})$, $\bar{R}$ be as above.*

(1) *The ring $\bar{R}$ is a strict complete topologically left coherent and linearly compact ring.*

(2) *The functor*

$$X \rightsquigarrow \mathrm{STC}({}_RR)(s(\hat{R}), X)$$

defines an equivalence

$$\mathrm{STC}({}_RR) \to \mathrm{STC}({}_{\bar{R}}\bar{R}).$$

The canonical ring homomorphism

$$R \to \mathrm{TF}({}_RR)(R, R)^{op} : x \rightsquigarrow \rho_x : r \rightsquigarrow rx$$

induces the canonical ring homomorphism

$$R \to \bar{R} = \mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))^{op} : x \rightsquigarrow s(\hat{\rho}_x).$$

Proposition 6.8. *R as in Theorem 6.7. The canonical ring homomorphism*

$$R \to \bar{R} = \mathrm{STC}({}_RR)(s(\hat{R}), s(\hat{R}))^{op} : x \rightsquigarrow s(\hat{\rho}_x)$$

is continuous.

Proof. A typical neighborhood of $\bar{R}$ is of the form $\mathrm{STC}({}_RR)(s(\hat{R}), Z)$ where Z is a special open submodule of $s(\hat{R})$. It is enough to show that the left ideal

$$\mathfrak{a} := \{x \in R;\ s(\hat{\rho}_x) \in \mathrm{STC}({}_RR)(s(\hat{R}), Z)\}$$

is open in R. But $x \in \mathfrak{a}$ iff $s(\hat{\rho}_x)(s(\hat{R})) \subseteq Z$ which is the same as $\hat{\rho}_x(\hat{R}) \subseteq Z$. Since $R \xrightarrow{\text{can}} \hat{R} : 1 \rightsquigarrow \hat{1}$ is continuous with dense image and since

$$\begin{array}{ccc} R & \xrightarrow{\rho_x} & R \\ \downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle \text{can}} \\ \hat{R} & \xrightarrow{\hat{\rho}_x} & \hat{R} \end{array}$$

commutes, this latter condition means that $R\hat{x} \subseteq Z$ where $\hat{x}$ is the image of x under the canonical map

$$f : R \to s(\hat{R}) : x \rightsquigarrow \hat{x} = x\hat{1}.$$

But f is continuous since $s(\hat{R})$ is a topological R-module, and I just calculated that $\mathfrak{a} = f^{-1}(Z)$. Since Z is open in $s(\hat{R})$ the left ideal $\mathfrak{a}$ is open in R.

In the situation of the preceding theorem I call $\bar{R}$ the strict coherent completion of R. It is not clear to me at this time in which way the assignment $R \rightsquigarrow \bar{R}$ is a functor.

VII. Topological Morita Theorems

The problem of this Section is to determine the relationship between two strict complete topologically left coherent and linearly compact rings R and S when one knows that the categories $\mathrm{STC}(R)$ and $\mathrm{STC}(S)$ are equivalent. In other words, what can be derived from an equivalence $\mathfrak{A}^{op} \simeq \mathrm{STC}(R)$ where R is as above and $\mathfrak{A}$ a Grothendieck category? Assume thus that $\mathfrak{A}$ is a Grothendieck category and R a strict complete topologically left coherent and linearly compact ring. Assume that E is an injective cogenerator of $\mathfrak{A}$ and that $h : R \to \mathfrak{A}(E, E)$ is a ring isomorphism (no topology here). Identify $R = \mathfrak{A}(E, E)$ via h. I want to reconstruct the category $\mathfrak{N}$ used in Theorem 3.4. Let $\mathfrak{N}$ be the full subcategory of $\mathfrak{A}$ of all those N such that $\mathfrak{A}(N, E) \in \mathrm{Coh}(\mathrm{Dis}\ R)$ and that there is a monomorphism $N \to E^k$, some $k \in \mathbb{N}$. Remark here that $\mathfrak{A}(N, E)$ is a left $R = \mathfrak{A}(E, E)$-module.

Lemma 7.1. *Situation as described above. Assume that for each special open left ideal* $\mathfrak{a}$ *of* R *there is a subobject* $N \subseteq E$ *with* $\mathfrak{a} = \mathfrak{A}(E/N, E)$. *Then the functor* $A \rightsquigarrow \mathfrak{A}(A, E)$ *induces an equivalence*

$$\mathfrak{N}^{op} \to \mathrm{Coh}(\mathrm{Dis}\ R).$$

Proof. (1) By definition of $\mathfrak{N}$ and since E is an injective cogenerator of $\mathfrak{A}$ the functor $\mathfrak{A}(-, E)$ maps $\mathfrak{N}^{op}$ into $\mathrm{Coh}(\mathrm{Dis}\ R)$ and is indeed an embedding.

(2) The same proof as the one of Proposition 3.3, (1) shows that for $A \in \mathfrak{A}$ and $N \in \mathfrak{N}$ the map

$$\mathfrak{A}(A, N) \to \mathrm{hom}_R(\mathfrak{A}(N, E), \mathfrak{A}(\mathfrak{A}, E)) : f \rightsquigarrow \mathfrak{A}(f, E)$$

is bijective. Hence $\mathfrak{A}(-, E)$ is a full embedding.

(3) Let $X \in \mathrm{Coh}(\mathrm{Dis}\ R)$. Then there is a short exact sequence

$$\coprod_{i \in I} R/\mathfrak{a}_i \xrightarrow{g} \coprod_{j \in J} R/\mathfrak{b}_j \longrightarrow X \longrightarrow 0$$

where the $\mathfrak{a}_i$, $i \in I$ finite, and $\mathfrak{b}_j$, $j \in J$ finite, are special open left ideals of R. By assumption there are subobjects M_i resp. N_j of E with

$$R/\mathfrak{a}_i = \mathfrak{A}(M_i, E) \qquad \text{and} \qquad R/\mathfrak{b}_j = \mathfrak{A}(N_j, E).$$

The objects $M := \coprod_{i\in I} M_i$ and $N := \coprod_{j\in J} N_j$ are obviously in $\mathfrak{N}$. The function

$$g : \coprod_i R/\mathfrak{a}_i \cong \coprod_i \mathfrak{A}(M_i, E) \cong \mathfrak{A}(M, E) \to \coprod_j R/\mathfrak{b}_j$$
$$\cong \coprod_j \mathfrak{A}(N_j, E) \cong \mathfrak{A}(N, E)$$

is of the form $g = \mathfrak{A}(f, E), f : N \to M$, by (2). Let $K = \ker f$. Then

$$\mathfrak{A}(\ker f, E) \cong \operatorname{coker} \mathfrak{A}(f, E) = \operatorname{coker} g \cong X.$$

Also $\ker f \subseteq N \subseteq E^k$, some $k \in \mathbb{N}$. Hence $K \in \mathfrak{N}$, and $X \cong \mathfrak{A}(K, E)$. This implies that

$$\mathfrak{A}(-, E) : \mathfrak{N}^{op} \to \operatorname{Coh}(\operatorname{Dis} R)$$

is an equivalence. ||

The preceding lemma implies in particular that $\mathfrak{N}$ is Abelian, and that indeed $\mathfrak{N}$ is a full, skeletal-small subcategory of $\mathfrak{A}$ closed under finite limits and colimits.

LEMMA 7.2. *Assumptions as in the preceding lemma.*

(1) *If $N \in \mathfrak{N}$ and if $N' \subseteq N$ is a subobject of N such that N/N' can be embedded into some E^k, $k \in \mathbb{N}$, then $N' \in \mathfrak{N}$.*

(2) *If $A \in \mathfrak{A}$ and if $\mathfrak{A}(A, E) \in \operatorname{Coh}(\operatorname{Dis} R)$, then $A \in \mathfrak{N}$.*

Proof. (2) Follows at once from the preceding lemma and the second part of its proof.

(1) Since N/N' can be embedded into E^k, $k \in \mathbb{N}$, $\mathfrak{A}(N/N', E)$ is finitely generated. It is also a submodule of the coherent module $\mathfrak{A}(N, E)$. Hence $\mathfrak{A}(N/N', E)$ is coherent in Dis R and then $\mathfrak{A}(N', E)$ is coherent too. Moreover, $N' \subseteq N \subseteq E^1$, some $1 \in \mathbb{N}$. Hence $N' \in \mathfrak{N}$.

