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Optimality conditions for proper efficient solutions of vector set-valued optimization ☆

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Abstract

Based on the concept of an epiderivative for a set-valued map introduced in J. Nanchang Univ. 25 (2001) 122–130, in this paper, we present a few necessary and sufficient conditions for a Henig efficient solution, a globally proper efficient solution, a positive properly efficient solution, an f-efficient solution and a strongly efficient solution, respectively, to a vector set-valued optimization problem with constraints.

Keywords: Set-valued vector optimization; Proper efficiency; Epiderivative; Optimality conditions

1. Introduction

More and more attention has been paid to sufficient and necessary conditions for an efficient solution in vector set-valued optimization. As early as in 1981, the concept of a contingent derivative of a set-valued map was firstly introduced by Aubin [1]. With the concept of a contingent derivative, Aubin and Ekeland [2], Corley [3], and Luc [4] derived

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sufficient and necessary conditions for a strongly efficient solution, a weakly efficient solution and a locally efficient solution, respectively. But Aubin and Ekeland [2] made a strong assumption that the graph of F is convex, and Corley [3] did not unify the necessary and sufficient optimality conditions. To overcome those disadvantages, Jahn and Rauh [5] introduced a contingent epiderivative of a set-valued map and derived the corresponding optimality conditions. Subsequently, a generalized contingent epiderivative of a set-valued map was introduced by Chen and Jahn [6]. Based on a vector variational inequality and the concept of a contingent epiderivative of a set-valued map introduced in Jahn and Rauh [5], Liu and Gong [7] obtained a necessary and sufficient condition for some kinds of properly efficiency in vector set-valued optimization. Gong and Dong [8] simplified the optimality conditions for properly efficient solutions of an unconstrained optimization problem in term of a radial contingent derivative of a set-valued map introduced in [9]. In light of the notions of a contingent epiderivative in [5] and a radial contingent derivative of set-valued map in [9], Song et al. [10] introduced a new concept of epiderivative for a set-valued map and obtained a few necessary and sufficient conditions for a weakly efficient solution, a strongly efficient solution, a Henig efficient solution, a super-efficient solution and a Benson efficient solution respectively in unconstrained vector set-valued optimization.

Meanwhile, many authors studied vector set-valued optimization problems with constraints. Corley [3] gave optimality conditions for an efficient solution and a weakly efficient solution of a constrained problem in terms of the Clarke derivative and the contingent derivative. Luc [4] presented necessary conditions for a local weakly efficient solution and sufficient conditions for a locally efficient solution of a constrained problem in terms of the contingent derivative when the functions are upper semi-differentiable. Amahroq and Taa [11] gave sufficient conditions for a locally efficient solution to a constrained problem with g-paraconvex data in term of the contingent derivative. Taa [9] derived several necessary conditions for a local weakly efficient solution, and sufficient condition for a weakly efficient solution and an efficient solution of a constrained problem in terms of radial contingent derivative. Ling [12] got some necessary and sufficient conditions for a super-efficient solution to a constrained problem under the assumption that the ordering cone has a bounded base in terms of contingent derivatives. Gätz and Jahn [13] provided a few necessary and sufficient conditions for a weakly efficient solution to a constrained problem in terms of contingent epiderivatives.

It is well known that weak efficient solution is a kind of extremely efficient solutions in vector optimization. The concepts of proper efficient solutions are of great importance in vector optimization (see [19–23]). In this paper, based on the concept of an epiderivative for a set-valued map introduced in [10], we present a few necessary and sufficient conditions for a Henig efficient solution, a globally proper efficient solution, an f-efficient solution and a strongly efficient solution, respectively, to a vector set-valued optimization problem with constraints. The paper is organized as follows. The next section reviews some concepts. Section 3 presents a number of optimality conditions for a vector set-valued optimization problem with constraints.

2. Preliminaries and definitions

Throughout this paper, let X and Y be two real normed spaces, Y^* be the topological dual space of Y, and C be a closed convex pointed cone in Y. The cone C induces a partially ordering of Y. Let C^* be the dual cone of cone C, defined by

$$C^* := \left\{ f \in Y^*:\ f(y) \geqslant 0 \text{ for all } y \in C \right\}.$$

Denote the quasi-interior of C by $C^\sharp$, i.e.,

$$C^\sharp := \left\{ f \in Y^*:\ f(y) > 0 \text{ for all } y \in C \backslash \{\theta\} \right\}.$$

Denote the cone hull of A by

$$\operatorname{cone}(A) := \bigcup \{\lambda A:\ \lambda \geqslant 0\}.$$

Denote the closure of A by $\operatorname{cl}(A)$ and the interior of A by $\operatorname{int}(A)$.

A nonempty convex subset B of the convex cone C is called a base of C if $C = \operatorname{cone}(B)$ and $\theta \notin \operatorname{cl}(B)$.

We know that $C^\sharp \neq \emptyset$ if and only if C has a base (see [14]).

In fact, if $C^\sharp \neq \emptyset$, then we can choose $f \in C^\sharp$. It is easy to see that set $\{y \in C:\ f(y) = 1\}$ is a base of C. If B is a base of C, since $0 \notin \operatorname{cl}(B)$, by a known separation argument there exists some $f \in C^\sharp$ (see [15]).

Denote the closed unit ball of Y by U. Suppose that C has a base B. Let $\delta := \inf\{\|b\|:\ b \in B\}$ and

$$C_\varepsilon(B) := \operatorname{cone}(B + \varepsilon U) \quad \text{for all } 0 < \varepsilon < \delta.$$

It is clear that $\delta > 0$, $\operatorname{cl}(C_\varepsilon(B))$ is a closed convex pointed cone and $C \backslash \{\theta\} \subset \operatorname{int} C_\varepsilon(B)$ for all $0 < \varepsilon < \delta$ (see [16]).

Let $F : X \to 2^Y$ be a set-valued map, i.e., $F(x)$ is a set in Y for each $x \in X$. The set

$$\operatorname{dom}(F) := \left\{ x \in X,\ F(x) \neq \emptyset \right\}$$

is called the domain of F. The set

$$\operatorname{graph}(F) := \left\{ (x, y) \in X \times Y:\ x \in \operatorname{dom}(F),\ y \in F(x) \right\}$$

is called the graph of F. The set

$$\operatorname{epi}(F) := \left\{ (x, y) \in X \times Y:\ x \in \operatorname{dom}(F),\ y \in F(x) + C \right\}$$

is called the epigraph of F.

Let us recall some concepts.

