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Stabilized plethysms for the classical Lie groups

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ABSTRACT

The plethysms of the Weyl characters associated to a classical Lie group by the symmetric functions stabilize in large rank. In the case of a power sum plethysm, we prove that the coefficients of the decomposition of this stabilized form on the basis of Weyl characters are branching coefficients which can be determined by a simple algorithm. This generalizes in particular some classical results by Littlewood on the power sum plethysms of Schur functions. We also establish explicit formulas for the outer multiplicities appearing in the decomposition of the tensor square of any irreducible finite-dimensional module into its symmetric and antisymmetric parts. These multiplicities can notably be expressed in terms of the Littlewood–Richardson coefficients.

1. Introduction

This paper is concerned with the plethysms of the Weyl characters associated to complex classical Lie groups $GL_n(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ or $SO_{2n}(\mathbb{C})$ by the symmetric functions. Let G be one of the previous complex classical Lie groups. We write $\mathfrak{g}$ for the Lie algebra of G. Since we consider only Lie groups and Lie algebras over $\mathbb{C}$ in the sequel, we drop the symbol $\mathbb{C}$ in G and $\mathfrak{g}$ and simply write $G = GL_n, SO_{2n+1}, Sp_{2n}, SO_{2n}$ and $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$. Given λ a partition, we denote by $s_\lambda^{\mathfrak{g}}$ the Weyl character of the $\mathfrak{g}$-module $V^{\mathfrak{g}}(\lambda)$. The modules $V^{\mathfrak{g}}(\lambda)$ almost coincide with the finite-dimensional $\mathfrak{g}$-modules but there are some exceptions (see Section 2). Consider f a symmetric function of degree d and suppose $n \geqslant dl(\lambda)$ where $l(\lambda)$ is the number of non-zero parts of λ. Write $\mathcal{P}_n$ for the set of partitions with at most n parts. It follows from results by Littlewood [9] that the plethysm $f \circ s_\lambda^{\mathfrak{g}}$ of the Weyl character $s_\lambda^{\mathfrak{g}}$ by f decomposes on the basis $\{s_\mu^{\mathfrak{g}} \mid \mu \in \mathcal{P}_n\}$ with coefficients which do not depend on n. When $f = p_\ell$ is the power sum of degree ℓ, we establish that

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the coefficients so obtained are branching coefficients corresponding to the restriction to certain Levi subgroups (Theorem 4.5.1). Suppose $n \geqslant \ell l(\lambda)$ and set

$$p_\ell \circ s_\lambda^{\mathfrak{g}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{g}} s_\mu^{\mathfrak{g}}.$$

For $\mathfrak{g} = \mathfrak{gl}_n$, it is well known, by an algorithm due to Littlewood [7], that the coefficients $a_{\lambda,\mu}^{\mathfrak{gl}_n,\ell}$ can, up to a sign, be expressed as a sum of products of Littlewood–Richardson coefficients. They are then obtained from the ℓ-quotient of the partition μ. We give a similar algorithm for computing the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{g}}$ when $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$. This algorithm was originally introduced in [6] to decompose the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ on the basis of Weyl characters for any integers $n \geqslant 2$ and $\ell \geqslant 1$ (that is, with no restrictive conditions on the rank n). Although similar procedures also exist for $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$ when ℓ is odd, our method failed for the even power sum plethysms on the Weyl characters of type C_n or D_n. In the present paper, we show that this difficulty can be overcome by considering stabilized power sum plethysms, i.e. by assuming that $n \geqslant \ell l(\lambda)$. Under this hypothesis, one has indeed $a_{\lambda,\mu}^{\ell,\mathfrak{so}_{2n+1}} = a_{\lambda,\mu}^{\ell,\mathfrak{so}_{2n}}$ and $a_{\lambda,\mu}^{\ell,\mathfrak{sp}_{2n}} = (-1)^{|\lambda|}(-1)^{\ell-1} a_{\lambda',\mu'}^{\ell,\mathfrak{so}_{2n+1}}$. So it suffices to consider the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}_{2n+1}}$ for which there exists an algorithm in both cases ℓ even and ℓ odd. As a consequence, we obtain that the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{g}}$ can be expressed as branching coefficients.

In Proposition 5.2.1, we use our expression of the coefficients $a_{\lambda,\mu}^{2,\mathfrak{g}}$ as branching coefficients, to derive explicit formulas giving the decompositions of the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ in their irreducible components when $n \geqslant 2l(\lambda)$. The corresponding multiplicities can then be expressed in terms of the Littlewood–Richardson coefficients and give an alternative to analogous formulas introduced without a complete proof by Littlewood in [9].

The paper is organized as follows. In Section 2, we recall some basics on the representation theory of the classical Lie groups. Section 3 is concerned with plethysms $f \circ s_\lambda^{\mathfrak{g}}$ and their stabilization in large rank. Most of the material of this section can be found in [7–10]. In Section 4, we describe the algorithm of [6] which permits to compute the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ for any positive integer ℓ. We then state Theorem 4.5.1 which gives the promised expression of the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{g}}$ as branching coefficients corresponding to the restrictions to Levi subgroups. Finally, in Section 5, we express the multiplicities $a_{\lambda,\mu}^{\mathfrak{g},2}$ in terms of the Littlewood–Richardson coefficients.

2. Background on classical Lie groups

2.1. Root systems and Weyl groups

In the sequel G is one of the complex Lie groups Sp_{2n}, SO_{2n+1} or SO_{2n} and $\mathfrak{g}$ is its Lie algebra. We follow the convention of [5] to realize G as a subgroup of GL_N and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ where

$$N = \begin{cases} n & \text{when } G = GL_n, \\ 2n & \text{when } G = Sp_{2n}, \\ 2n+1 & \text{when } G = SO_{2n+1}, \\ 2n & \text{when } G = SO_{2n}. \end{cases}$$

Let d_N be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in I_n = \{1, \dots, n\}$, write ε_i for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_{n-i+1}$ for any diagonal matrix D whose (i,i)-coefficient is δ_i. Then $(\varepsilon_1, \dots, \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$ (the real part of $\mathfrak{h}^*$). Let $(\cdot,\cdot)$ be the corresponding nondegenerate symmetric bilinear form defined on $\mathfrak{h}_{\mathbb{R}}^*$. Write R for the root system associated to G. For any $\alpha \in R$ we set $\alpha^\vee = \frac{\alpha}{(\alpha,\alpha)}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in R} \mathfrak{g}_\alpha$. We take for the set of positive roots:

$$\begin{cases} R^+ = \{\varepsilon_j - \varepsilon_i \text{ with } 1 \leqslant i < j \leqslant n\} & \text{for the root system } A_{n-1}, \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_i \text{ with } 1 \leqslant i \leqslant n\} & \text{for the root system } B_n, \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leqslant i < j \leqslant n\} \cup \{2\varepsilon_i \text{ with } 1 \leqslant i \leqslant n\} & \text{for the root system } C_n, \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leqslant i < j \leqslant n\} & \text{for the root system } D_n. \end{cases}$$