LEMMA 7.3. *Assumptions as in Lemma 7.1. Assume in addition that for each subobject $A \subseteq E$ the left ideal $\mathfrak{A}(E/A, E)$ of $R = \mathfrak{A}(E, E)$ is special closed. Then $\mathfrak{N}$ is finitely closed.*

Proof. (1) Let $N \in \mathfrak{N}$ and $M \subseteq N$ a subobject of N. One has to show that $M \in \mathfrak{N}$. But $N \subseteq E^k$, some $k \in \mathbb{N}$. I show the assertion by induction on k.

(2) $k = 1$: In this case $M \subseteq N \subseteq E$. I show first that $\mathfrak{A}(E/N, E)$ is special open in $R = \mathfrak{A}(E, E)$. But by assumption $\mathfrak{A}(E/N, E)$ is special closed in $\mathfrak{A}(E, E) = R$, and by definition of $\mathfrak{N}$ the module $\mathfrak{A}(E, E)/\mathfrak{A}(E/N, E) =$

$\mathfrak{A}(N, E)$ lies in Coh(Dis R). Since R is strict this implies that $\mathfrak{A}(E/N, E)$ is special open. Since $M \subseteq N$ there is the surjection

$$\mathfrak{A}(N, E) \xrightarrow{f} \mathfrak{A}(M, E).$$

Since R is strict and $\mathfrak{A}(E/M, E)$ is special closed in R the module $R/\mathfrak{A}(E/M, E) = \mathfrak{A}(M, E)$ lies in STC(R). Since $\mathfrak{A}(N, E)$ is discrete and f is a surjection in STC(R) the module $\mathfrak{A}(M, E)$ is itself discrete, hence $\mathfrak{A}(M, E) \in$ Coh(Dis R). Hence $M \in \mathfrak{N}$.

(3) The conclusion: Assume that $M \subseteq N \subseteq E^k$, $k \geqslant 2$. With the canonical injection and projection there is the exact sequence

$$0 \longrightarrow E^{k-1} \xrightarrow{f} E^k \xrightarrow{g} E \longrightarrow 0$$

which first gives rise to the exact sequence

$$0 \to f^{-1}(N) \to N \to g(N) \to 0.$$

By Lemma 7.2 $g(N) \in \mathfrak{N}$ because $N \in \mathfrak{N}$. Since $\mathfrak{N}$ is Abelian this implies $f^{-1}(N) \in \mathfrak{N}$. One also obtains the exact sequence

$$0 \to f^{-1}(M) \to M \to g(M) \to 0$$

with

$$f^{-1}(M) \subseteq f^{-1}(N) \subseteq E^{k-1} \qquad \text{and} \qquad g(M) \subseteq g(N) \subseteq E.$$

By the induction hypothesis $f^{-1}(M)$ and $g(M)$ lie in $\mathfrak{N}$. There results the exact sequence

$$(*) \qquad 0 \to \mathfrak{A}(g(M), E) \to \mathfrak{A}(M, E) \to \mathfrak{A}(f^{-1}(M), E) \to 0$$

where the outer modules are coherent in Dis R. For the middle module there is the surjection

$$\mathfrak{A}(N, E) \to \mathfrak{A}(M, E) \quad \text{with} \quad \mathfrak{A}(N, E) \in \text{Dis } R, \quad \text{hence} \quad \mathfrak{A}(M, E) \in \text{Dis } R.$$

The exact sequence (*) implies $\mathfrak{A}(M, E) \in$ Coh(Dis R), i.e., $M \in \mathfrak{N}$.

The preceding two lemmas show the main part of the following

THEOREM 7.4. *Let $\mathfrak{A}$ be a Grothendieck category and R a strict complete topologically left coherent and linearly compact ring. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ and* STC(R) *are dual to each other.*

(2) *There are an injective cogenerator E of $\mathfrak{A}$ and a ring isomorphism $h : R \to \mathfrak{A}(E, E)$ (Identify $R = \mathfrak{A}(E, E)$ via h) with the following properties:*

(i) *The subobjects A of all E^k, $k \in \mathbb{N}$, form a set of generators of $\mathfrak{A}$.*

(ii) *For each subobject $A \subseteq E$ the left ideal $\mathfrak{A}(E/A, E)$ of $R = \mathfrak{A}(E, E)$ is special closed, and for each special open ideal $\mathfrak{a}$ of R there is a subobject A of E with $\mathfrak{a} = \mathfrak{A}(E/A, E)$.*

Proof. (1) $\Rightarrow$ (2). W.l.o.g. one may assume $\mathfrak{A} = \mathrm{STC}(R)^{op}$. The injective cogenerator R of $\mathrm{STC}(R)^{op}$ has all the desired properties.

(2) $\Rightarrow$ (1). (a) The preceding two lemmas show that the full subcategory $\mathfrak{N}$ of $\mathfrak{A}$ of all those N which admit a monomorphism $N \to E^k$, some $k \in \mathbb{N}$, and satisfy $\mathfrak{A}(N, E) \in \mathrm{Coh}(\mathrm{Dis}\, R)$ is a full, skeletal-small, and finitely closed subcategory of $\mathfrak{A}$.

(b) I show that E is the sum (supremum) of its subobjects which lie in $\mathfrak{N}$. But $R \cong \lim_{\mathfrak{a}} R/\mathfrak{a}$ where $\mathfrak{a}$ runs over the special open left ideals $\mathfrak{a}$ of R. These $\mathfrak{a}$ are of the form $\mathfrak{a} = \mathfrak{A}(E/N_{\mathfrak{a}}, E)$. Hence

$$\begin{aligned} R &\cong \lim_{\mathfrak{a}} R/\mathfrak{a} \cong \lim_{\mathfrak{a}} \mathfrak{A}(E, E)/\mathfrak{A}(E/N_{\mathfrak{a}}, E) \\ &\cong \lim_{\mathfrak{a}} \mathfrak{A}(N_{\mathfrak{a}}, E) \cong \mathfrak{A}(\operatorname{colim} N_{\mathfrak{a}}, E) \end{aligned}$$

Since $\operatorname{colim} N_{\mathfrak{a}}$ is a subobject of E and E is an injective cogenerator this implies $\operatorname{colim} N_{\mathfrak{a}} \cong E$, i.e., $\bigcup_{\mathfrak{a}} N_{\mathfrak{a}} = E$. The assertion follows since the $N_{\mathfrak{a}}$ lie in $\mathfrak{N}$.

(c) Since $\mathfrak{N}$ is finitely closed the set $\{N; N \subseteq A, N \in \mathfrak{N}\}$ is directed for each object $A \in \mathfrak{A}$. I conclude that in particular the set

$$\left\{ N_1 \coprod \cdots \coprod N_k\, ;\, N_i \in \mathfrak{N}, N_i \subseteq E \right\}$$

is a directed set of subobjects of E^k in $\mathfrak{N}$. From (b) one obtains that

$$\bigcup \left(N_1 \coprod \cdots \coprod N_k\right) = \left(\bigcup N_1\right) \coprod \cdots \coprod \left(\bigcup N_k\right) = E \coprod \cdots \coprod E = E^k,$$

hence also E^k is the supremum of its subobjects which lie in $\mathfrak{N}$. Let now $A \subseteq E^k$. Since

$$E^k = \bigcup \{N; N \subseteq E^k, N \in \mathfrak{N}\},$$

and since $\{N; N \subseteq E^k, N \in \mathfrak{N}\}$ is directed the Grothendieck (AB 5) condition implies

$$A = \bigcup \{A \cap N; N \subseteq E^k, N \in \mathfrak{N}\}.$$

The objects $A \cap N$ are in $\mathfrak{N}$ because $\mathfrak{N}$ is closed under subobjects. Hence also A is the supremum of its subobjects in $\mathfrak{N}$. But the set of all these A is a set of generators for $\mathfrak{A}$. The preceding calculation then allows the conclusion that $\mathfrak{N}$ generates $\mathfrak{A}$.