Definition 2.1. Let A be a nonempty subset of X and $x_0 \in A$. The contingent cone $T(A, x_0)$ to A at x_0 is the set of all $h \in X$ for which there exist a sequence $\{t_n\}$ of positive real numbers and a sequence $\{x_n\}$ in A such that

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0) = h.$$

Definition 2.2. Let A be a nonempty subset of X and $x_0 \in A$. The radial cone $R(A, x_0)$ to A at x_0 is the set of all $h \in X$ for which there exist a sequence $\{t_n\}$ of positive real numbers and a sequence $\{x_n\}$ in A such that

$$\lim_{n\to\infty} t_n(x_n - x_0) = h.$$

Definition 2.3. Let A be a nonempty subset of X and $x_0 \in A$. The Clarke tangent cone $C(A, x_0)$ to A at x_0 is the set of all $h \in X$, for which for every sequence $\{x_n\}$ in A converging to x_0 and for every sequence $\{t_n\}$ of positive real numbers converging to 0, there is a sequence of elements $\{v_n\}$ in X converging to h such that

$$x_n + t_n v_n \in A \quad \text{for all } n.$$

Remark 2.1 (see [3]). (a) $T(A, x_0)$ is a closed cone.
(b) $C(A, x_0)$ is a closed convex cone.
(c) $C(A, x_0) \subset T(A, x_0) \subset R(A, x_0)$.
(d) If A is convex, then the three sets in (c) coincide and $A - x_0 \subset T(A, x_0)$.

To simplify optimality conditions, Song et al. [10] introduced the following concept of an epiderivative which differentiates from the contingent epiderivative introduced by [5].

Definition 2.4. Let $(x_0, y_0) \in \operatorname{graph}(F)$. The contingent epiderivative $DF(x_0, y_0)$ of F at (x_0, y_0) is a set-valued map from X to Y defined by

$$\operatorname{graph}\big(DF(x_0, y_0)\big) = T\big(\operatorname{epi}(F), (x_0, y_0)\big).$$

According to Definition 2.1, $y \in DF(x_0, y_0)(x)$ if and only if there exist a sequence $\{(x_n, y_n)\}$ in $\operatorname{epi}(F)$ and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0) \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0) = (x, y).$$

The contingent epiderivative introduced by [5] is single-valued and its epigraph equals to the contingent cone to epigraph of F at (x_0, y_0), while the above contingent epiderivative is set-valued and its graph equals to the contingent cone to epigraph of F at (x_0, y_0).

Definition 2.5. Let $(x_0, y_0) \in \operatorname{graph}(F)$. The Clarke tangent epiderivative $CF(x_0, y_0)$ of F at (x_0, y_0) is the set-valued map from X to Y defined by

$$\operatorname{graph}\big(CF(x_0, y_0)\big) = C\big(\operatorname{epi}(F), (x_0, y_0)\big).$$

Definition 2.6. Let $(x_0, y_0) \in \operatorname{graph}(F)$. The radial epiderivative $RF(x_0, y_0)$ of F at (x_0, y_0) is the set-valued map from X to Y defined by

$$\operatorname{graph}\big(RF(x_0, y_0)\big) = R\big(\operatorname{epi}(F), (x_0, y_0)\big).$$

Due to Definition 2.2, $y \in RF(x_0, y_0)(x)$ if and only if there exist a sequence $\{(x_n, y_n)\}$ in $\operatorname{epi}(F)$ and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0) = (x, y).$$

Definition 2.7. Let $(x_0, y_0) \in \text{graph}(F)$. The Y-epiderivative $YF(x_0, y_0)$ of F at (x_0, y_0) is the set-valued map from X to Y defined by $y \in YF(x_0, y_0)(x)$ if there exist a sequence $\{(x_n, y_n)\}$ in $\text{epi}(F)$ and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} y_n = y_0 \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0) = (x, y).$$

Suppose that $S \subset X$ is a convex subset of $\text{dom}(F)$. Thus, F is C-convex on S if, for any $x_1, x_2 \in S$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F\big(\lambda x_1 + (1 - \lambda)x_2\big) + C.$$

It is well known that if F is C-convex on S, then $\text{epi}(F)$ is a convex subset in $X \times Y$.

Remark 2.2. Let $(x_0, y_0) \in \text{graph}(F)$.

(a) The set-valued maps $DF(x_0, y_0)$, $CF(x_0, y_0)$, $RF(x_0, y_0)$, and $YF(x_0, y_0)$ are positive homogeneous with closed graphs.

(b) $\text{graph}(CF(x_0, y_0)) \subset \text{graph}(DF(x_0, y_0)) \subset \text{graph}(YF(x_0, y_0)) \subset \text{graph}(RF(x_0, y_0))$.

(c) Whenever $\text{epi}(F)$ is starshaped at (x_0, y_0), the four sets in (b) coincide. Especially, when F is C-convex, the four sets in (b) coincide.

(d) The set-valued map $CF(x_0, y_0)$ is a close convex process (see [2]).

The following proposition is crucial in the sequel.

Proposition 2.1. *Let* $(x_0, y_0) \in \text{graph}(F)$. *Then*

(i) $S - x_0 \subset \text{dom}\, RF(x_0, y_0)$ *for all* $S \subset \text{dom}(F)$;
(ii) $F(x) - y_0 \subset RF(x_0, y_0)(x - x_0)$ *for all* $x \in \text{dom}(F)$.

Proof. It suffices to prove (ii). Since $(x_0, y_0) \in \text{graph}(F)$, $\text{dom}(F) \neq \emptyset$. For $x \in \text{dom}(F)$, let $y \in F(x)$. Set $t_n = 1$, $x_n = x$, $y_n = y$. So,

$$t_n(x_n - x_0, y_n - y_0) = (x - x_0, y - y_0).$$

Since $(x_n, y_n) = (x, y) \in \text{graph}(F) \subset \text{epi}(F)$, by Definition 2.6, we have

$$y - y_0 \subset RF(x_0, y_0)(x - x_0).$$

Hence, (ii) holds. $\square$

Now we consider the following constrained vector set-valued optimization problem (SVOP):

$$\begin{array}{ll} \min & F(x) \\ \text{s.t.} & x \in S,\ G(x) \cap -D \neq \emptyset, \end{array}$$

where $\text{dom}(F) = S$, G is a set valued map from X to Z, and D is a nonempty pointed closed convex cone in real linear normed space Z. The cone D introduces a partial order in Z.

Set

$$A=\{x\in S:\ G(x)\cap -D\neq\emptyset\}\quad\text{and}\quad F(A)=\bigcup\{F(x):\ x\in A\}.$$

Definition 2.8. A triple $(x,y,z)\in S\times Y\times Z$ is said to be feasible if $x\in \operatorname{dom}F\cap\operatorname{dom}G$, $y\in F(x)$, and $z\in G(x)\cap -D$.

In the following definitions, we always assume that $x_0\in A$ and $y_0\in F(x_0)$.