For any $i \in I_n$, we write $\bar{\bar{i}} = i$ and $|i| = |\bar{i}| = i$. The Weyl group W of the Lie group G is the subgroup of the permutation group of the set $J_n = \{\bar{n}, \ldots, \bar{2}, \bar{1}, 1, 2, \ldots, n\}$ generated by the permutations

$$\begin{cases} s_i = (i, i+1)(\bar{i}, \overline{i+1}), & i = 1, \ldots, n-1 \text{ and } s_0 = (1, \bar{1}) \text{ for the root systems } B_n \text{ and } C_n, \\ s_i = (i, i+1)(\bar{i}, \overline{i+1}), & i = 1, \ldots, n-1 \text{ and } s_0' = (1, \bar{2})(2, \bar{1}) \text{ for the root system } D_n \end{cases}$$

where for $a \neq b$, (a, b) is the simple transposition which switches a and b. We identify the subgroup of W generated by $s_i = (i, i+1)(\bar{i}, \overline{i+1})$, $i = 1, \ldots, n-1$, with the symmetric group S_n. We denote by l the length function corresponding to the above set of generators. For any $w \in W$, we set $\varepsilon(w) = (-1)^{l(w)}$. The action of $w \in W$ on $\beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$w \cdot (\beta_1, \ldots, \beta_n) = \left(\beta_1^{w^{-1}}, \ldots, \beta_n^{w^{-1}}\right)$$

where $\beta_i^w = \beta_{w(i)}$ if $w(i) \in \{1, \ldots, n\}$ and $\beta_i^w = -\beta_{w(\bar{i})}$ otherwise. We denote by ρ the half sum of the positive roots of R^+. For any $x \in J_n$, we set $\bar{\bar{x}} = x$ and $|x| = x$ if x is unbarred, $|x| = \bar{x}$ otherwise.

A partition of length m is a finite weakly *increasing* sequence of nonnegative integers. The terms of the sequence are called the parts of the partition. Customarily, we identify two such sequences if they only differ in the number of zero occurring as parts. However, there are situations where we distinguish such sequences (see Proposition 3.2.2 and Section 4.4). We hope the context makes clear what we mean. The *length* of the partition λ is the number of non-zero parts in λ and is denoted by $l(\lambda)$. Denote by $\mathcal{P}_m$ the set of partitions with length at most m. By way of the identification mentioned above, we have $\mathcal{P}_m \subset \mathcal{P}_n$ if $m \leqslant n$; set $\mathcal{P} = \bigcup_{m \geqslant 0} \mathcal{P}_m$. For $\lambda \in \mathcal{P}$, write λ' for the conjugate partition of λ.

For $G = Sp_{2n}$ or SO_{2n+1} and $\lambda \in \mathcal{P}_n$, denote by $V^{\mathfrak{g}}(\lambda)$ the irreducible finite-dimensional representation of G of highest weight λ. For $G = SO_{2n}$, we define $V^{\mathfrak{so}_{2n}}(\lambda)$ similarly when $\lambda_1 = 0$ and we write $V^{\mathfrak{so}_{2n}}(\lambda)$ for the direct sum of the two irreducible representations of highest weights $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\bar{\lambda} = (-\lambda_1, \lambda_2, \ldots, \lambda_n)$ when $\lambda_1 \neq 0$. This means that $V^{\mathfrak{so}_{2n}}(\lambda)$ is in fact the irreducible representation of O_{2n} associated to the partition λ restricted to SO_{2n}.

We shall also need the irreducible rational representations of GL_n. They are indexed by the n-tuples

$$(\gamma^-, \gamma^+) = \left(-\gamma_q^-, \ldots, -\gamma_1^-, \gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+\right) \tag{1}$$

where $\gamma^+ = (\gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+)$ and $\gamma^- = (\gamma_1^-, \ldots, \gamma_q^-)$ are partitions of length at most p and q, respectively and such that $p + q = n$. Write $\widetilde{\mathcal{P}}_n$ for the set of such n-tuples and denote also by $V^{\mathfrak{gl}_n}(\gamma)$ the irreducible rational representation of $\mathfrak{gl}_n$ of highest weight $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$. For any $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$, we set $|\gamma| = \sum \gamma_i^- + \sum \gamma_i^+$.

Write $s_\lambda^{\mathfrak{gl}_n}$ for the Weyl character (Schur function) of the finite-dimensional $\mathfrak{gl}_n$-module $V^{\mathfrak{gl}_n}(\lambda)$ of highest weight $\lambda \in \mathcal{P}_n$. The character ring of the polynomial representations of GL_n is $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{sym}}$ the ring of symmetric functions in n variables. Here, each variable x_i, $i = 1, \ldots, n$, can be identified with the formal exponential e^{ε_i}.

For any $\lambda \in \mathcal{P}_n$, we denote by $s_\lambda^{\mathfrak{g}}$ the Weyl character of $V^{\mathfrak{g}}(\lambda)$. Let $\mathcal{R}^{\mathfrak{g}}$ be the $\mathbb{Z}$-algebra with basis $\{s_\lambda^{\mathfrak{g}} \mid \lambda \in \mathcal{P}_n\}$. For $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, we have $s_\lambda^{\mathfrak{g}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathcal{H}_n}$ where $\mathcal{H}_n$ is the Weyl group of type B_n acting on the Laurent polynomials as the permutation of variables and exchanging x_i in x_i^{-1}. These Weyl characters form a basis of $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathcal{H}_n}$, thus $\mathcal{R}^{\mathfrak{g}} \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathcal{H}_n}$.

Consider P a parabolic subgroup of G and L its Levi subgroup. Write $\mathfrak{l}$ for the Levi algebra associated to L. We denote by P_L^+ the set of dominant weights corresponding to L. For any partition $\lambda \in \mathcal{P}_n$ and $\gamma \in P_L^+$, write $[V^{\mathfrak{g}}(\lambda) : V^{\mathfrak{l}}(\gamma)]$ for the branching coefficient giving the multiplicity of $V^{\mathfrak{l}}(\gamma)$ (the irreducible representation of L of highest weight γ) in the restriction of $V^{\mathfrak{g}}(\lambda)$ to L.