(d) The preceding parts (a), (b), (c) show that the triple $\mathfrak{A}, \mathfrak{N}, E$ satisfies the assumptions of Theorem 3.4. One obtains the duality

$$\mathfrak{A}^{op} \to \mathrm{STC}(\mathfrak{A}(E, E) : A \rightsquigarrow \mathfrak{A}(A, E)$$

where one takes the $(E, \mathfrak{N})$-topology on $\mathfrak{A}(E, E)$ and $\mathfrak{A}(A, E)$. Theorem 7.4 will be shown once I have shown that the $\mathfrak{N}$-topology on $\mathfrak{A}(E, E) = R$ is the same as the original topology. But if $\mathfrak{a} \subseteq R$ is a special open left ideal, then $\mathfrak{a} = \mathfrak{A}(E/N_{\mathfrak{a}}, E)$ where $N_{\mathfrak{a}} \subseteq E$. By definition of $\mathfrak{N}$ this $N_{\mathfrak{a}}$ lies in $\mathfrak{N}$, hence $\mathfrak{a} = \mathfrak{A}(E/N_{\mathfrak{a}}, E)$ is open w.r.t. the $\mathfrak{N}$-topology. Let now $\mathfrak{A}(E/N, E)$, $N \in \mathfrak{N}$, be special open w.r.t. the $\mathfrak{N}$-topology. By assumption (ii) $\mathfrak{A}(E/N, E)$ is special closed w.r.t. the original topology, and also

$$R/\mathfrak{A}(E/N, E) = \mathfrak{A}(N, E) \in \mathrm{Coh}(\mathrm{Dis}\, R).$$

Since R is strict this implies that $\mathfrak{A}(E/N, E)$ is open w.r.t. the original topology. Hence the two topologies on $R = \mathfrak{A}(E, E)$ coincide. There results the equivalence

$$\mathfrak{A}^{op} \to \mathrm{STC}(R) : A \rightsquigarrow \mathfrak{A}(A, E). \quad \|$$

The preceding theorem implies the following topological Morita theorem. I formulate this for right modules for simplicity (no ring antiisomorphisms).

THEOREM 7.5. (*Topological Morita theorem*) *Let R and S be strict complete topologically right coherent and linearly compact rings. Then the following assertions are equivalent*:

(1) *The categories* $\mathrm{STC}(R_R)$ *and* $\mathrm{STC}(S_S)$ *are equivalent.*

(2) *There is a projective generator P in* $\mathrm{STC}(S_S)$ *and a ring ismorphism* $h: R \to \mathrm{STC}(S_S)(P, P)$ [*identify* $R = \mathrm{STC}(S_S)(P, P)$ *via* h] *with the following properties*:

(i) *The quotient objects of all P^k, $k \in \mathbb{N}$, form a family of cogenerators of* $\mathrm{STC}(S_S)$.

(ii) *For each special closed submodule Y of P the right ideal* $\mathrm{CTC}(S_S)(P, Y)$ *of* $R = \mathrm{STC}(S_S)(P, P)$ *is special closed, and for each special open right ideal* $\mathfrak{a}$ *of R there is a special closed submodule Y of P with* $\mathfrak{a} = \mathrm{CTC}(S_S)(P, Y)$.

If (1) *and* (2) *are satisfied the equivalence is given by*

$$\mathrm{STC}(S_S) \to \mathrm{STC}(R_R) : Y \rightsquigarrow \mathrm{STC}(S_S)(P, Y). \quad \|$$

The usual Morita theorem for modules is not a special case of the preceding theorem. For Theorem 7.5 classifies (up to equivalence) all co-Grothendieck categories whereas the usual Morita theorem classifies Grothendieck categories of modules.

VIII. Examples

By choosing special categories $\mathfrak{A}$, $\mathfrak{N}$, and a special E in Theorem 3.4 one obtains more specific duality theorems.

(1) *The locally noetherian case*: This case has been treated by J.-E. Roos in [11]. The propositions 3.7 and 5.24 connect the results of this paper with those of Roos.

(2) *The discrete case*: If $\mathfrak{A}$, $\mathfrak{N}$, E are as in Theorem 3.4, then $E \in \mathfrak{N}$ iff $\mathfrak{A}(E, E)$ is discrete. Proposition 4.1 shows that in this case $R := \mathfrak{A}(E, E)$ is a left coherent and algebraically linearly compact ring. This means that the ordered set

$$\{\emptyset\} \cup \{x + \mathfrak{a};\quad x \in R,\quad \mathfrak{a} \subseteq R \text{ finitely generated left ideal}\}$$

is closed under intersections and compact, and that all annihilator left ideals $(0 : x)$, $x \in R$, are finitely generated. The categories $\operatorname{Dis} R$ and $\operatorname{mod} R$ coincide. If, on the other side, R is any algebraically left coherent and linearly compact ring, then R, with the discrete topology, is a strict complete topologically left coherent and linearly compact ring. The category $\operatorname{STC}(R)$ is a co-Grothendieck category with the discrete projective generator $R \in \operatorname{Coh}(\operatorname{Mod} R) \subset \operatorname{STC}(R)$. The modules in $\operatorname{Coh}(\operatorname{mod} R)$ are exactly the finitely presented R-left modules. The objects of $\operatorname{CTC}(R)$ are the complete linear topological R-left modules X with a basis of open submodules X' such that X/X' is finitely presented (=coherent).

Proposition. 8.1. *Let R be an algebraically left coherent and linearly compact ring. Each finitely presented R-left module X admits a projective cover $f : P \to X$ (in* $\operatorname{mod} R$*). The module P is itself finitely generated, i.e., coherent.*

Proof. The module X has a projective cover $f : P \to X$ in $\operatorname{STC}(R)$ since $\operatorname{STC}(R)$ is a co-Grothendieck category. Since X is finitely generated there is a surjection $g : R^k \to X$ which is a morphism in $\operatorname{STC}(R)$ too. Because f is a projective cover the map g factorizes as

$$R^k \xrightarrow{e} P \xrightarrow{f} X$$

with a surjection e. Since $\operatorname{Coh}(\operatorname{mod} R)$ is closed under quotient objects in $\operatorname{STC}(R)$ the module P lies itself in $\operatorname{Coh}(\operatorname{mod} R)$. Since e is an epimorphism in $\operatorname{STC}(R)$ and since P is projective in $\operatorname{STC}(R)$ the morphism e splits. Hence P is finitely generated projective in $\operatorname{mod} R$ as a direct summand of R^k. Finally *I* show that f is also essential in $\operatorname{mod} R$. Let P' be any submodule of

P with $P' + \ker f = P$. Since P is finitely generated there is a finitely generated submodule P'' of P' with $P'' + \ker f = P$. As a finitely generated submodule of the coherent module P the module P'' is itself coherent, hence a subobject of P in $\mathrm{STC}(R)$. Since f is essential in $\mathrm{STC}(R)$ this implies $P'' = P$, hence $P' = P$. Thus $f : P \to X$ is a projective cover in $\mathrm{mod}\, R$, and P is finitely generated, i.e. coherent. ||

(3) *The module case.* I apply Theorem 3.4 to the situation of 1.7. Let R be a ring and let $\mathfrak{N}$ be the full subcategory of $\mathrm{mod}\, R_R$ consisting of all submodules of finitely generated R-right modules. Assume that E is an injective R-right module with the property that for each right ideal $\mathfrak{a}$ of R there is a finite family $(x_i\, ;\, i \in I)$ of elements of E with $\mathfrak{a} = \bigcap_{i\in I} (0 : x_i)$. By 1.7 one knows that E is an injective cogenerator of $\mathrm{mod}\, R_R$, and that for each $N \in \mathfrak{N}$ there is a short exact sequence $0 \to N \to E^k$, $k \in \mathbb{N}$. The functor $A \rightsquigarrow \hom_R(A, E)$ induces the equivalence

$$(\mathrm{mod}\, R_R)^{op} \to \mathrm{STC}(\hom_R(E, E)).$$

The topological modules $\hom_R(A, E)$ have the basis $\hom_R(A/A', E)$ where A' runs over those submodules of A which are contained in some finitely generated R-right module. If A' itself is finitely generated, say $A' = a_1 R + \cdots + a_m R$, then

$$\hom_R(A/A', E) = \{f : A \to E; f(a_i) = 0, i = 1,..., m\}.$$

This shows that the topology of $\hom_R(A, E)$ defined here is a refinement of the so-called finite topology. The two topologies coincide if R_R is noetherian. The above duality maps R_R onto $E = \hom_R(R, E)$. Hence with $S = \mathit{End} E$ the module E is an S-R-bimodule, and as an S-module is contained in $\mathrm{STC}(S)$. Since the functor $\hom_R(-, E)$ is a duality one has the string of R-isomorphisms

$$\begin{aligned} M &\cong \hom(R_R, M_R) \cong \mathrm{STC}(S)(\hom_R(M, E), (\hom_R(R, E)) \\ &\cong \mathrm{STC}(S)(\hom_R(M, E), E) \end{aligned}$$

for all R-right modules M. This means, in other words, that the functor $\mathrm{STC}(S)(-, E)$ is the quasiinverse of $\hom_R(-, E)$. We thus obtain the following

THEOREM 8.2. *Let R be a ring and E an injective R-right module such that for each right ideal $\mathfrak{a}$ of R there is a finite family $(x_i\, ;\, i \in I)$ of elements in E with $\mathfrak{a} = \bigcap_{i\in I} (0 : x_i)$. Define $S = \mathit{End}_R E$.*

(1) *With the above defined topology the ring S is a strict complete topologically left coherent and linearly compact ring, and with the canonical S-left module structure and the discrete topology E is in* $\mathrm{STC}(S)$.