Definition 2.9 (see [16]). Suppose that C has a base B. A pair (x_0,y_0) is called a Henig efficient pair of (SVOP) if for some $0<\varepsilon<\delta$,

$$\bigl(F(A)-y_0\bigr)\cap -\operatorname{int}C_\varepsilon(B)=\emptyset.$$

Definition 2.10. Let $f\in C^*\backslash\{\theta_{Y^*}\}$. A pair (x_0,y_0) is called an f-efficient pair of (SVOP) if

$$f\bigl(F(x)-y_0\bigr)\geqslant 0\quad\text{for all } x\in A.$$

Definition 2.11. A pair (x_0,y_0) is called a strongly efficient pair of (SVOP) if

$$F(A)\subset y_0+C.$$

Definition 2.12. A pair (x_0,y_0) is called a super-efficient pair of (SVOP) if there exists a constant $M>0$ such that

$$\operatorname{cone}\bigl(F(A)-y_0\bigr)\cap(U-C)\subset MU.$$

Definition 2.13 (see [18]). A pair (x_0,y_0) is called a globally proper efficient pair of (SVOP) if there exists a pointed convex cone $H\subset Y$ with $C\backslash\{\theta\}\subset\operatorname{int}H$, such that

$$\bigl(F(A)-y_0\bigr)\cap -H=\{\theta\}.$$

Definition 2.14. A pair (x_0,y_0) is called a positive properly efficient pair of (SVOP) if there is $f\in C^\sharp$ such that

$$f\bigl(F(x)-y_0\bigr)\geqslant 0\quad\text{for all } x\in A.$$

3. Set-valued epiderivative and optimality conditions

In this section, by using the concept of a set-valued epiderivative introduced in [10], we give optimality conditions for various kinds of properly efficient pairs to (SVOP).

In the sequel, couple (F,G) is a set valued map from X into $Y\times Z$ defined by

$$(F,G)(x)=\bigl(F(x)\times G(x)\bigr).$$

We make an assumption (C): For any $u\in D^*\backslash\{\theta_{Z^*}\}$, there exists $x\in A$ such that

$$u\bigl(G(x)\bigr)\cap -\operatorname{int}R_+\neq\emptyset,$$

where $R_+=\{r\in R:\ r\geqslant 0\}$ and $A=\{x\in S:\ G(x)\cap -D\neq\emptyset\}$.

Remark 3.1. It is easy to show that assumption (C) is weaker than the Slater constraint condition.

Now we give optimality conditions for a Henig efficient solution to (SVOP). Firstly, a necessary condition is given in a general setting.

Theorem 3.1. *Suppose that B is a base of C and* $\operatorname{int} D \neq \emptyset$*. Let* $(x_0, y_0) \in \operatorname{graph}(F)$ *and* $\delta := \inf\{\|b\|\colon b \in B\}$*. If* (x_0, y_0) *is a Henig efficient pair of* (SVOP)*, then for some* $0 < \varepsilon < \delta$ *and for any* $z_0 \in G(x_0) \cap -D$*,*

$$\big[Y(F,G)(x_0,y_0,z_0)(x) + (\theta, z_0)\big] \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big) = \emptyset \tag{1}$$

for all $x \in \operatorname{dom} Y(F,G)(x_0,y_0,z_0)$.

Proof. Suppose that (x_0, y_0) is a Henig efficient pair of (SVOP); then there exists $0 < \varepsilon < \delta$ such that

$$\big(F(A) - y_0\big) \cap -\operatorname{int} C_\varepsilon(B) = \emptyset. \tag{2}$$

If, for ε above, there exist $x \in \operatorname{dom} Y(F,G)(x_0,y_0,z_0)$ and $(y,z) \in Y \times Z$ such that

$$(y, z + z_0) \in \big[Y(F,G)(x_0,y_0,z_0)(x) + (\theta, z_0)\big] \cap -\operatorname{int}\big(C_\varepsilon(B) \times D\big), \tag{3}$$

then $(y,z) \in Y(F,G)(x_0,y_0,z_0)(x)$ and $y \in -\operatorname{int} C_\varepsilon(B)$.

By Definition 2.7, there exist a sequence $\{(x_n, y_n, z_n)\}$ in $\operatorname{epi}(F,G)$ and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty}(y_n, z_n) = (y_0, z_0) \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0, z_n - z_0) = (x, y, z).$$

Since $t_n > 0$ and $C_\varepsilon(B)$ is a cone, there exists N_1 such that

$$y_n - y_0 \in -\operatorname{int} C_\varepsilon(B) \quad \text{for all } n \geqslant N_1.$$

Similarly, since D is a cone, $z + z_0 \in -\operatorname{int} D$ and $\lim_{n\to\infty} t_n(z_n - z_0) = z$, there exists N_2 such that

$$t_n(z_n - z_0) + z_0 \in -\operatorname{int} D \quad \text{for all } n \geqslant N_2. \tag{4}$$

Moreover, there exists $N \geqslant \max(N_1, N_2)$ such that $t_N > 1$. Otherwise, because $\lim_{n\to\infty} y_n = y_0$ and $\lim_{n\to\infty} t_n(y_n - y_0) = \theta = y$, it contradicts $y \in -\operatorname{int} C_\varepsilon(B)$. It follows from (4) that

$$t_N(z_N - z_0) + z_0 = t_N\left(z_N - \left(1 - \frac{1}{t_N}\right)z_0\right) \in -\operatorname{int} D,$$

and hence, $z_N - (1 - 1/t_N)z_0 \in -\operatorname{int} D$. Since $t_N > 1$ and $z_0 \in -D$, $(1 - 1/t_N)z_0 \in -D$. Thus,

$$z_N \in -D - \operatorname{int} D = -\operatorname{int} D.$$

Since $(x_n, y_n, z_n) \in \operatorname{epi}(F,G)$ for all $n \in N$, there are $y_n' \in F(x_n)$ with $y_n \in y_n' + C$ and $z_n' \in G(x_n)$ with $z_n \in z_n' + D$. Thus,

$$y_N' \in y_N - C \subset y_0 - \operatorname{int} C_\varepsilon(B) - C = y_0 - \operatorname{int} C_\varepsilon(B)$$

and

$$z'_N \in z_N - D \subset -\operatorname{int} D - D = -\operatorname{int} D.$$

Hence, (x_N, y'_N, z'_N) is a feasible triple, but $y'_N \in y_0 - \operatorname{int} C_\varepsilon(B)$, which contradicts (2). Therefore, (1) holds. $\square$

Let C be a convex cone with base B. Denote

$$C^\Delta(B) = \{f \in C^*\colon \inf\{f(b)\colon b \in B\} > 0\}.$$

By the separation theorem, $C^\Delta(B) \neq \emptyset$. Clearly (see [17,25]),

$$C^\sharp \supset C^\Delta(B) \supset C^* + C^\Delta(B).$$

To obtain necessary and sufficient conditions for a Henig efficient solution to (SVOP), we give the following lemma.