2.2. Universal characters

For each Lie algebra $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$ and any partition $\nu \in \mathcal{P}_N$, we denote by $V^{\mathfrak{gl}_N}(\nu)\downarrow_{\mathfrak{g}}^{\mathfrak{gl}_N}$ the restriction of $V^{\mathfrak{gl}_N}(\nu)$ to $\mathfrak{g}$. Set

$$V^{\mathfrak{gl}_N}(\nu)\downarrow_{\mathfrak{so}_N}^{\mathfrak{gl}_N} = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{so}_N}(\lambda)^{\oplus b_{\nu,\lambda}^{\mathfrak{so}_N}} \quad \text{and} \quad V^{\mathfrak{gl}_{2n}}(\nu)\downarrow_{\mathfrak{sp}_{2n}}^{\mathfrak{gl}_{2n}} = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{sp}_{2n}}(\lambda)^{\oplus b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}}.$$

This makes in particular appear the branching coefficients $b_{\nu,\lambda}^{\mathfrak{so}_N}$ and $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}$. The restriction map $r^{\mathfrak{g}}$ is defined by setting

$$r^{\mathfrak{g}}:\begin{cases}\mathbb{Z}[x_1,\dots,x_N]^{\mathrm{sym}}\to\mathcal{R}^{\mathfrak{g}},\\ s_\nu^{\mathfrak{gl}_N}\mapsto \mathrm{char}(V^{\mathfrak{gl}_N}(\nu)\downarrow_{\mathfrak{g}}^{\mathfrak{gl}_N}).\end{cases}$$

We have then

$$r^{\mathfrak{g}}\big(s_\nu^{\mathfrak{gl}_N}\big)=\begin{cases}s_\nu^{\mathfrak{gl}_N}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}) & \text{when } N=2n,\\ s_\nu^{\mathfrak{gl}_N}(x_1,\dots,x_n,1,x_n^{-1},\dots,x_1^{-1}) & \text{when } N=2n+1.\end{cases}$$

Let $\mathcal{P}_n^{(2)}$ and $\mathcal{P}_n^{(1,1)}$ be the subsets of $\mathcal{P}_n$ containing the partitions with even length rows and the partitions with even length columns, respectively. When $\nu\in\mathcal{P}_n$ we have the following formulas for the branching coefficients $b_{\nu,\lambda}^{\mathfrak{so}_N}$ and $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}$:

Proposition 2.2.1. *(See [8, Appendix, p. 295].) Consider $\lambda,\nu\in\mathcal{P}_n$. Then*:

1. $b_{\nu,\lambda}^{\mathfrak{so}_{2n+1}}=b_{\nu,\lambda}^{\mathfrak{so}_{2n}}=\sum_{\gamma\in\mathcal{P}_n^{(2)}}c_{\lambda,\gamma}^{\nu}$,
2. $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}=\sum_{\gamma\in\mathcal{P}_n^{(1,1)}}c_{\lambda,\gamma}^{\nu}$,

where $c_{\gamma,\lambda}^{\nu}$ is the n-independent multiplicity of $s_\nu^{\mathfrak{gl}_n}$ in the Schur functions product $s_\lambda^{\mathfrak{gl}_n}s_\gamma^{\mathfrak{gl}_n}$.

Remarks. (i) Note that the equality $b_{\nu,\lambda}^{\mathfrak{so}_{2n+1}}=b_{\nu,\lambda}^{\mathfrak{so}_{2n}}$ becomes false in general when $\nu\notin\mathcal{P}_n$.

(ii) By the above proposition we have for any $\nu\in\mathcal{P}_m$ with $m\leqslant n$

$$r^{\mathfrak{sp}_{2n}}\big(s_\nu^{\mathfrak{gl}_{2n}}\big)=\sum_{\lambda\in\mathcal{P}_m}\sum_{\gamma\in\mathcal{P}_m^{(1,1)}}c_{\lambda,\gamma}^{\nu}s_\lambda^{\mathfrak{sp}_{2n}}\quad\text{and}\quad r^{\mathfrak{so}_N}\big(s_\nu^{\mathfrak{gl}_N}\big)=\sum_{\lambda\in\mathcal{P}_m}\sum_{\gamma\in\mathcal{P}_m^{(2)}}c_{\lambda,\gamma}^{\nu}s_\lambda^{\mathfrak{so}_N}. \tag{2}$$

By Proposition 1.5.3 in [4], one has also for any $\lambda\in\mathcal{P}_m$

$$\begin{aligned} s_\lambda^{\mathfrak{sp}_{2n}}&=\sum_{\nu\in\mathcal{P}_m,\,\nu\subset\lambda,\,|\nu|\equiv|\lambda| \pmod 2}(-1)^{\frac{|\nu|-|\lambda|}{2}}\sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)}c_{\nu,\Gamma(\alpha)}^{\lambda}r^{\mathfrak{sp}_{2n}}\big(s_\nu^{\mathfrak{gl}_{2n}}\big),\\ s_\lambda^{\mathfrak{so}_N}&=\sum_{\nu\in\mathcal{P}_m,\,\nu\subset\lambda,\,|\nu|\equiv|\lambda| \pmod 2}(-1)^{\frac{|\nu|-|\lambda|}{2}}\sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)}c_{\nu,\Gamma'(\alpha)}^{\lambda}r^{\mathfrak{so}_N}\big(s_\nu^{\mathfrak{gl}_N}\big)\end{aligned} \tag{3}$$

where $\Gamma(\alpha)=(\alpha_1-1,\dots,\alpha_s-1\mid\alpha_1,\dots,\alpha_s)$ in the Frobenius notation for the partitions. Observe that the coefficients appearing in the decompositions (2) and (3) do not depend on the rank n considered. Moreover they coincide for the orthogonal types B_n and D_n.