(2) *There are the quasiinverse dualities*

$$\mathrm{mod}\ R_R \underset{\hom_{S,\mathrm{cont}}(-,E)}{\overset{\hom_R(-,E)}{\rightleftarrows}} \mathrm{STC}({}_SS).$$

The topology on $\hom_R(A, E)$, $A \in \mathrm{mod}\ R_R$, *is the one defined above. The* hom-*functor in* $\mathrm{STC}({}_SS)$ *is denoted by* $\hom_{S,\mathrm{cont}}$. *The R-right module structure of* $\hom_{S,\mathrm{cont}}(X, E)$, $X \in \mathrm{STC}({}_SR)$, *is induced by that of E.* ||

COROLLARY 8.3. *In the situation of the preceding theorem one has*

$$R^{op} \cong \hom_{S,\mathrm{cont}}(E, E), \qquad \text{i.e.,}$$

the endomorphisms of E in $\mathrm{STC}({}_SS)$ *are exactly the right multiplications with elements of R.* ||

Since R_R is a projective generator of finite type in mod R_R the module ${}_SE$ is an injective cogenerator of $\mathrm{STC}({}_SS)$ of cofinite type. The Morita theorem for Abelian categories ([5], p. 405, cor. 1) selects the module categories among the Grothendieck categories. For co-Grothendieck categories one obtains the following

PROPOSITION. 8.4. *Let S be a strict complete topologically left coherent and linearly compact ring. Then* $\mathrm{STC}({}_SS)$ *is dual to a module category* mod R_R *iff* $\mathrm{STC}(S)$ *admits an injective cogenerator of cofinite type. If* ${}_SE$ *is such a cogenerator and if* $R = \mathrm{STC}(E, E)$, *then a duality is given by*

$$\mathrm{STC}({}_SS)^{op} \to \mathrm{mod}\ R_R : X \rightsquigarrow \hom_{S,\mathrm{cont}}(X, E). \quad ||$$

Remember here that any co-Grothendieck category $\mathfrak{A}$ is of the form $\mathrm{STC}({}_SS)$.

If in Theorem 8.2 R is right noetherian the modules $\hom_R(A, E)$ have the finite topology with the basis of neighborhoods of 0 consisting of the submodules $\{f : A \to E;\ f(a_i) = 0,\ i \in I\}$ where $(a_i\ ;\ i \in I)$ runs over all finite families of elements of A. In particular E as a left S-module has the basis $\{x \in E;\ r_ix = 0,\ i \in I\}$ where $(r_i\ ;\ i \in I)$ runs over all finite families of elements of R. If R_R is noetherian a suitable E is the coproduct of a representative system of indecomposable injective R-right modules. For general R the problem of finding suitable E's deserves detailed attention in order to make the duality functor more concrete.

(4) *Duality theory for generalized quasi-Frobenius rings.* A generalized quasi-Frobenius ring R_R is, by 1.8, a ring where, in Theorem 8.2, one can choose $E = R_R$. The following theorem characterizes the situation in more detail.

THEOREM 8.5. *Let R be a ring.*

(A) *The following assertions are equivalent:*

(a) *R is a generalized right quasi-Frobenius ring,* i.e., *R_R is injective and each right ideal of R is the annihilator of a finite set of elements of R.*

(b) (i) *R is algebraically left coherent and linearly compact.*

(ii) *If $(\mathfrak{a}_i\,;\, i \in I)$, I directed, is a decreasing family of finitely generated left ideals of R with $\bigcap_i \mathfrak{a}_i = 0$, then $\mathfrak{a}_i = 0$ for some $i \in I$.*

(iii) *${}_RR$ is injective* w.r.t. *finitely generated left ideals,* i.e.,

$$Ext_R{}^1(R/\mathfrak{a}, R) = 0$$

for each finitely generated left ideal $\mathfrak{a}$.

(iv) *The right annihilator $(0 : \mathfrak{a})$ of any proper finitely generated left ideal $\mathfrak{a}$ of R is not zero.*

(B) *If the equivalent conditions of (1) are satisfied one obtains the quasi-inverse dualities*

$$\operatorname{mod} R_R \underset{\hom_{R,\mathrm{cont}}(-,R)}{\overset{\hom_R(-,R)}{\rightleftarrows}} \mathrm{STC}({}_RR)$$

Proof. (1) If R is a generalized right quasi-Frobenius ring, then one can choose $E = R_R$ in Theorem 8.2 according to 1.8. Theorem 8.2 furnishes part (2) of the theorem, and also that $R \cong EndR_R$ is algebraically left linearly compact and coherent. Since R_R is of finite type in mod R_R the module ${}_RR$ is of cofinite type in $\mathrm{STC}({}_RR)$ by duality. Since the subobjects of ${}_RR$ in $\mathrm{STC}({}_RR)$ are exactly the coherent, i.e., here finitely generated R-left ideals, this is equivalent with (A)(b)(ii). Since R_R is a projective generator of mod R_R the module ${}_RR$ is an injective cogenerator of $\mathrm{STC}({}_RR)$, and in particular of Coh(dis ${}_RR$). This implies (A)(b)(iii) and (iv).

(2) I show the implication (b) $\Rightarrow$ (a) of part (A) of the theorem. For this purpose I show first that ${}_RR$ is an injective cogenerator of cofinite type in $\mathrm{STC}({}_RR)$.

(3) The module ${}_RR$ is of cofinite type in $\mathrm{STC}({}_RR)$. This follows directly from the definition and (ii).

(4) The module ${}_RR$ is injective in Coh(mod ${}_RR$), i.e., $Ext^1{}_R(M, R) = 0$ for each finitely presented, i.e. coherent, left R-module M. For M is finitely generated, i.e., $M = Rx_1 + \cdots + Rx_k$. I show the assertion by induction on k. For $k = 1$ the assertion follows from (A)(b)(iii). In general one has the exact sequence

$$0 \to M' = Rx_1 + \cdots + Rx_{k-1} \to M \to M/M' \to 0$$

where M', M, M/M' are coherent. But $Ext^1{}_R(M', R)$ and $Ext^1(M/M', R) = 0$ because M' resp. M/M' have $k - 1$ resp. one generators. The long exact sequence for Ext shows $Ext^1{}_R(M, R) = 0$.

(5) The module ${}_RR$ is injective in $\mathrm{STC}({}_RR)$. For let $f : X \to Y$ be an injection $\mathrm{STC}({}_RR)$, and $h : X \to {}_RR$ a continuous linear map. The map h factorizes as

$$X \longrightarrow X' \xrightarrow{h'} R$$

where X' is coherent in mod R and h' is injective because R is discrete. Let the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

be a push-out. Since f is injective so is f'. It is enough to lift h' to Y'. But $0 = \bigcap Y''$ where Y'' runs over all special open submodules of Y'' of Y'. Then $0 = \bigcap f'^{-1}(Y'')$ where the $f'^{-1}(Y'')$ are a directed decreasing family of finitely generated submodules of X'. But $h' : X' \to R$ is injective and R is of cofinite type, hence $f'^{-1}(Y'') = 0$ for some Y''. This means that the map $X' \xrightarrow{f'} Y' \xrightarrow{\text{can}} Y'/Y''$ is still injective where now both X' and Y'/Y'' are in Coh(mod ${}_RR$). Hence h' can be lifted to Y'/Y'' by 4). This implies that h can be lifted to Y, and ${}_RR$ is injective in $\mathrm{STC}({}_RR)$.