Lemma 3.1. *For any* $\varepsilon \in (0, \delta)$, $C_\varepsilon(B)^* \backslash \{\theta_{Y^*}\} \subset C^\Delta(B)$.

Proof. For any $f \in C_\varepsilon(B)^* \backslash \{\theta_{Y^*}\}$,

$$f(c) > 0 \quad \text{for all } c \in \operatorname{int} C_\varepsilon(B).$$

Since $C \backslash \{\theta\} \subset \operatorname{int} C_\varepsilon(B)$, $f(b) > 0$ for all $b \in B$.

Let $\eta = \inf\{f(b) \mid b \in B\}$. Suppose that $\eta \leqslant 0$; then there exists $b_n \in B$ with

$$f(b_n) < \frac{1}{n} \quad \text{for every } n \in N.$$

Fixed $u \in U$ with $f(u) > 0$, then

$$f(b_n - \varepsilon u) = f(b_n) - \varepsilon f(u) < 0 \quad \text{for sufficiently large } n \in N.$$

But $f \in C_\varepsilon(B)^* \backslash \{\theta_{Y^*}\}$ and $b_n - \varepsilon u \in B + \varepsilon U \in C_\varepsilon(B)$, thus $f(b_n - \varepsilon u) \geqslant 0$. This is a contradiction. Hence, $\eta > 0$. Therefore, $f \in C^\Delta(B)$. $\square$

Lemma 3.2. *For any* $f \in C^\Delta(B)$, *there exists* $0 < \varepsilon < \delta$ *with* $f \in C_\varepsilon(B)^* \backslash \{\theta_{Y^*}\}$.

Proof. Let $f \in C^\Delta(B)$. Thus, $\eta = \inf\{f(b) \mid b \in B\} > 0$. So $f \neq \theta_{Y^*}$.

Let $\varepsilon \in (0, \min(\eta/(2\|f\|), \delta))$, $C_\varepsilon(B) = \operatorname{cl}(\operatorname{cone}(B + \varepsilon U))$, and $V = B + \varepsilon U$. Then, for any $y \in V$, there exist $b \in B$ and $u \in U$ such that $y = b + \varepsilon u$. So,

$$f(y) = f(b) + \varepsilon f(u) \geqslant f(b) - \varepsilon \|f\| \geqslant \frac{\eta}{2} > 0.$$

Hence, $f(y) \geqslant 0$ for all $y \in C_\varepsilon(B)$, that is, $f \in C_\varepsilon(B)^*$. Therefore, $f \in C_\varepsilon(B)^* \backslash \{\theta_{Y^*}\}$. $\square$

Lemma 3.3. (i) $\operatorname{int} C^* \subset C^\Delta(B)$, *where* $\operatorname{int} C^*$ *is the interior of* C^* *in* Y^* *with respect to the norm of* Y^*.

(ii) *If* B *is bounded, then* $\operatorname{int} C^* = C^\Delta(B)$.

Now, we can give a Fritz John type necessary condition for a Henig efficient solution to (SVOP) in a general setting.

Proposition 3.1. *Let B be a base of C and* $\operatorname{int} D \neq \emptyset$. *If (x_0, y_0) is a Henig efficient pair of* (SVOP)*, then, for any $z_0 \in G(x_0) \cap -D$, there exist $f \in C^{\Delta}(B) \cup \{\theta_{Y^*}\}$ and $u \in D^*$, both dependent on z_0 and not both being zero functionals, such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} C(F, G)(x_0, y_0, z_0)$ and $(y, z) \in C(F, G)(x_0, y_0, z_0)(x)$.

Proof. Let $z_0 \in G(x_0) \cap -D$ and define

$$Q = \left[\bigcup_{x \in \Omega} C(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0) \right],$$

where $\Omega = \operatorname{dom} C(F, G)(x_0, y_0, z_0)$.

We first show that Q is convex by showing that $Q_1 = Q - (\theta, z_0)$ is convex. Let $(y_1, z_1), (y_2, z_2) \in Q_1$. Then, there exist $x_1, x_2 \in \Omega$ such that

$$(y_i, z_i) \in C(F, G)(x_0, y_0, z_0)(x_i), \quad i = 1, 2,$$

and thus,

$$(x_i, y_i, z_i) \in C\big(\operatorname{epi}(F, G), (x_0, y_0, z_0)\big), \quad i = 1, 2.$$

But $C(\operatorname{epi}(F, G), (x_0, y_0, z_0))$ is a convex cone, therefore,

$$\lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) \in C\big(\operatorname{epi}(F, G), (x_0, y_0, z_0)\big)$$

for all $\lambda \in [0, 1]$, that is,

$$\big(\lambda y_1 + (1 - \lambda) y_2, \lambda z_1 + (1 - \lambda) z_2\big) \in C(F, G)(x_0, y_0, z_0)\big(\lambda x_1 + (1 - \lambda) x_2\big)$$

for all $\lambda \in [0, 1]$. It follows that Q_1 and its translate Q are convex.

By Theorem 3.1 and Remark 2.2(b), it is easy to show that $Q \cap -\operatorname{int}(C_\varepsilon(B) \times D) = \emptyset$. By the separation theorem, there exist $f \in Y^*$ and $u \in Z^*$, not both zero functionals, and a real number ξ such that

$$f(y) + u(z) \geqslant \xi \quad \text{for all } (y, z) \in Q \tag{5}$$

and

$$f(y) + u(z) < \xi \quad \text{for all } (y, z) \in -\operatorname{int}\big(C_\varepsilon(B) \times D\big). \tag{6}$$

But since $(y, z) \in -\operatorname{int}(C_\varepsilon(B) \times D)$ can be made as close as possible to (θ, θ), from (6), the continuity of f and u leads to that $\xi \geqslant 0$. Since z can be made as close as possible to θ in (6), by the continuity of u, we have

$$f(y) \leqslant \xi \quad \text{for all } y \in -\operatorname{int} C_\varepsilon(B).$$

Since $C_\varepsilon(B)$ is a cone, $f(y) \leqslant 0$ for all $y \in -\operatorname{int} C_\varepsilon(B)$. Thus, $f(y) \geqslant 0$ for all $y \in C_\varepsilon(B)$, that is, $f \in C_\varepsilon(B)^*$. Similarly, we can easily get that $u \in D^*$.

By Lemma 3.1, $f \in C^{\Delta}(B) \cup \{\theta_{Y^*}\}$.

From (5) and the fact $(\theta, z_0) \in Q$, we get $u(z_0) \geqslant 0$. But $z_0 \in -D$ and $u \in D^*$, so $u(z_0) \leqslant 0$. Thus, $u(z_0) = 0$. Let $x \in \operatorname{dom} C(F, G)(x_0, y_0, z_0)$ and $(y, z) \in C(F, G)(x_0,$

$y_0, z_0)(x)$. Since $C(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0) \subset Q$ and $u(z_0) = 0$, from (5), we know that $f(y) + u(z) \geqslant 0$.