As suggested by the above decompositions, the manipulation of the Weyl characters is simplified by working with infinitely many variables. In [4], Koike and Terada have introduced a universal character ring for the classical Lie groups. This ring is the ring $\Lambda=\mathbb{Z}[x_1,\dots,x_n,\dots]^{\mathrm{sym}}$ of symmetric functions in countably many variables. It is equipped with three natural $\mathbb{Z}$-bases indexed by partitions, namely

$$\mathcal{B}^{\mathfrak{gl}}=\big\{\mathrm{s}_\lambda^{\mathfrak{gl}}\bigm|\lambda\in\mathcal{P}\big\},\qquad\mathcal{B}^{\mathfrak{sp}}=\big\{\mathrm{s}_\lambda^{\mathfrak{sp}}\bigm|\lambda\in\mathcal{P}\big\}\quad\text{and}\quad\mathcal{B}^{\mathfrak{so}}=\big\{\mathrm{s}_\lambda^{\mathfrak{so}}\bigm|\lambda\in\mathcal{P}\big\}. \tag{4}$$

We have then

$$\mathrm{s}_\nu^{\mathfrak{gl}}=\sum_{\lambda\in\mathcal{P}}\sum_{\gamma\in\mathcal{P}^{(2)}}c_{\lambda,\gamma}^{\nu}\mathrm{s}_\lambda^{\mathfrak{so}}\quad\text{and}\quad\mathrm{s}_\nu^{\mathfrak{gl}}=\sum_{\lambda\in\mathcal{P}}\sum_{\gamma\in\mathcal{P}^{(1,1)}}c_{\lambda,\gamma}^{\nu}\mathrm{s}_\lambda^{\mathfrak{sp}}, \tag{5}$$

$$\mathrm{s}_\lambda^{\mathfrak{sp}}=\sum_{\nu\in\mathcal{P},\,\nu\subset\lambda,\,|\nu|\equiv|\lambda| \pmod 2}(-1)^{\frac{|\nu|-|\lambda|}{2}}\sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)}c_{\nu,\Gamma(\alpha)}^{\lambda}\mathrm{s}_\nu^{\mathfrak{gl}}, \tag{6}$$

$$\mathrm{s}_\lambda^{\mathfrak{so}}=\sum_{\nu\in\mathcal{P},\,\nu\subset\lambda,\,|\nu|\equiv|\lambda| \pmod 2}(-1)^{\frac{|\nu|-|\lambda|}{2}}\sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)}c_{\nu,\Gamma'(\alpha)}^{\lambda}\mathrm{s}_\nu^{\mathfrak{gl}}. \tag{7}$$

In the sequel we will write for short

$$b^{\mathfrak{so}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}^{(2)}} c^{\nu}_{\lambda,\gamma}, \qquad b^{\mathfrak{sp}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}^{(1,1)}} c^{\nu}_{\lambda,\gamma}, \qquad r^{\mathfrak{so}}_{\lambda,\nu} = \sum_{\alpha} c^{\lambda}_{\nu,\Gamma'(\alpha)} \quad \text{and} \quad r^{\mathfrak{sp}}_{\lambda,\nu} = \sum_{\alpha} c^{\lambda}_{\nu,\Gamma(\alpha)}. \tag{8}$$

We denote by ω the linear involution defined on Λ by $\omega(\mathrm{s}^{\mathfrak{gl}}_{\lambda}) = \mathrm{s}^{\mathfrak{gl}}_{\lambda'}$. Then we have by Theorem 2.3.2 of [4]

$$\omega\big(\mathrm{s}^{\mathfrak{so}}_{\lambda}\big) = \mathrm{s}^{\mathfrak{sp}}_{\lambda'}. \tag{9}$$

Write $\pi_n : \mathbb{Z}[x_1,\ldots,x_n,\ldots]^{\mathrm{sym}} \to \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{sym}}$ for the ring homomorphism obtained by specializing each variable x_i, $i > n$, at 0. Then $\pi_n(\mathrm{s}^{\mathfrak{gl}}_{\lambda}) = s^{\mathfrak{gl}_n}_{\lambda}$ if $\lambda \in \mathcal{P}_n$. Let $\pi^{\mathfrak{sp}_{2n}}$ and $\pi^{\mathfrak{so}_N}$ be the specialization homomorphisms defined by setting $\pi^{\mathfrak{sp}_{2n}} = r^{\mathfrak{sp}_{2n}} \circ \pi_{2n}$ and $\pi^{\mathfrak{so}_N} = r^{\mathfrak{so}_N} \circ \pi_N$. For any partition $\lambda \in \mathcal{P}_n$ one has $s^{\mathfrak{sp}_{2n}}_{\lambda} = \pi^{\mathfrak{sp}_{2n}}(\mathrm{s}^{\mathfrak{sp}}_{\lambda})$ and $s^{\mathfrak{so}_N}_{\lambda} = \pi^{\mathfrak{so}_N}(\mathrm{s}^{\mathfrak{so}}_{\lambda})$. We shall also need the following proposition (see [3] and [4]).

Proposition 2.2.2. *Consider a Lie algebra $\mathfrak{g}$ of type $X_n \in \{B_n, C_n, D_n\}$. Let $\lambda \in \mathcal{P}_r$ and $\mu \in \mathcal{P}_s$. Suppose $n \geqslant r + s$ and set*

$$V^{\mathfrak{g}}(\lambda) \otimes V^{\mathfrak{g}}(\mu) = \bigoplus_{\nu\in\mathcal{P}_n} V^{\mathfrak{g}}(\nu)^{\oplus d^{\nu}_{\lambda,\mu}}.$$

Then the coefficients $d^{\nu}_{\lambda,\mu}$ neither depend on the rank n of $\mathfrak{g}$ nor on its type B, C or D. Moreover we have

$$d^{\nu}_{\lambda,\mu} = \sum_{\xi,\sigma,\tau} c^{\lambda}_{\xi,\sigma} c^{\mu}_{\xi,\tau} c^{\nu}_{\sigma,\tau}.$$

Remarks. (i) The previous proposition implies the decompositions $\mathrm{s}^{\mathfrak{sp}}_{\lambda} \times \mathrm{s}^{\mathfrak{sp}}_{\mu} = \sum_{\nu\in\mathcal{P}} d^{\nu}_{\lambda,\mu} \mathrm{s}^{\mathfrak{sp}}_{\nu}$ and $\mathrm{s}^{\mathfrak{so}}_{\lambda} \times \mathrm{s}^{\mathfrak{so}}_{\mu} = \sum_{\nu\in\mathcal{P}} d^{\nu}_{\lambda,\mu} \mathrm{s}^{\mathfrak{so}}_{\nu}$ for any $\lambda, \mu \in \mathcal{P}$, in the ring Λ.

(ii) The analogous result for $\mathfrak{g} = \mathfrak{gl}_n$ is well known: the outer multiplicities $c^{\nu}_{\lambda,\mu}$ appearing in the decomposition of $V^{\mathfrak{gl}_n}(\lambda) \otimes V^{\mathfrak{gl}_n}(\mu)$ do not depend on n provided $n \geqslant r + s$.