(6) I show that ${}_RR$ is a cogenerator of $\mathrm{STC}({}_RR)$. The condition (b)(iv) means that for every finitely generated left ideal $\mathfrak{a}$ of R with $\mathfrak{a} \neq R$ there is a nonzero map $R/\mathfrak{a} \to R$. Now let $f : M \to N$ be a nonzero map in Coh(mod R), and let $0 \neq x \in \mathrm{Im} f$. Since $Rx \simeq R/(0 : x)$ there is a nonzero map $Rx \to R$ which can be extended to a nonzero map $h : N \to R$ since R is injective in Coh(mod R). Hence $hf \neq 0$, and ${}_RR$ is a cogenerator of Coh(mod R). Finally let $f : X \to Y$ be any nonzero map in $\mathrm{STC}(R)$. Then there is an epimorphism $Y \xrightarrow{e} Y'$ with discrete Y' in $\mathrm{STC}({}_RR)$ such that $ef \neq 0$. The map ef factorizes as $X \xrightarrow{\text{can}} X' \xrightarrow{f'} Y'$ where X' is discrete, "can" is surjective and $f' \neq 0$ since $ef \neq 0$. By the preceding considerations there is a map $h : Y' \to R$ with $hf' \neq 0$. Then $hef = hf'$ can $\neq 0$, hence ${}_RR$ is a cogenerator in $\mathrm{STC}({}_RR)$.

(7) The preceding considerations show that ${}_RR$ is an injective cogenerator of cofinite type in $\mathrm{STC}({}_RR)$. The Morita theorem shows that

$$\mathrm{STC}({}_RR)^{op} \to \mathrm{mod}\ R_R : X \rightsquigarrow \hom_{R\,\mathrm{cont}}(X, R)$$

is an equivalence (see [5], p. 405, cor. 1). Since ${}_RR$ is a projective generator of STC$({}_RR)$ (6.6), R_R is an injective cogenerator of mod R_R. Let finally M_R be any finitely generated R-right module, and let ${R_R}^k \to M$ be surjective. By duality this goes into the injection $M^* \to {}_RR^k$ where M^* is the dual of M under the above duality. This means that M^* is discrete, hence in Coh(mod ${}_RR$). In particular there is a surjection ${}_RR^1 \to M^*$ which by duality gives the injection $M \to {R_R}^1$. Hence each finitely generated R-right module M is contained in a free module and R_R is an injective cogenerator. This means that R is a generalized right quasi-Frobenius ring.

I do not know at this moment whether a generalized right quasi-Frobenius ring is already quasi-Frobenius. The preceding theorem should be compared with the results in [10].

(5) *Duality theory for spectral categories.* A Grothendieck category $\mathfrak{A}$ is called a spectral category if every morphism in $\mathfrak{A}$ splits ([6]). A Grothendieck category $\mathfrak{A}$ is spectral iff each object of $\mathfrak{A}$ is projective resp. injective. If U is a generator of $\mathfrak{A}$ one obtains the full-faithful functor

$$\mathfrak{A} \to \text{mod } \mathfrak{A}(U, U)_{\mathfrak{A}(U,U)} : A \rightsquigarrow \mathfrak{A}(U, A)$$

where the action of $\mathfrak{A}(U, U)$ on $\mathfrak{A}(U, A)$ is the composition of morphisms. If R is any ring let $\mathfrak{S}(R_R)$ be the full subcategory of mod R_R consisting of all direct summands of products of copies of R.

Proposition. 8.6. ([6]), *Theorems* 2.1, 2.2)

(1) *If $\mathfrak{A}$ is a spectral category with generator U, then $\mathfrak{A}(U, U)$ is a right self-injective, regular ring, and the functor $A \rightsquigarrow \mathfrak{A}(U, A)$ induces the equivalence*

$$\mathfrak{A} \to \mathfrak{S}(\mathfrak{A}(U, U)_{\mathfrak{A}(U,U)}).$$

(2) *If R is a right self-injective, regular ring, then $\mathfrak{S}(R_R)$ is a spectral category with generator R_R.* ▯

For spectral categories one also obtains a nice duality theory. If $\mathfrak{A}$ is a spectral category with the generator U then U is also an injective cogenerator. The full subcategory $\mathfrak{N}$ of $\mathfrak{A}$ consisting of all direct summands of some U^k, $k \in \mathbb{N}$, is skeletal-small, finitely closed and generates $\mathfrak{A}$ since $U \in \mathfrak{N}$. Theorem 3.4 is thus applicable to the triple $\mathfrak{A}$, $\mathfrak{N}$, U. If $\mathfrak{A} = \mathfrak{S}(R_R)$ where R_R is a regular, right self-injective ring, then one can take $R = U$, and $\mathfrak{N}$ is simply the category $PF(R_R)$ of all finitely generated projective R-right modules. In detail one obtains

Theorem 8.7. *Let R be a regular ring.*

(i) *R is right self-injective iff R is algebraically left linearly compact.*

(ii) *If R has the properties of (i), then one obtains the quasiinverse dualities*

$$\mathfrak{S}(R_R) \underset{\hom_{R,\mathrm{cont}}(-,R)}{\overset{\hom_R(-,R)}{\rightleftarrows}} \mathrm{STC}({}_RR)$$

Proof. (1) Assume that R is right self-injective. Then $\mathfrak{S}(R_R)$ is a spectral category. With $E = R_R$ and $\mathfrak{N} = PF(R_R)$ Theorem 3.4 is applicable and shows that $R = \hom(R_R, R_R)$ is algebraically left coherent and linearly compact. Moreover

$$\mathfrak{S}(R_R)^{op} \to \mathrm{STC}({}_RR) : M \rightsquigarrow \hom_R(M, R)$$

is an equivalence. It is again trivial to see that the quasiinverse of this duality is given by $X \rightsquigarrow \hom_{R,\mathrm{cont}}(X, R)$.

(2) Now assume that R is algebraically left linearly compact. Since R is regular it is left coherent, hence $\mathrm{STC}({}_RR)$ is a co-Grothendieck category with the projective generator ${}_RR$. I show that each object of $\mathrm{STC}({}_RR)$ is projective. But $\mathrm{Coh}(\mathrm{mod}\ {}_RR) = PF({}_RR)$ where $PF({}_RR)$ is the category of all finitely generated projective R-left modules. This follows from the regularity of R by [2], section 2, ex. 18. Hence every surjection in $PF({}_RR)$ splits. Since $PF({}_RR)$ cogenerates the co-Grothendieck category $\mathrm{STC}({}_RR)$ an object Z of $\mathrm{STC}({}_RR)$ is projective iff for every surjection $f : X \to Y$ in $\mathrm{STC}({}_RR)$ with $X \in PF({}_RR)$ and every $g : Z \to Y$ there is an $h : Z \to X$ with $fh = g$ ([7], ch. 1). But since X is discrete and f is surjective in $\mathrm{STC}({}_RR)$ the module Y is also discrete, i.e., lies in $\mathrm{Coh}(\mathrm{mod}\ {}_RR) = PF({}_RR)$. This implies that f splits, i.e., has a section s, and then $h = sg$ has the desired property. Hence all objects of $\mathrm{STC}({}_RR)$ are projective. From this one concludes that $\mathrm{STC}({}_RR)^{op}$ is a spectral category with the generator R. In particular $\hom({}_RR, {}_RR)^{op} \simeq R$. is right self-injective by 8.6. ||

PROPOSITION. 8.8. *Let R be a regular, algebraically left linearly compact ring. The objects of* $\mathrm{STC}({}_RR)$ *are exactly the modules $s(Y)$ where Y is a special closed submodule of some R^I, I a set. Here R^I has the product topology and s is the right adjoint of the inclusion* $\mathrm{STC}({}_RR) \subset \mathrm{CTC}({}_RR)$.