This completes the proof. □

Let F_A and G_A denote F and G restricted on A, respectively.

The following theorem is a sufficient condition involving multiplier functional for a Henig efficient solution of (SVOP).

Theorem 3.2. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$ *and* B *be a base of* C. *If there exist* $z_0 \in G(x_0) \cap -D$, $f \in C^{\Delta}(B)$, *and* $u \in D^*$ *such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} R(F_A, G_A)(x_0, y_0, z_0)$ *and* $(y, z) \in R(F_A, G_A)(x_0, y_0, z_0)(x)$, *then* (x_0, y_0) *is a Henig efficient pair of* (SVOP).

Proof. Since $f \in C^{\Delta}(B)$, by Lemma 3.2, there exists $\varepsilon \in (0, \delta)$ such that $f \in C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\}$. Then $(y_0 - \operatorname{int} C_\varepsilon(B)) \cap F(A) = \emptyset$. Otherwise, there exist $x' \in A$ and $y' \in F(x')$ such that

$$y' - y_0 \in -\operatorname{int} C_\varepsilon(B).$$

Since $x' \in A$, there exists $Z' \in G(x') \cap -D$. By Proposition 2.1, we have

$$(y' - y_0, z' - z_0) \in R(F_A, G_A)(x_0, y_0, z_0)(x' - x_0).$$

Thus,

$$f(y' - y_0) + u(z' - z_0) \geqslant 0. \tag{7}$$

Since $y' - y_0 \in -\operatorname{int} C_\varepsilon(B)$ and $f \in C_\varepsilon(B)^* \setminus \{\theta_{Y^*}\}$, $f(y' - y_0) < 0$. Since $z' \in G(x') \cap -D$, $u(z_0) = 0$, and $u \in D^*$, $u(z' - z_0) \leqslant 0$. Thus, $f(y' - y_0) + u(z' - z_0) < 0$, which contradicts (7). Hence, (x_0, y_0) is a Henig efficient pair of (SVOP). □

The following corollary gives a necessary and sufficient conditions for a Henig efficient solution of (SVOP) when G satisfies assumption (C).

Corollary 3.1. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$, B *be a base of* C *and* $\operatorname{int} D \neq \emptyset$. *Suppose that* F *is* C*-convex and* G *is* D*-convex. If* G *satisfies assumption* (C), *then* (x_0, y_0) *is a Henig efficient pair of* (SVOP) *if and only if there exist* $z_0 \in G(x_0) \cap -D$, $f \in C^{\Delta}(B)$, *and* $u \in D^*$, *such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0 \tag{8}$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ *and* $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

Proof. Suppose that (x_0, y_0) is a Henig efficient pair of (SVOP). Thus, $G(x_0) \cap -D \neq \emptyset$. Let $z_0 \in G(x_0) \cap -D$. By Proposition 3.1 and Remark 2.2(c), there exist $f \in C^{\Delta}(B) \cup \{\theta_{Y^*}\}$ and $u \in D^*$, which satisfy (8) and are not both zero functionals.

If $f = \theta_{Y^*}$, then $u \in D^*\backslash\{\theta_{Z^*}\}$. Since G satisfies assumption (C), there exists $x \in A$ such that $u(G(x)) \cap -\operatorname{int} R_+ \neq \emptyset$, that is, there exists $z \in G(x)$ with $u(z) < 0$.

Since $x \in A$, there exists $y \in F(x)$. By Proposition 2.1,

$$(y - y_0, z - z_0) \in D(F, G)(x_0, y_0, z_0)(x - x_0).$$

So, $u(z - z_0) \geqslant 0$, that is, $u(z) \geqslant 0$, which is a contradiction. Hence, $f \neq \theta_{Y^*}$, that is, $f \in C^{\Delta}(B)$.

Conversely, it follows directly from Theorem 3.2 and Remark 2.2. □

For simplicity, we denote the set of super-efficient points of E by $E_S(E, C)$ and the set of all Henig efficient points of E by $E_H(E, C)$.

To give optimality conditions for a super-efficient solution to (SVOP), let us recall the following lemma.

Lemma 3.4. *If C has a bounded base B and if A is a nonempty subset of X, then $E_S(F(A), C) = E_H(F(A), C)$.*

Under the assumption that C has a bounded base B, the super-efficiency equals to the Henig efficiency. Hence, we can also give a necessary and sufficient condition for a super-efficient solution of (SVOP).

Corollary 3.2. *Let $(x_0, y_0) \in \operatorname{graph}(F)$. Suppose that C has a bounded base B and $\operatorname{int} D \neq \emptyset$. Suppose that F is C-convex and G is D-convex and G satisfies assumption* (C). *Then (x_0, y_0) is a super-efficient pair of* (SVOP) *if and only if there exist $z_0 \in G(x_0) \cap -D$, $f \in \operatorname{int} C^*$, and $u \in D^*$ such that*

$$u(z_0) = 0 \quad \text{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ and $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

The above result is different from Theorems 3.3 and 3.4 in [12] because we use the epiderivative introduced in [10].

Next we give optimality conditions for a globally proper efficient solution to (SVOP). Firstly, we provide a necessary condition in a general setting.

Theorem 3.3. *Let $(x_0, y_0) \in \operatorname{graph}(F)$ and $\operatorname{int} D \neq \emptyset$. Suppose that (x_0, y_0) is a globally proper efficient pair of* (SVOP). *If H is a pointed convex cone which satisfies Definition* 2.13 (*that is, $C\backslash\{\theta\} \subset \operatorname{int} H$, and $(F(A) - y_0) \cap -H = \{\theta\}$*), *then for any $z_0 \in G(x_0) \cap -D$,*

$$\left[Y(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0)\right] \cap -\operatorname{int}(H \times D) = \emptyset$$

for all $x \in \operatorname{dom} Y(F, G)(x_0, y_0, z_0)$.

Proof. Suppose that (x_0, y_0) is a globally proper efficient pair of (SVOP), then there exists a convex point cone $H \subset Y$, such that $C\backslash\{\theta\} \subset \operatorname{int} H$ and

$$\big(F(A) - y_0\big) \cap -H = \{\theta\}. \tag{9}$$

If there exist $x \in \operatorname{dom} Y(F,G)(x_0,y_0,z_0)$ and $(y,z) \in Y \times Z$ such that

$$(y, z+z_0) \in \big[Y(F,G)(x_0,y_0,z_0)(x) + (\theta, z_0)\big] \cap -\operatorname{int}(H \times D), \tag{10}$$

then $(y,z) \in Y(F,G)(x_0,y_0,z_0)(x)$.