3. Plethysms and stabilized plethysms

3.1. Plethysms on the Weyl characters

Consider $f \in \Lambda$ and $s^{\mathfrak{g}}_{\lambda}$ the Weyl character for $\mathfrak{g}$ associated to $\lambda \in \mathcal{P}_n$. Set $s^{\mathfrak{g}}_{\lambda} = \sum_{\beta\in\mathbb{Z}^n} a_{\beta} x^{\beta}$. As in the case of ordinary plethysms on symmetric functions (see [10, p. 135]), one defines the set of variables y_i such that

$$\prod_i (1 + t y_i) = \prod_{\beta} \big(1 + t x^{\beta}\big)^{a_{\beta}}.$$

Then the plethysm of the Weyl character $s^{\mathfrak{g}}_{\lambda}$ by the symmetric function f is defined by $f \circ s^{\mathfrak{g}}_{\lambda} = f(y_1, y_2, \ldots)$. In the sequel, we will focus on the power sum plethysms ψ_{ℓ} where ℓ is a positive integer. They are defined from the identity $\psi_{\ell}(s^{\mathfrak{g}}_{\lambda}) = p_{\ell} \circ s^{\mathfrak{g}}_{\lambda} = s^{\mathfrak{g}}_{\lambda}(x^{\ell}_1, \ldots, x^{\ell}_n)$. In particular, the map ψ_{ℓ} is linear on $\mathcal{R}^{\mathfrak{g}}$. The characters of the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ can be expressed as plethysms by the complete and elementary symmetric functions h_2 and e_2. More precisely we have

$$h_2 \circ s^{\mathfrak{g}}_{\lambda} = \mathrm{char}\big(S^2\big(V^{\mathfrak{g}}(\lambda)\big)\big) \quad \text{and} \quad e_2 \circ s^{\mathfrak{g}}_{\lambda} = \mathrm{char}\big(\Lambda^2\big(V^{\mathfrak{g}}(\lambda)\big)\big).$$

From the identities $h_2 = \frac{1}{2}(p_1^2 + p_2)$ and $e_2 = \frac{1}{2}(p_1^2 - p_2)$, we derive the relations

$$h_2 \circ s^{\mathfrak{g}}_{\lambda} = \frac{1}{2}\big(\big(s^{\mathfrak{g}}_{\lambda}\big)^2 + \psi_2\big(s^{\mathfrak{g}}_{\lambda}\big)\big) \quad \text{and} \quad e_2 \circ s^{\mathfrak{g}}_{\lambda} = \frac{1}{2}\big(\big(s^{\mathfrak{g}}_{\lambda}\big)^2 - \psi_2\big(s^{\mathfrak{g}}_{\lambda}\big)\big). \tag{10}$$

3.2. Stabilized plethysms on the Schur functions

Given $(\mu^{(0)},\dots,\mu^{(\ell-1)})$ an ℓ-tuple of partitions, we write $c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}$ for coefficient of $s_{\lambda}^{\mathfrak{gl}_n}$ in the product $s_{\mu^{(0)}}^{\mathfrak{gl}_n}\cdots s_{\mu^{(\ell-1)}}^{\mathfrak{gl}_n}$ which is independent of n if n is sufficiently large. For any partition $\lambda\in\mathcal{P}_n$, the plethysm $\psi_\ell(s_{\lambda}^{\mathfrak{gl}_n})$ decomposes on the basis of Schur functions on the form

$$\psi_\ell\big(s_{\lambda}^{\mathfrak{gl}_n}\big)=\sum_{|\mu|=\ell|\lambda|}\varepsilon(\mu)c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}s_{\mu}^{\mathfrak{gl}_n}. \tag{11}$$

Here $\varepsilon(\mu)\in\{-1,0,1\}$ and $\mu/\ell=(\mu^{(0)},\dots,\mu^{(\ell-1)})$ are respectively the ℓ-sign and the ℓ-quotient of the partition μ. We now briefly recall the algorithm which permits to obtain the sign $\varepsilon(\mu)$ and the ℓ-tuple of partitions μ/ℓ. Our description slightly differs from that which can be usually found in the literature (see [10, Example 8, p. 12]). This is because we have made our notation consistent with Section 4.

We now regard μ as an n-terms sequence by supplying an appropriate number of zeroes. Set $\rho_n=(1,2,\dots,n)$ and $I_n=\{1,2,\dots,n\}$. For any $k\in\{0,\dots,\ell-1\}$ consider the ordered sequences in the increasing order

$$I^{(k)}=(i\in I_n\mid \mu_i+i\equiv k \bmod \ell)\quad\text{and}\quad J^{(k)}=(i\in I_n\mid i\equiv k \bmod \ell).$$

Set $r_k=\operatorname{card}(I^{(k)})$ and write $I^{(k)}=(i_1^{(k)},\dots,i_{r_k}^{(k)})$.

1. If there exists $k\in\{0,\dots,\ell-1\}$ such that $\operatorname{card}(I^{(k)})\neq\operatorname{card}(J^{(k)})$ (i.e. the ℓ-core of μ is nontrivial) then $\varepsilon(\mu)=0$.
2. Otherwise (i.e. the ℓ-core of μ is trivial), let $\sigma_0\in S_n$ be the permutation mapping $I^{(k)}$ to $J^{(k)}$ for any $k=0,\dots,\ell-1$. Then we have $\varepsilon(\mu)=\varepsilon(\sigma_0)$ and $\mu/\ell=(\mu^{(0)},\dots,\mu^{(\ell-1)})$ where

$$\mu^{(0)}=\left(\frac{\mu_i+i}{\ell}\;\Big|\; i\in I^{(0)}\right)-(1,2,\dots,r_0)\in\mathbb{Z}^{r_0}$$

and for any $k\in\{1,\dots,\ell-1\}$

$$\mu^{(k)}=\left(\frac{\mu_i+i+\ell-k}{\ell}\;\Big|\; i\in I^{(k)}\right)-(1,2,\dots,r_k)\in\mathbb{Z}^{r_k}. \tag{12}$$

Remarks. (i) Even in the case where μ has a nontrivial ℓ-core κ, the ℓ-quotient $\mu/\ell=(\mu^{(0)},\dots,\mu^{(\ell-1)})$ of μ can also be obtained by the formula in step 2. However, in this case, one has $|\mu|-\ell\sum_{k=0}^{\ell-1}|\mu^{(k)}|=|\kappa|>0$, so that, under the condition $|\mu|=\ell|\lambda|$ for the summation in (11), one also has $c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}=0$.

(ii) Set $n=q\ell+r$ where q and r are respectively the quotient and the remainder of the division of n by ℓ. Then we have $\operatorname{card}(J^{(k)})=q+1$ for any $k\in\{1,\dots,r\}$ and $\operatorname{card}(J^{(k)})=q$ for any $k\in\{0,r+1,\dots,\ell-1\}$. Hence in (12), we have $r_k=q+1$ for any $k\in\{1,\dots,r\}$ and $r_k=q$ for any $k\in\{0,r+1,\dots,\ell-1\}$.