Proof. By 5.21 R^I with the product topology lies in $\mathrm{CTC}({}_RR)$. The same is true for any special closed submodule Y of R^I, hence $s(Y) \in \mathrm{STC}({}_RR)$ (5.22). Let on the other side $X \in \mathrm{STC}({}_RR)$. Then $X \simeq \lim_{X'} X/X'$ where X' runs over the special open submodules of X. There results the continuous injection $X \to \prod_{X'} X/X'$. But the X/X' are finitely generated projective and thus contained in some finitely generated free module. Hence there is a continuous injection $f: X \to R^I$, some set I. The continuous bijection

$$X \to f(X) =: Y$$

induces the isomorphism $X = s(X) \to s(Y)$. ||

IX. Representation of Grothendieck Categories as Functor Categories

In [5], p. 353, P. Gabriel has shown that for any small Abelian category $\mathfrak{B}$ the category $\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab})$ of all left exact, additive functors from $\mathfrak{B}$ to the category $\mathscr{Ab}$ of Abelian groups is a Grothendieck category. In [11], Section 2, prop. 2, J.-E. Roos has improved this result by showing that a category $\mathfrak{A}$ is a locally coherent Grothendieck category iff it is of the form $\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab})$ with a small abelian category $\mathfrak{B}$. In this section I am going to show a similar result for arbitrary Grothendieck categories. If $\mathfrak{C}$ and $\mathfrak{D}$ are additive categories $[\mathfrak{C}, \mathfrak{D}]$ denotes the category of additive functors from $\mathfrak{C}$ to $\mathfrak{D}$.

Let R be a ring and $\mathfrak{B}$ a full, skeletal-small linearly compact subcategory of mod ${}_R R$. I know that $\mathfrak{B}$ is a full and dense subcategory of $\hat{\mathfrak{B}}$ (5.17). By [12] Lemma 1.7, one obtains the full embedding

$$\hat{\mathfrak{B}}^{op} \to [\mathfrak{B}, \mathscr{Ab}] : X \rightsquigarrow \hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$$

where $[\mathfrak{B}, \mathscr{Ab}]$ denotes the additive functors from $\mathfrak{B}$ to $\mathscr{Ab}$. I characterize those additive functors from $\mathfrak{B}$ to $\mathscr{Ab}$ which lie in the image of the above full embedding.

PROPOSITION. 9.1. *Situation as above. Let $F: \mathfrak{B} \to \mathscr{Ab}$ be an additive functor. The following assertions are equivalent*:

(1) *F is isomorphic to some $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$, $X \in \hat{\mathfrak{B}}$.*

(2) *F is a strict filtered colimit of representable functors.*

(3) *F is left exact, and commutes with filtered intersections.*

The assertion of (2) means that there is an inverse system

$$(B_i, f_{ij}: B_j \to B_i, i \leqslant j \in I),$$

I directed, with epimorphisms f_{ij} such that $F \cong \mathrm{colim}_I \mathfrak{B}(B_i, -)$. The second assertion of (3) means that $F(\bigcap_i B_i) = \bigcap_i FB_i$ whenever $(B_i; i \in I)$, I directed, is a decreasing family of submodules in $\mathfrak{B}$ of some $B \in \mathfrak{B}$. Remark that when F is left exact one can identify FB_i with its image under the canonical injection $FB_i \to FB$, and that then both $F(\bigcap_i B_i)$ and $\bigcap_i FB_i$ are subgroups of FB. The condition (3) is equivalent with the requirement that F maps monic pullbacks onto pullbacks. A monic pullback is a pullback $(C \xrightarrow{g_i} B_i \xrightarrow{f_i} B;\ i \in I)$, I any set, $f_i g_i = f_j g_j$, all $i, j \in I$, such that all f_i are monomorphisms. This implies that also all g_i are monomorphisms, and that, upon identification of B_i with its image under f_i, one has $C = \bigcap_i B_i$.

Proof. (1) $\Rightarrow$ (3). The functor

$$\mathfrak{B}(X, -) : \hat{\mathfrak{B}} \to \mathcal{A}b$$

is left exact. Also the injection $\mathfrak{B} \to \hat{\mathfrak{B}}$ is left exact by Theorem 5.21, hence $\hat{\mathfrak{B}}(X, -)\mid_{\mathfrak{B}}$ is left exact. It is trivial that $\hat{\mathfrak{B}}(X, -)\mid_{\mathfrak{B}}$ preserves intersections.

(3) $\Rightarrow$ (2). Let $F : \mathfrak{B} \to \mathcal{A}b$ be a left exact additive functor which preserves intersections (or filtered intersections, that is the same for a left exact functor). In particular $F \in \operatorname{Lex}(\mathfrak{B}, \mathcal{A}b)$.

(a) I show first that the image of a morphism $\mathfrak{B}(B, -) \xrightarrow{h} F$ in $\operatorname{Lex}(\mathfrak{B}, \mathcal{A}b)$ is representable. It is enough to show that the kernel of h is representable. By Yoneda's lemma h is given as

$$h_C : \mathfrak{B}(B, C) \to FC : f \rightsquigarrow (Ff)(x)$$

where $x = h_B(1_B)$. For $B' \subseteq B$ I identify FB' with its canonical image under $FB' \to FB$ (F left exact.), hence $FB' \subseteq FB$. Then Let $B_0 := \bigcap_{B'} B'$ where B' runs over all submodules B' of B in $\mathfrak{B}$ with $x \in FB'$. By assumption (3) one has

$$FB_0 = F\left(\bigcap B'\right) = \bigcap FB',$$

hence $x \in FB_0$. Let $f : B \to B/B_0$ be the canonical map. I show that

$$0 \longrightarrow \mathfrak{B}(B/B_0, -) \xrightarrow{\mathfrak{B}(f,-)} \mathfrak{B}(B, -) \xrightarrow{\ h\ } F$$

is exact in $\operatorname{Lex}(\mathfrak{B}, \mathcal{A}b)$ (or in $[\mathfrak{B}, \mathcal{A}b]$), i.e , that for each $C \in \mathfrak{B}$

$$(*) \qquad 0 \longrightarrow \mathfrak{B}(B/B_0, C) \xrightarrow{\mathfrak{B}(f,C)} \mathfrak{B}(B, C) \xrightarrow{\ h_C\ } F$$

is exact in $\mathcal{A}b$. Let $g : B/B_0 \to C$ be any linear map. Then

$$h_C(\mathfrak{B}(f, C)(g)) = F(gf)(x) = (Fg)(Ff)(x) = 0$$

since

$$0 \longrightarrow FB_0 \longrightarrow FB \xrightarrow{Ff} F(B/B_0)$$

is exact and $x \in FB_0$, so $(Ff)(x) = 0$. Let on the other side $g \in \mathfrak{B}(B, C)$ with $(Fg)(x) = h_C(g) = 0$. Since

$$0 \longrightarrow \ker g \longrightarrow B \xrightarrow{g} C$$

is exact, so is

$$0 \longrightarrow F \ker g \longrightarrow FB \xrightarrow{Fg} FC,$$

hence $x \in F \ker g$. This implies $B_0 \subseteq \ker g$, i.e. g can be factorized as

$$B \xrightarrow{f} B/B_0 \xrightarrow{g'} C.$$

Hence $g = \mathfrak{B}(f, C)(g')$ which implies the exactness of (*).

(b) The calculation of (a) and the fact that the representable functors are closed under finite coproducts show that the representable subfunctors of F form a directed set I whose union ($=$ colimit) in $\mathrm{Lex}(\mathfrak{B}, \mathscr{A}b)$ is F. For each $i \in I$ choose a $B_i \in \mathfrak{B}$ with $\mathfrak{B}(B_i, -) \cong i$. The inclusions $i \subseteq j$ come from epimorphisms $f_{ij} : B_j \to B_i$. Hence

$$(B_i, f_{ij} : B_j \to B_i, i \subseteq j, i \in I)$$

is a strict inverse system in $\mathfrak{B}$, and

$$\operatorname*{colim}_I \mathfrak{B}(B_i, -) \cong \operatorname*{colim}_I i \cong F.$$

It does not matter whether one takes this colimit in $\mathrm{Lex}(\mathfrak{B}, \mathscr{A}b)$ or in $[\mathfrak{B}, \mathscr{A}b]$ because I is directed.