By Definition 2.7, there exists a sequence $\{(x_n,y_n,z_n)\}$ in $\operatorname{epi}(F,G)$ and a sequence $\{t_n\}$ of positive real numbers with

$$\lim_{n\to\infty}(y_n,z_n) = (y_0,z_0) \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0, z_n - z_0) = (x,y,z).$$

By (11), $y \in -\operatorname{int} H$. Thus, there exists N_1 such that

$$y_n - y_0 \in -\operatorname{int} H \quad \text{for all } n \geqslant N_1,$$

since $t_n > 0$ and H is a cone. Similarly, from the proof of Theorem 3.1, there exits $N \geqslant N_1$ such that

$$z_N \in -\operatorname{int} D.$$

Since $(x_n,y_n,z_n) \in \operatorname{epi}(F,G)$ for all $n \in N$, there are $y_n' \in F(x_n)$ with $y_n \in y_n' + C$ and $z_n' \in G(x_n)$ with $z_n \in z_n' + D$. Thus,

$$y_N' \in y_N - C \subset y_0 - \operatorname{int} H - C = y_0 - \operatorname{int} H$$

and

$$z_N' \in z_N - D \subset -\operatorname{int} D - D = -\operatorname{int} D.$$

Hence, (x_N, y_N', z_N') is a feasible triple, but $y_N' \in y_0 - \operatorname{int} H$, which contradicts (10). So, (9) fulfills. □

The following proposition is a Fritz John necessary condition for a globally proper efficient solution to (SVOP).

Proposition 3.2. *Let* $(x_0,y_0) \in \operatorname{graph}(F)$ *and* $\operatorname{int} D \neq \emptyset$. *If* (x_0,y_0) *is a globally proper efficient pair of* (SVOP), *then, for any* $z_0 \in G(x_0) \cap -D$, *there exist* $f \in C^\sharp \cup \{\theta_{Y^*}\}$ *and* $u \in D^*$, *both dependent on* z_0, *but not both being zero functionals, such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} C(F,G)(x_0,y_0,z_0)$ *and* $(y,z) \in C(F,G)(x_0,y_0,z_0)(x)$.

Proof. Suppose that (x_0,y_0) is a globally proper efficient pair of (SVOP); then there exists a convex point cone $H \subset Y$, such that $C \setminus \{\theta\} \subset \operatorname{int} H$ and $(F(A) - y_0) \cap -H = \{\theta\})$. Let $z_0 \in G(x_0) \cap -D$ and define

$$Q = \left[\bigcup_{x\in\Omega} C(F,G)(x_0,y_0,z_0)(x) + (\theta, z_0)\right],$$

where $\Omega = \operatorname{dom} C(F,G)(x_0,y_0,z_0)$.

It is not hard to show that Q is convex.

We can easily show that $Q \cap -\operatorname{int}(H \times D) = \emptyset$, from Theorem 3.3 and Remark 2.2(b). So by the separation theorem, there exist $f \in Y^*$ and $u \in Z^*$, not both zero functionals, and a real number ξ such that

$$f(y) + u(z) \geqslant \xi \quad \text{for all } (y, z) \in Q \tag{11}$$

and

$$f(y) + u(z) < \xi \quad \text{for all } (y, z) \in -\operatorname{int}(H \times D). \tag{12}$$

From the proof of Proposition 3.1, we get that $f \in H^*$, $\xi \geqslant 0$, and $u \in D^*$. If $f \neq \theta_{Y^*}$, then $f(y) > 0$ for all $y \in \operatorname{int} H$. Since $C \setminus \{0\} \subset \operatorname{int} H$, $f(c) > 0$ for all $c \in C \setminus \{\theta\}$, hence $f \in C^\sharp$.

Therefore, $f \in C^\sharp \cup \{\theta_{Y^*}\}$. Since $(\theta, z_0) \in Q$, from (12) we get $u(z_0) \geqslant 0$, but $z_0 \in -D$ and $u \in D^*$ so $u(z_0) \leqslant 0$. Thus, $u(z_0) = 0$. Finally, let $x \in \operatorname{dom} C(F, G)(x_0, y_0, z_0)$ and $(y, z) \in C(F, G)(x_0, y_0, z_0)(x)$; since $C(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0) \subset Q$ and $u(z_0) = 0$, from (12), we know,

$$f(y) + u(z) \geqslant 0.$$

This completes the proof. □

Lemma 3.5 (see [14]). *A positive properly efficient pair of* (SVOP) *must be a globally proper efficient pair of* (SVOP).

By applying Lemma 3.5, we can give a sufficient condition involving multiplier functionals for a globally proper efficient solution of (SVOP).

Theorem 3.4. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$. *Suppose that there exist* $z_0 \in G(x_0) \cap -D$, $f \in C^\sharp$, *and* $u \in D^*$ *such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} R(F_A, G_A)(x_0, y_0, z_0)$ *and* $(y, z) \in R(F_A, G_A)(x_0, y_0, z_0)(x)$. *Then* (x_0, y_0) *is a positive properly efficient pair of* (SVOP). *Therefore,* (x_0, y_0) *is a globally proper efficient pair of* (SVOP).

Proof. Suppose the assumption is satisfied. If (x_0, y_0) is not a positive proper efficient pair of (SVOP), then, for any $f \in C^\sharp$, there exist $x' \in A$ and $y' \in F(x')$ such that

$$f(y' - y_0) < 0.$$

Since $x' \in A$, there exists $z' \in G(x') \cap -D$. From Proposition 2.1, we have

$$(y' - y_0, z' - z_0) \in R(F_A, G_A)(x_0, y_0, z_0)(x' - x_0).$$

Thus, $f(y' - y_0) + u(z' - z_0) \geqslant 0$. Therefore,

$$u(z' - z_0) > 0. \tag{13}$$

Since $z' \in G(x') \cap -D$, $u(z_0) = 0$, and $u \in D^*$, $u(z' - z_0) \leqslant 0$, which contradicts (14). Thus, there is $f \in C^\sharp$ such that $f(F(x) - y_0) \geqslant 0$ for all $x \in A$.

Hence, (x_0, y_0) is a positive proper efficient pair of (SVOP). Therefore, (x_0, y_0) is a globally proper efficient pair of (SVOP). $\square$

The following corollary gives a necessary and sufficient condition for a globally proper efficient solution of (SVOP).