Example 3.2.1. Consider $\mu=(1,2,3,4,4,4,6,6)$ and take $\ell=3$. We have $\mu+\rho_8=(2,4,6,8,9,10,13,14)$. Thus $I^{(0)}=(3,5)$, $I^{(1)}=(2,6,7)$, $I^{(2)}=(1,4,8)$ and $J^{(0)}=(3,6)$, $J^{(1)}=(1,4,7)$, $J^{(2)}=(2,5,8)$. Then $\mu^{(0)}=(1,1)$, $\mu^{(1)}=(1,2,2)$ and $\mu^{(2)}=(0,1,2)$. Moreover

$$\sigma_0=\begin{pmatrix}1&2&3&4&5&6&7&8\\2&1&3&5&6&4&7&8\end{pmatrix}.$$

Hence $\varepsilon(\mu)=-1$.

Proposition 3.2.2. *Consider $\mu\in\mathcal{P}_n$ such that $\varepsilon(\mu)\neq 0$ and set $\mu/\ell=(\mu^{(0)},\dots,\mu^{(\ell-1)})$. Let $\nu\in\mathcal{P}_{n+1}$ be the partition obtained by adding in μ a part 0. Then $\varepsilon(\nu)=\varepsilon(\mu)$ and we have $\nu/\ell=(\nu^{(0)},\dots,\nu^{(\ell-1)})$ where*

$\nu^{(0)} = \mu^{(\ell-1)}$, $\nu^{(k)} = \mu^{(k-1)}$ for any $k \in \{2, \dots, \ell-1\}$ and $\nu^{(1)} = (0, \mu^{(0)})$ is obtained by adding a part 0 in $\mu^{(0)}$. On the other hand, if $\mu \in \mathcal{P}_n$ is such that $\varepsilon(\mu) = 0$, then we have $\varepsilon(\nu) = 0$ where $\nu = (0, \mu)$.

Proof. Let us slightly abuse the notation and write $I^{(k)}(\mu)$, $J^{(k)}(\mu)$, $I^{(k)}(\nu)$, $J^{(k)}(\nu)$, $k = 0, \dots, \ell-1$, for the sequences defined from μ and ν by applying the previous procedure. Then, we have

$$\begin{cases} I^{(1)}(\nu) = \{1\} \cup \big(I^{(0)}(\mu) + 1\big), \quad I^{(0)}(\nu) = \big(I^{(\ell-1)}(\mu) + 1\big), \\ I^{(k)}(\nu) = \big(I^{(k-1)}(\mu) + 1\big) \quad \text{for } k = 2, \dots, \ell-1. \end{cases} \tag{13}$$

Here by $(I^{(k-1)}(\mu) + 1)$, we mean the sequence obtained by adding 1 to the integers of $I^{(k-1)}(\mu)$. We have also $I_{n+1} = \{1\} \cup (I_n + 1)$. This gives

$$\begin{cases} J^{(1)}(\nu) = \{1\} \cup \big(J^{(0)}(\mu) + 1\big), \quad J^{(0)}(\nu) = \big(J^{(\ell-1)}(\mu) + 1\big), \\ J^{(k)}(\nu) = \big(J^{(k-1)}(\mu) + 1\big) \quad \text{for } k = 2, \dots, \ell-1. \end{cases} \tag{14}$$

Set $n = q\ell + r$ as in the previous remark. We will assume that $r \neq \ell - 1$ so that q and $r+1$ are respectively the quotient and the remainder of the division of $n+1$ by ℓ. The case $r = \ell-1$ is similar. We have then $\mathrm{card}(J^{(k)}(\mu)) = \mathrm{card}(I^{(k)}(\mu)) = q+1$ for $k \in \{1, \dots, r\}$ and $\mathrm{card}(J^{(k)}(\mu)) = \mathrm{card}(I^{(k)}(\mu)) = q$ for $k \in \{0, r+1, \dots, \ell-1\}$. By (13) and (14), this implies that $\mathrm{card}(J^{(k)}(\nu)) = \mathrm{card}(I^{(k)}(\nu)) = q+1$ for $k \in \{1, \dots, r+1\}$ and $\mathrm{card}(J^{(k)}(\nu)) = \mathrm{card}(I^{(k)}(\nu)) = q$ for $k \in \{0, r+2, \dots, \ell-1\}$. Thus $\varepsilon(\nu) \neq 0$. Let us write $\sigma_0(\mu)$ and $\sigma_0(\nu)$ for the elements of S_n and S_{n+1} described in the algorithm above such that $\varepsilon(\mu) = \varepsilon(\sigma_0(\mu))$ and $\varepsilon(\nu) = \varepsilon(\sigma_0(\nu))$, respectively. We have $\sigma_0(\nu)(1) = 1$ and for any $k = 2, \dots, n+1$, $\sigma_0(\nu)(k) = \sigma_0(\mu)(k-1) + 1$. Thus $\varepsilon(\mu) = \varepsilon(\nu)$. We then easily deduce $\nu^{(0)}, \dots, \nu^{(\ell-1)}$ from (12) and (13). □

Remarks. (i) The decomposition (11) does not depend on the rank n considered provided $n \geqslant \ell l(\lambda)$. In fact, we have $a_{\lambda,\mu}^{\ell,\mathfrak{gl}} = a_{\lambda,\mu}^{\ell,\mathfrak{gl}_n}$ for such an n. Indeed, it follows easily from Proposition 3.2.2 that $a_{\lambda,\mu}^{\ell,\mathfrak{gl}_n} = \varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}$ does not depend on n so long as $n \geqslant \max\{l(\mu), l(\lambda)\}$: if $n \geqslant l(\lambda)$, Proposition 3.2.2 implies that $\varepsilon(\mu)$ is not modified when parts equal to 0 are added to μ; moreover if $\varepsilon(\mu) \neq 0$, the partitions $\mu^{(k)}$, $k \in \{0, \dots, \ell-1\}$, are also not modified up to a cyclic shift of the indices and an adjustment of the number of parts equal to 0. Thus it suffices to show that, under the assumption $n \geqslant \ell l(\lambda)$, the inequality $n \geqslant \max\{l(\mu), l(\lambda)\}$ holds for all μ appearing in (11) with non-zero coefficients $\varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}$. When $\varepsilon(\mu) \neq 0$ in (11), we must have by classical properties of the coefficients $c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}$, $l(\mu^{(k)}) \leqslant l(\lambda)$ for $k = 0, \dots, \ell-1$. By the previous algorithm, there is an integer $k \in \{0, \dots, \ell-1\}$ such that $l(\mu) \leqslant \ell \cdot l(\mu^{(k)})$. Thus $l(\mu) \leqslant \ell \cdot l(\lambda)$ and the assumption $n \geqslant \ell \cdot l(\lambda)$ suffices to guarantee that $n \geqslant \max\{l(\mu), l(\lambda)\}$.