(2) $\Rightarrow$ (1). Assume that $F \cong \mathrm{colim}_I \mathfrak{B}(B_i, -)$ where (B_i, f_{ij}), I directed, is a strict inverse system in $\mathfrak{B}$. Let $X = \lim B_i$, algebraic limit with the limit topology. One knows (5.11) that the canonical projections $p_i : X \to B_i$ are surjective, that X has the $\ker p_i$, $i \in I$, as basis, and that $X \in \hat{\mathfrak{B}}$. For $B \in \mathfrak{B}$ (with the discrete topology) it follows that

$$\hat{\mathfrak{B}}(X, B) = \hat{\mathfrak{B}}(\lim B_i, B) \cong \mathrm{colim}\, \hat{\mathfrak{B}}(B_i, B) = \mathrm{colim}\, \mathfrak{B}(B_i, B).$$

The first isomorphism holds because, if $f : X \to B$ is continuous and linear, then $\ker f$ is open, hence contains some $\ker p_i$ and thus f can be factorized as

$$X \xrightarrow{p_i} B_i \longrightarrow B.$$

These calculations imply $F \cong \hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$. ||

Since under the above circumstances $\mathfrak{B}$ is also dense in $\tilde{\mathfrak{B}}$ (5.23) one also obtains the full embedding

$$\tilde{\mathfrak{B}}^{op} \to [\mathfrak{B}, \mathscr{A}b] : X \rightsquigarrow \tilde{\mathfrak{B}}(X, -)|_{\mathfrak{B}}.$$

I shall characterize those functors F which are isomorphic to some $\tilde{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$. Obviously, these F must have the equivalent properties of Proposition 9.1. A morphism $f: G \to F$ in $[\mathfrak{B}, \mathscr{Ab}]$ is called special if for each morphism $\mathfrak{B}(B, -) \to F$ the pullback $\mathfrak{B}(B, -) \times_F G$ is also representable. If F has the equivalent properties of Proposition 9.1 this is the same as to require that for each representable subfunctor $\mathfrak{B}(B, -) \subseteq F$ the inverse image $f^{-1}(\mathfrak{B}(B, -) \simeq \mathfrak{B}(B, -) \times_F G$ is representable. A subfunctor $G \subseteq F$ is called special if the injection $G \to F$ is special. A functor $F \in [\mathfrak{B}, \mathscr{Ab}]$ is called strict if it satisfies the following conditions:

(i) F satisfies the equivalent conditions of Proposition 9.1.

(ii) If G is a special subfunctor of F and also a subfunctor of some representable functor, then G is representable.

The strict functors are obviously left exact by Proposition 9.1. Let then $S\,\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab})$ be the full subcategory of $\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab})$ of all strict functors.

THEOREM 9.2. *Let $\mathfrak{B}$ be a full, skeletal-small and linearly compact subcategory of* mod R, *R a ring. Then the functor $X \rightsquigarrow \tilde{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$ induces an equivalence*

$$(\hat{\mathfrak{B}})^{op} \to S\,\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab}).$$

Proof. (1) Let $F \in S\,\mathrm{Lex}(\mathfrak{B}, \mathscr{Ab})$. By Proposition 9.1 there is an X in $\hat{\mathfrak{B}}$ with $F \simeq \hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$. I show that X is strict, i.e. $X \in \tilde{\mathfrak{B}}$. Let Y be a special closed submodule of X. Then $\hat{\mathfrak{B}}(X/Y, -)|_{\mathfrak{B}} \subseteq \hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$. I show that $\hat{\mathfrak{B}}(X/Y, -)|_{\mathfrak{B}}$ is a special subfunctor of $\hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$. For let

$$\mathfrak{B}(B, -) = \hat{\mathfrak{B}}(B, -)|_{\mathfrak{B}} \to \hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}$$

be an injection. This gives rise to the surjection $X \to B$ in $\hat{\mathfrak{B}}$. Hence w.l.o.g. $B = X/X'$ where X' is a special open submodule of X. The co-Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X/Y \\ \downarrow & & \downarrow \\ X/X' & \longrightarrow & X/X' + Y \end{array}$$

gives rise to the Cartesian diagram of functors

$$\begin{array}{ccc} \hat{\mathfrak{B}}(X/X' + Y, -)|_{\mathfrak{B}} & \longrightarrow & \hat{\mathfrak{B}}(X/X', -)|_{\mathfrak{B}}\,. \\ \downarrow & & \downarrow \\ \hat{\mathfrak{B}}(X/Y, -)|_{\mathfrak{B}} & \longrightarrow & \hat{\mathfrak{B}}(X, -)|_{\mathfrak{B}}\,. \end{array}$$

But $X/X' + Y$ is discrete and in $\mathfrak{B}$, hence

$$\hat{\mathfrak{B}}(X/X' + Y, -)\,|_{\mathfrak{B}} = \mathfrak{B}(X/X' + Y, -)$$

is representable. This shows that the inverse image of the representable subfunctor $\mathfrak{B}(B, -)$ of $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$ is representable, hence $\hat{\mathfrak{B}}(X/Y, -)\,|_{\mathfrak{B}}$ is a special subfunctor of $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$. Now assume in addition that $(X/Y)_{\mathrm{dis}}$, i.e., X/Y with the discrete topology, lies in $\mathfrak{B}$. Then $\hat{\mathfrak{B}}(X/Y, -)\,|_{\mathfrak{B}}$ is both a special subfunctor of $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}} \simeq F$ and a subfunctor of the representable functor $\mathfrak{B}((X/Y)_{\mathrm{dis}}, -)$. Since F is strict this implies that $\hat{\mathfrak{B}}(X/Y, -)\,|_{\mathfrak{B}}$ is representable. Hence X/Y is discrete and Y is open in X. Thus $X \in \tilde{\mathfrak{B}}$.

(2) I show now that for $X \in \tilde{\mathfrak{B}}$ the functor $\tilde{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}} = \hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$ is strict. Let G be a special subfunctor of $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$. Since $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$ is the directed colimit of its representable subfunctors and since the intersections of G with these representable subfunctors are again representable because G is special, G itself is the directed colimit of representable subfunctors. By Proposition 9.1 $G \simeq \hat{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}}$ where $Z \in \hat{\mathfrak{B}}$. The injection

$$\hat{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}} \to \hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$$

gives rise to the surjection $f : X \to Z$ in $\hat{\mathfrak{B}}$. I show that f is open. But let X' be a special open submodule of X'. Then $\mathfrak{B}(X/X', -)$ is a representable subfunctor of $\hat{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$, hence

$$\mathfrak{B}(X/X', -) \cap \hat{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}} \simeq \hat{\mathfrak{B}}(Z/f(X'), -)\,|_{\mathfrak{B}}$$

is representable since $G \simeq \hat{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}}$ is special. Hence $Z/f(X')$ is discrete, i.e., $f(X')$ is open in Z. Thus f is an open surjection which implies $X/\ker f \simeq Z$ in $\hat{\mathfrak{B}}$. But $X \in \tilde{\mathfrak{B}}$, hence $X/\ker f$ and $Z \in \tilde{\mathfrak{B}}$ and $G \simeq \tilde{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}}$. Now assume in addition that G is a subfunctor of some representable functor. Then there is an injection

$$\tilde{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}} \to \mathfrak{B}(B, -) = \tilde{\mathfrak{B}}(B, -)\,|_{\mathfrak{B}}$$

giving rise to a surjection $B \to Z$ in $\tilde{\mathfrak{B}}$. This implies that Z is discrete and

$$G \simeq \tilde{\mathfrak{B}}(Z, -)\,|_{\mathfrak{B}} = \mathfrak{B}(Z, -)$$

is representable. Hence $\tilde{\mathfrak{B}}(X, -)\,|_{\mathfrak{B}}$ is strict. ||