Corollary 3.3. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$ *and* $\operatorname{int} D \neq \emptyset$. *Suppose that F is C-convex, G is D-convex, and G satisfies assumption* (C). *Then,* (x_0, y_0) *is a globally proper efficient pair of* (SVOP) *if and only if there exist* $z_0 \in G(x_0) \cap -D$, $f \in C^\sharp$, *and* $u \in D^*$ *such that*

$$u(z_0) = 0 \quad \text{and} \quad f(y) + u(z) \geqslant 0 \tag{14}$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ *and* $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

Proof. Suppose that (x_0, y_0) is a globally proper efficient pair of (SVOP). Thus, $G(x_0) \cap -D \neq \emptyset$. Let $z_0 \in G(x_0) \cap -D$. By Proposition 3.2 and Remark 2.2(c), there exist $f \in C^\sharp \cup \{\theta_{Y^*}\}$ and $u \in D^*$, which satisfy (14) and are not both zero functionals.

If $f = \theta_{Y^*}$, then $u \in D^* \backslash \{\theta_{Z^*}\}$. Since G satisfies assumption (C), there exists $x \in A$ such that $u(G(x)) \cap -\operatorname{int} R_+ \neq \emptyset$. So, there exists $z \in G(x)$ with $u(z) < 0$.

Since $x \in A$, there exists $y \in F(x)$. By Proposition 2.1,

$$(y - y_0, z - z_0) \in D(F, G)(x_0, y_0, z_0)(x - x_0).$$

Hence, $u(z - z_0) \geqslant 0$, that is, $u(z) \geqslant 0$, which is a contradiction. Therefore, $f \neq \theta_{Y^*}$, that is, $f \in C^\sharp$.

Conversely, it follows directly form Theorem 3.4 and Remark 2.2. $\square$

The concept of an f-efficient solution is of great importance in scalarization of vector set-valued optimization (see [24]). The following theorem gives a necessary condition for an f-efficient solution to (SVOP) in a general setting.

Let I be the identical mapping from Z into Z, i.e., $I(z) = z$ for all $z \in Z$.

Theorem 3.5. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$ *and* $\operatorname{int} D \neq \emptyset$. *Suppose that* (x_0, y_0) *is an f-efficient pair of* (SVOP). *Then for any* $z_0 \in G(x_0) \cap -D$,

$$(f, I)\big[Y(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0)\big] \cap -\operatorname{int}(R_+ \times D) = \emptyset \tag{15}$$

for all $x \in \operatorname{dom} Y(F, G)(x_0, y_0, z_0)$.

Proof. If (15) does not hold, then there exist $x \in \operatorname{dom} Y(F, G)(x_0, y_0, z_0)$ and $(y, z) \in Y \times Z$ such that

$$(f, I)(y, z + z_0) \in -\operatorname{int}(R_+ \times D) \quad \text{and} \quad (y, z) \in Y(F, G)(x_0, y_0, z_0)(x),$$

that is, $f(y) < 0$ and $z + z_0 \in -\operatorname{int} D$.

By Definition 2.7, there exist a sequence $\{(x_n, y_n, z_n)\}$ in $\operatorname{epi}(F, G)$ and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} (y_n, z_n) = (y_0, z_0) \quad \text{and} \quad \lim_{n\to\infty} t_n(x_n - x_0, y_n - x_0, z_n - z_0) = (x, y, z).$$

By $f(y) < 0$ and the continuity of f, there exists N_1 such that

$$f\big(t_n(y_n - y_0)\big) < 0 \quad \text{for all } n \in N_1.$$

Because of $t_n > 0$ and the linearity of f, we have

$$f(y_n - y_0) < 0 \quad \text{for all } n \geqslant N_1,$$

that is, $f(y_n) - f(y_0) < 0$ for all $n \geqslant N_1$. Similarly, since D is a cone, $z + z_0 \in -\operatorname{int} D$ and $\lim_{n\to\infty} t_n(z_n - z_0) = z$, there exists N_2 such that

$$t_n(z_n - z_0) + z_0 \in -\operatorname{int} D \quad \text{for all } n \geqslant N_2. \tag{16}$$

Similarly, from the proof of Theorem 3.1, there exits $N \geqslant \max(N_1, N_2)$ such that

$$z_N \in -\operatorname{int} D.$$

Since $(x_n, y_n, z_n) \in \operatorname{epi}(F, G)$ for all $n \in N$, there are $y'_n \in F(x_n)$ with $y_n \in y'_n + C$ and $z'_n \in G(x_n)$ with $z_n \in z'_n + D$. Thus,

$$y'_N \in y_N - C$$

and

$$z'_N \in z_N - D \subset -\operatorname{int} D - D = -\operatorname{int} D.$$

So, $f(y'_N) \leqslant f(y_N) < f(y_0)$. But (x_N, y'_N, z'_N) is a feasible triple. Therefore, (x_0, y_0) is not an f-efficient pair of (SVOP) which is a contradiction. □

The following proposition gives a Fritz John necessary condition for an f-efficient solution to (SVOP).

Proposition 3.3. *Let $(x_0, y_0) \in \operatorname{graph}(F)$, $f \in C^*\backslash\{\theta_{Y^*}\}$, and $\operatorname{int} D \neq \emptyset$. If (x_0, y_0) is an f-efficient pair of* (SVOP), *then, for any $z_0 \in G(x_0) \cap -D$, there exist $\alpha \geqslant 0$ and $u \in D^*$, both dependent on z_0, but not both being zero functionals, such that*

$$u(z_0) = 0 \quad \text{and} \quad \alpha f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} C(F, G)(x_0, y_0, z_0)$ and $(y, z) \in C(F, G)(x_0, y_0, z_0)(x)$.

Proof. Let $z_0 \in G(x_0) \cap -D$ and define

$$Q = \left[\bigcup_{x\in\Omega} C(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0)\right],$$

where $\Omega = \operatorname{dom} C(F, G)(x_0, y_0, z_0)$.

It can be shown that Q is convex.

We can easily show that $Q \cap -\operatorname{int}(R_+ \times D) = \emptyset$, from Theorem 3.5 and Remark 2.2(b). Since the continuity and the linearity of f and I, $(f, I)Q$ is a convex set. So by the separation theorem, there exist $\alpha \in R$ and $u \in Z^*$, not both zero functionals, and a real number ξ such that

$$\alpha f(y) + u(z) \geqslant \xi \quad \text{for all } (y, z) \in Q \tag{17}$$

and

$$\alpha\beta + u(z) < \xi \quad \text{for all } (\beta, z) \in -\operatorname{int}(R_+ \times D). \tag{18}$$

But since $(\beta, z) \in -\operatorname{int}(R_+ \times D)$ can be made arbitrarily close to (θ, θ), the continuity of f and u give from (18) that $\xi \geqslant 0$. z in (18) can be arbitrarily close to θ, by the continuity of u; then

$$\alpha\beta \leqslant \xi \quad \text{for all } \beta \in -\operatorname{int} R_+.$$

Since R_+ is a cone, we have $\alpha \geqslant 0$. Similarly, we can easily get that $u \in D^*$.