(ii) When $n \geqslant \ell l(\lambda)$, we write for short $a_{\lambda,\mu}^{\ell,\mathfrak{gl}} = \varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}$. Then $a_{\lambda,\mu}^{\ell,\mathfrak{gl}} \neq 0$ only if $l(\mu) \leqslant \ell l(\lambda)$.

Proposition 3.2.3. *Consider $f \in \Lambda$ with degree d and $\lambda \in \mathcal{P}_n$. Then the coefficients of the expansion of $f \circ s_{\lambda}^{\mathfrak{gl}_n}$ on the basis of Schur functions do not depend on n provided $n \geqslant dl(\lambda)$.*

Proof. The subspace Λ^d of polynomials in Λ with degree d is generated by the Newton polynomials $p_\beta = p_{\beta_1} \cdots p_{\beta_k}$, such that $\beta_i \in \mathbb{N}$ and $\beta_1 + \cdots + \beta_k = d$. So it suffices to prove the proposition for $f = p_\beta$. Since, the map $g \mapsto g \circ s_{\lambda}^{\mathfrak{gl}_n}$ is a ring homomorphism from Λ to Λ_n, we have $p_\beta \circ s_{\lambda}^{\mathfrak{gl}_n} = p_{\beta_1} \circ s_{\lambda}^{\mathfrak{gl}_n} \times \cdots \times p_{\beta_k} \circ s_{\lambda}^{\mathfrak{gl}_n}$. By Remark (i) above, the expansion of $p_{\beta_i} \circ s_{\lambda}^{\mathfrak{gl}_n}$ stabilizes for $n \geqslant \beta_i l(\lambda)$ and makes appear Schur functions indexed by partitions with no more than $\beta_i l(\lambda)$ non-zero parts. Using classical properties of the Littlewood–Richardson coefficients, we obtain that the decomposition of $p_\beta \circ s_{\lambda}^{\mathfrak{gl}_n}$ involves Schur functions associated to partitions with at most $\sum_{i=1}^{k} \beta_i l(\lambda) = dl(\lambda)$ non-zero parts and stabilizes for $n \geqslant dl(\lambda)$. □

3.3. Stabilized plethysms on the Weyl characters

Lemma 3.3.1. *Consider λ a partition, ℓ a positive integer and $\mathfrak{g}$ an orthogonal or symplectic Lie algebra with rank n.*

- *The coefficients of the expansion of the plethysm $p_\ell \circ s_\lambda^{\mathfrak{g}}$ on the basis of Weyl characters do not depend on n provided $n \geqslant \ell l(\lambda)$.*
- *In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.*
- *For any $n \geqslant \ell l(\lambda)$, set*

$$p_\ell \circ s_\lambda^{\mathfrak{so}_N} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\ell,\mathfrak{so}} s_\mu^{\mathfrak{so}_N} \quad \text{and} \quad p_\ell \circ s_\lambda^{\mathfrak{sp}_{2n}} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\ell,\mathfrak{sp}} s_\mu^{\mathfrak{sp}_{2n}}.$$

We have

$$a_{\lambda,\mu}^{\ell,\mathfrak{so}} = \sum_{\nu \in \mathcal{P}_m,\, |\nu| \leqslant |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P},\, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}} a_{\nu,\delta}^{\ell,\mathfrak{gl}} b_{\delta,\mu}^{\mathfrak{so}},$$
$$a_{\lambda,\mu}^{\ell,\mathfrak{sp}} = \sum_{\nu \in \mathcal{P}_m,\, |\nu| \leqslant |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P},\, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{sp}} a_{\nu,\delta}^{\ell,\mathfrak{gl}} b_{\delta,\mu}^{\mathfrak{sp}}.$$

Proof. We have $n \geqslant \ell l(\lambda)$. Hence, the decomposition $s_\lambda^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}_m,\, |\nu| \leqslant |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} r_{\lambda,\nu}^{\mathfrak{so}} r^{\mathfrak{so}_N}(s_\nu^{\mathfrak{gl}_N})$ holds. Since ψ_ℓ and $r^{\mathfrak{so}_N}$ commute, this gives

$$p_\ell \circ s_\lambda^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}_m,\, |\nu| \leqslant |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P},\, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}} a_{\nu,\delta}^{\ell,\mathfrak{gl}} r^{\mathfrak{so}_N}\left(s_\delta^{\mathfrak{gl}_N}\right)$$
$$= \sum_{\mu \in \mathcal{P}_n} \sum_{\nu \in \mathcal{P}_m,\, |\nu| \leqslant |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P},\, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}} a_{\nu,\delta}^{\ell,\mathfrak{gl}} b_{\delta,\mu}^{\mathfrak{so}} s_\mu^{\mathfrak{so}_N}.$$

Indeed, we have $l(\delta) \leqslant \ell l(\nu) \leqslant \ell l(\lambda) \leqslant n$, thus $r^{\mathfrak{so}_N}(s_\delta^{\mathfrak{gl}_N}) = \sum_u b_{\delta,\mu}^{\mathfrak{so}} s_\mu^{\mathfrak{so}_N}$. This yields the desired expression for the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$. In particular they do not depend on n and coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$. The proof is similar for $\mathfrak{g} = \mathfrak{sp}_{2n}$. □

Proposition 3.3.2. *Consider $f \in \Lambda$ with degree d and $\lambda \in \mathcal{P}_n$. Then the coefficients of the expansion of $f \circ s_\lambda^{\mathfrak{g}}$ on the basis of Weyl characters do not depend on n provided $n \geqslant dl(\lambda)$. In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.*

Proof. The proposition follows from Lemma 3.3.1 by similar arguments to those of Proposition 3.2.3. □

Observe that we have

$$r^{\mathfrak{g}} \circ \psi_\ell = \psi_\ell \circ r^{\mathfrak{g}}$$

for $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ where ψ_ℓ is an operator on Λ in the left-hand side whereas ψ_ℓ is an operator on $R^{\mathfrak{g}}$ on the right-hand side. According to the previous lemma, we have then the decompositions

$$p_\ell \circ \mathrm{s}_\lambda^{\mathfrak{gl}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{gl}} \mathrm{s}_\mu^{\mathfrak{gl}}, \qquad p_\ell \circ \mathrm{s}_\lambda^{\mathfrak{sp}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{sp}} \mathrm{s}_\mu^{\mathfrak{sp}} \quad \text{and} \quad p_\ell \circ \mathrm{s}_\lambda^{\mathfrak{so}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{so}} \mathrm{s}_\mu^{\mathfrak{so}}.$$