As an obvious corollary one obtains

PROPOSITION 9.3. *Situation as in Theorem* 9.2. *Then* $\mathfrak{B}$ *is artinian iff*

$$S\,\mathrm{Lex}(\mathfrak{B}, Ab) = \mathrm{Lex}(\mathfrak{B}, Ab).$$

Proof. If

$$S\,\mathrm{Lex}(\mathfrak{B}, Ab) = \mathrm{Lex}(\mathfrak{B}, Ab)$$

then $\tilde{\mathfrak{B}} = \hat{\mathfrak{B}}$ by 9.1 and 9.2. Hence $\mathfrak{B}$ is artinian by 5.24. If $\mathfrak{B}$ is artinian, then $\tilde{\mathfrak{B}} = \hat{\mathfrak{B}}$ by 5.24, and obviously every left exact functor preserves intersections. Hence every left exact additive functor is strict, and

$$S\,\mathrm{Lex}(\mathfrak{B}, Ab) = \mathrm{Lex}(\mathfrak{B}, Ab). \quad \|$$

Remark 9.4. If in the above situation $\mathfrak{B}$ is artinian then one can show by a relatively short proof that

$$(\hat{\mathfrak{B}})^{op} \to \mathrm{Lex}(\mathfrak{B}, \mathscr{A}b)$$

is an equivalence. Since $\mathrm{Lex}(\mathfrak{B}, \mathscr{A}b)$ is a Grothendieck category ([G], p. 353) this implies that $\hat{\mathfrak{B}}$ is a co-Grothendieck category. This saves a great deal of work done in section 5. I do not know whether a similar argument is possible for linearly compact $\mathfrak{B}$'s. It is obvious that one can define the category $S\,\mathrm{Lex}(\mathfrak{B}, \mathscr{A}b)$ for any skeletal-small Abelian category $\mathfrak{B}$ in which the lattices of subobjects are complete. At this moment I do not know a list of categorical properties of $\mathfrak{B}$ which insure that $S\,\mathrm{Lex}(\mathfrak{B}, Ab)$ is a Grothendieck category. As another corollary of Theorem 9.2 one obtains

THEOREM 9.5. (1) *If R is a strict complete topologically left coherent and linearly compact ring there is the duality*

$$\mathrm{STC}(R)^{op} \to S\,\mathrm{Lex}(\mathrm{Coh}(\mathrm{Dis}\,R), \mathscr{A}b) : X \rightsquigarrow \hom_{R,\mathrm{cont}}(X, -)_{\mathrm{res}}\,.$$

(2) *Any Grothendieck category is equivalent to a category*

$$S\,\mathrm{Lex}(\mathrm{Coh}(\mathrm{Dis}\,R), \mathscr{A}b)$$

where R is a strict complete topologically left coherent and linearly compact ring.

(3) *If $\mathfrak{A}$ is a Grothendieck category and of $\mathfrak{N}$ is a full, skeletal-small, finitely closed, and generating subcategory of $\mathfrak{A}$, then one obtains the equivalence*

$$\mathfrak{A} \to S\,\mathrm{Lex}(\mathfrak{N}^{op}, \mathscr{A}b) : A \rightsquigarrow \mathfrak{A}(-, A)\,|_{\mathfrak{N}}\,. \quad \|$$

APPENDIX (Added in Proof)

(1) If R is a complete topologically left coherent and linearly compact ring then R is automatically strict, i.e. $R \in STC(R)$.

(2) In order to avoid confusion with the classical notion of linearly compact module (Lefschetz, Zelinsky, Leptin) the algebraically (topologically) linearly compact modules of this paper (Definition 4.8) should be called algebraically (topologically) F-linearly compact ("F" for "finite"). The second half of Remark 4.7. is misleading. The theory of classically linearly compact modules is the special case of Section 5 where $\mathfrak{B}$ is also closed under taking subobjects.

(3) The results 5.22 and 5.23 show that a module $X \in \hat{\mathfrak{B}}$ is strict iff the topology of X is the finest topology $\mathfrak{T}$ on X such that the abstract module X, equipped with the topology $\mathfrak{T}$, lies in $\hat{\mathfrak{B}}$.

(4) The results 6.6 ff can be generalized as follows: Let R be a ring and $\mathfrak{B}$ a skeletal-small, linearly compact subcategory of mod R_R. By 5.23 the injection $\tilde{\mathfrak{B}} \to \text{mod } R_R$ preserves limits, and thus has a left adjoint $\sim$. Since the injection is also exact and since R_R is a projective generator of mod R_R the module $\tilde{R}$ is a projective generator of $\tilde{\mathfrak{B}}$, indeed

$$\tilde{\mathfrak{B}}(\tilde{R}, X) \cong \hom_R(R, X) \cong X \qquad \text{for all } X \in \tilde{\mathfrak{B}}.$$

In particular $\tilde{\mathfrak{B}}(\tilde{R}, \tilde{R}) \cong \tilde{R}$, hence $\tilde{R}$ is a ring in a canonical fashion. Now assume in addition that the modules in $\mathfrak{B}$ are finitely generated. Then, for each $B \in \mathfrak{B}$, there is an exact sequence in $\tilde{\mathfrak{B}}$

$$\tilde{R}^k \to B \to 0, \qquad k \in \mathbb{N}.$$

With the identifications $\tilde{R} = \tilde{\mathfrak{B}}(\tilde{R}, \tilde{R})$ and $X = \tilde{\mathfrak{B}}(\tilde{R}, X)$ the Theorems 3.3 and 3.4 imply $\mathfrak{B} = \text{Coh}(\text{Dis } \tilde{R}_{\tilde{R}})$ and $\tilde{\mathfrak{B}} = \text{STC}(\tilde{R}_{\tilde{R}})$.

(5) Let R be a complete topologically left coherent and linearly compact ring. The spectral category of STC(R) is a functor

$$P : \text{STC}(R) \to \text{Spec STC}(R),$$

which is universal w.r.t. the property that it transforms essential epimorphisms into isomorphisms ([6]). Then $(\text{Spec STC}(R))^{op}$ is a spectral category, and P is right exact and preserves strict filtered limits (loc. cit.).

THEOREM (Assumptions and Terminology as Above). *Then R/RaR is regular and right self-injective, and idempotents can be lifted from R/RaR to R. The category* STC(R/RaR) *is the spectral category of* STC(R), *and the spectral functor is given on the discrete modules by*

$$P : \text{Coh}(\text{Dis } R) \to PF(R/RaR) : X \rightsquigarrow X/RaX.$$

Here RaR and RaX denote the Jacobson radical.

(6) Assumptions of Section 9. A trivial extension of the proof of 9.1, applied to the dual situation of 9.5, (3), shows that an additive, left exact functor $F : \mathfrak{N}^{op} \to \mathscr{Ab}$ is strict iff it satisfies the following condition: If $N \in \mathfrak{N}$ and if $(N_i \,;\, i \in I)$, I directed, is an increasing family of subobjects of N with $\cup N_i = N$ then the canonical map

$$FN \to \lim_I FN_i$$

is a bijection. This has been shown by Jan-Erik Roos by means of the theory of sheaves w.r.t. Grothendieck topologies (unpublished manuscript). J.-E. Roos also gives a complete and elegant solution of the problem in Remark 9.4; in particular he shows that the strict left exact functors from $\mathfrak{N}^{op}$ to $\mathscr{Ab}$ are exactly the sheaves w.r.t. the canonical topology on $\mathfrak{N}$.

(7) The theorem 5.23 implies a generalization and a new simplified proof of several theorems of C. U. Jensen (On the vanishing of $\lim^{(n)}$, to appear in *J. Algebra*). Let R be a ring and let $\mathfrak{B}$ be a skeletal-small, full subcategory of mod R, closed under finite limits and colimits and intersection of subobjects. Let $S : \hat{\mathfrak{B}} \to \text{mod } R$ be the injection.

THEOREM (Assumptions as Above). *The following assertions are equivalent:*

(1) *The category $\mathfrak{B}$ is linearly compact.*

(2) *For each small category I which is filtered from below, for each $F \in \hat{\mathfrak{B}}^I$, and for each $n > 0$ the relation*

$$lim_I{}^{(n)} SF = 0$$

holds.

(3) *The assertion of* (2) *is true for directed ordered sets I, $F \in B^I$ and $n = 1$. Here* $\lim_I{}^{(n)}$ *is the n-th right derived functor of the functor*

$$\lim_I : (\text{mod } R)^I \to \text{mod } R.$$

The proof of this theorem is a trivial consequence of the fact that $\mathfrak{B}$ is a co-Grothendieck category, that mod R has exact direct products, and that the injection $\mathfrak{B} \to \text{mod } R$ preserves products and is exact.

(8) Several decisive results of this paper have independently been found by Remi Goblot (unpublished).

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