Since $(\theta, z_0) \in Q$, from (17) we get $u(z_0) \geqslant 0$. But $z_0 \in -D$ and $u \in D^*$, so $u(z_0) \leqslant 0$. Thus, $u(z_0) = 0$. Finally, let $x \in \operatorname{dom} C(F, G)(x_0, y_0, z_0)$ and $(y, z) \in C(F, G)(x_0, y_0, z_0)(x)$; since $C(F, G)(x_0, y_0, z_0)(x) + (\theta, z_0) \subset Q$ and $u(z_0) = 0$, from (17), we know that

$$\alpha f(y) + u(z) \geqslant 0.$$

This completes the proof. □

The following theorem is a sufficient condition involving multiplier functional for an f-efficient solution of (SVOP).

Theorem 3.6. *Let $f \in C^* \backslash \{\theta_{Y^*}\}$ and $(x_0, y_0) \in \operatorname{graph}(F)$. Suppose that there exist $z_0 \in G(x_0) \cap -D$ and $u \in D^*$ such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} R(F_A, G_A)(x_0, y_0, z_0)$ and $(y, z) \in R(F_A, G_A)(x_0, y_0, z_0)(x)$. Then (x_0, y_0) is an f-efficient pair of (SVOP).

Proof. If (x_0, y_0) is not an f-efficient pair of (SVOP), then there exist $x' \in A$ and $y' \in F(x')$ such that

$$f(y' - y_0) < 0.$$

Since $x' \in A$, there exists $z' \in G(x') \cap -D$. By Proposition 2.1, we have

$$(y' - y_0, z' - z_0) \in R(F_A, G_A)(x_0, y_0, z_0)(x' - x_0).$$

Thus, $f(y' - y_0) + u(z' - z_0) \geqslant 0$. Therefore,

$$u(z' - z_0) > 0. \tag{19}$$

Since $z' \in G(x') \cap -D$, $u(z_0) = 0$, and $u \in D^*$, $u(z' - z_0) \leqslant 0$, which contradicts (19). Hence, (x_0, y_0) is an f-efficient pair of (SVOP). □

The following corollary is a necessary and sufficient condition for an f-efficient solution to (SVOP) when G satisfies assumption (C).

Corollary 3.4. *Let $f \in C^*\backslash\{\theta_{Y^*}\}$, $(x_0, y_0) \in \operatorname{graph}(F)$ and $\operatorname{int} D \neq \emptyset$. Suppose that F is C-convex, G is D-convex and G satisfies assumption* (C). *Then (x_0, y_0) is an f-efficient pair of* (SVOP) *if and only if there exist $z_0 \in G(x_0) \cap -D$ and $u \in D^*$ such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ and $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

Proof. Suppose that (x_0, y_0) is an f-efficient pair of (SVOP). Thus, $G(x_0) \cap -D \neq \emptyset$. Let $z_0 \in G(x_0) \cap -D$. By Proposition 3.3 and Remark 2.2(c), there exist $\alpha \geqslant 0$ and $u' \in D^*$, not both zero functionals, such that

$$u'(z_0) = 0 \quad \text{and} \quad \alpha f(y) + u'(z) \geqslant 0 \tag{20}$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ and $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

Assume that $\alpha = 0$. Then $u' \in D^*\backslash\{\theta_{Z^*}\}$. Since G satisfies assumption (C), there exists $x' \in A$ such that $u'(G(x')) \cap -\operatorname{int} R_+ \neq \emptyset$, that is, there exists $z' \in G(x')$ with

$$u'(z') < 0. \tag{21}$$

Since $x' \in A$, there exists $y' \in F(x')$. By Proposition 2.1,

$$(y' - y_0, z' - z_0) \in D(F, G)(x_0, y_0, z_0)(x' - x_0).$$

So, $u'(z' - z_0) \geqslant 0$, that is, $u'(z') \geqslant 0$, which contradicts (21). Hence, $\alpha \neq 0$, that is, $\alpha > 0$. Therefore, dividing α in two sides of (20), we have

$$f(y) + u'(z) \geqslant 0 \quad \text{and} \quad \frac{u'(z_0)}{\alpha} = 0.$$

Let $u = u'/\alpha$. We have

$$u(z_0) = 0 \quad \text{and} \quad f(y) + u(z) \geqslant 0$$

for all $\in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ and $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

Conversely, it follows directly form Theorem 3.6 and Remark 2.2. □

To obtain necessary and sufficient conditions for a strongly efficient solution to (SVOP), we give the following lemma.

Lemma 3.6. *Let $(x_0, y_0) \in \operatorname{graph}(F)$. Then (x_0, y_0) is a strongly efficient pair of* (SVOP) *if and only if (x_0, y_0) is an f-efficient pair of* (SVOP) *for any $f \in C^*\backslash\{\theta_{Y^*}\}$.*

Proof. Suppose that (x_0, y_0) is a strong efficient pair of (SVOP), i.e.,

$$F(A) - y_0 \subset C.$$

For any $f \in C^* \setminus \{\theta_{Y^*}\}$, we have (see [15])

$$f\big(F(x) - y_0\big) \geqslant 0 \quad \text{for any } x \in A.$$

That is, (x_0, y_0) is an f-efficient pair of (SVOP). Conversely, for any $f \in C^* \setminus \{\theta_{Y^*}\}$, (x_0, y_0) is an f-efficient pair of (SVOP), i.e.,

$$f\big(F(x) - y_0\big) \geqslant 0 \quad \text{for any } x \in A.$$

From the duality of C, we have

$$F(x) - y_0 \subset C \quad \text{for any } x \in A.$$

Hence, (x_0, y_0) is a strong efficient pair of (SVOP). $\square$

As a direct consequence of Corollary 3.4 and Lemma 3.6, we can easily get the following result.

Corollary 3.5. *Let* $(x_0, y_0) \in \operatorname{graph}(F)$ *and* $\operatorname{int} D \neq \emptyset$. *Suppose that* F *is* C*-convex,* G *is* D*-convex, and* G *satisfies assumption* (C). *Then* (x_0, y_0) *is a strongly efficient pair of* (SVOP) *if and only if for any* $f \in C^* \backslash \{\theta_{Y^*}\}$, *there exist* $z_0 \in G(x_0) \cap -D$ *and* $u \in D^*$ *such that*

$$u(z_0) = 0 \quad \textit{and} \quad f(y) + u(z) \geqslant 0$$

for all $x \in \operatorname{dom} D(F, G)(x_0, y_0, z_0)$ *and* $(y, z) \in D(F, G)(x_0, y_0, z_0)(x)$.

It should be mentioned that each type of properly efficient solutions to a vector set-valued optimization problem with constraints can be characterized by the corresponding positive functional. With the approach used in this paper, we can derive several other optimality conditions for constrained vector set-valued optimization.

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