We shall need in Section 4.5 the following lemma:

Lemma 3.3.3. *Consider $f \in \Lambda$ and $\lambda \in \mathcal{P}$. Then*

- $\omega(f \circ \mathrm{s}_\lambda^{\mathfrak{g}}) = f \circ \omega(\mathrm{s}_\lambda^{\mathfrak{g}})$ *if* $|\lambda|$ *is even,*
- $\omega(f \circ \mathrm{s}_\lambda^{\mathfrak{g}}) = \omega(f) \circ \omega(\mathrm{s}_\lambda^{\mathfrak{g}})$ *if* $|\lambda|$ *is odd.*

Proof. From Example 1 of [10, p. 136] we have for any positive integer ℓ, $\omega(p_\ell \circ g) = p_\ell \circ \omega(g)$ if g is homogeneous of even degree and $\omega(p_\ell \circ g) = \omega(p_\ell) \circ \omega(g)$ if g is homogeneous of odd degree. Since ψ_ℓ is linear, this shows that $\omega(p_\ell \circ \mathrm{s}_\lambda^{\mathfrak{g}}) = p_\ell \circ \omega(\mathrm{s}_\lambda^{\mathfrak{g}})$ if $|\lambda|$ is even and $\omega(p_\ell \circ \mathrm{s}_\lambda^{\mathfrak{g}}) = \omega(p_\ell) \circ \omega(\mathrm{s}_\lambda^{\mathfrak{g}})$ if $|\lambda|$ is odd. Indeed, according to (6), $\mathrm{s}_\lambda^{\mathfrak{g}}$ is a sum of homogeneous functions of degrees equal to $|\lambda|$ modulo 2. The lemma then follows since the maps ω and $f \mapsto f \circ \mathrm{s}_\lambda^{\mathfrak{g}}$ are ring homomorphisms of Λ. $\square$

Remarks. (i) Since $\omega(p_\ell) = (-1)^{\ell-1} p_\ell$, one has by the previous lemma $a_{\lambda,\mu}^{\ell,\mathfrak{sp}} = a_{\lambda',\mu'}^{\ell,\mathfrak{so}}$ if $|\lambda|$ is even and $a_{\lambda,\mu}^{\ell,\mathfrak{sp}} = (-1)^{\ell-1} a_{\lambda',\mu'}^{\ell,\mathfrak{so}}$ otherwise. This can also be verified by using the explicit formulas of Lemma 3.3.1.

(ii) The coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$ are rather complicated to compute by using formulas of Lemma 3.3.1. We are going to see in the following section that they coincide with branching coefficients corresponding to restriction to certain Levi subgroups.

4. Power sum plethysms for Weyl characters of type B_n

4.1. Statement of the theorem

In Theorem 3.2.8 of [6], we have described an algorithm for computing the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ for any positive integer ℓ and any rank n. It notably permits to show that the decomposition of $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ on the basis of Weyl characters makes appear branching coefficients corresponding to the restriction to a Levi subgroup of $\mathfrak{so}_{2n+1}$. Surprisingly, similar algorithms for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ only exist when ℓ is odd. In particular, the coefficients of the decomposition of $p_\ell \circ s_\lambda^{\mathfrak{sp}_{2n}}$ and $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n}}$ on the basis of Weyl characters are not branching coefficients in general when ℓ is even. As we are going to see, this is nevertheless the case for the stabilized forms of these plethysms.

Theorems 3.2.8 and 3.2.10 of [6] can be reformulated as follows:

Theorem 4.1.1. *For any partition $\lambda \in \mathcal{P}_n$ and any positive integer ℓ we have*

$$p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}} = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu) \big[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu}) \big] s_\mu^{\mathfrak{so}_{2n+1}} \tag{15}$$

where

- $\varepsilon(\mu) \in \{-1, 0, 1\}$,
- $\mathfrak{g}_{\ell,\mu}$ *is the Lie algebra of a Levi subgroup* $G_{\ell,\mu}$ *of* SO_{2n+1},
- $\gamma_{\ell,\mu}$ *is a dominant weight for* $G_{\ell,\mu}$.

Moreover, $\varepsilon(\mu)$, $G_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ are determined from μ and ℓ by an algorithm which can be regarded as an analogue in type B_n of the computation of the ℓ-quotient μ/ℓ.

We now recall the algorithm which permits to determinate $\varepsilon(\mu)$, $G_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ in the above theorem. Set

$$J_n = \{\bar{n}, \ldots, \bar{1}, 1, \ldots, n\} \quad \text{and} \quad L_n = \{\overline{n-1}, \ldots, \bar{1}, 0, 1, \ldots, n\}.$$

Let η be the bijection from J_n to L_n defined by $\eta(x) = x + 1$ if $x < 0$ and $\eta(x) = x$ otherwise. For each element $w \in W$ (the Weyl group of $\mathfrak{so}_{2n+1}$), denote by $\widetilde{w}$ the bijection from J_n to L_n defined by $\widetilde{w} = \eta \circ w$. This means that $\widetilde{w}(x) = w(x)$ if $w(x) > 0$ and $\widetilde{w}(x) = w(x) + 1$ if $w(x) < 0$. In particular

w is determined by $\widetilde{w}$. For any $x \in L_n$, set $x^* = \bar{x} + 1$. The map $x \mapsto x^*$ is involutive from L_n to itself. Since $w(\bar{x}) = \overline{w(x)}$, we have also

$$\widetilde{w}(\bar{x}) = \widetilde{w}(x)^*. \tag{16}$$

Hence, $\widetilde{w}$ is determined by the images of any subset $U_n \subset J_n$ such that $\mathrm{card}(U_n) = n$ and $x \in U_n$ implies $\bar{x} \notin U_n$.

For any $k = 1, \dots, \ell$ set

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \bmod \ell) \quad \text{and} \quad J^{(k)} = (x \in L_n \mid x \equiv k \bmod \ell). \tag{17}$$

Note that $(J^{(k)})^* = J^{(l-k+1)}$. We also we keep the notation $I_n = \{1, 2, \dots, n\}$ and $\rho_n = (1, \dots, n)$ from Section 3.2.3.

Remark. Set $n = q\ell + r$ where q and r are respectively the quotient and the remainder of the division of n by ℓ. Then we have

$$\mathrm{card}\big(J^{(k)}\big) = \begin{cases} 2q & \text{for } r+1 \leqslant k \leqslant \ell - r \\ 2q+1 & \text{otherwise} \end{cases} \quad \text{when } r < \frac{\ell}{2},$$
$$\mathrm{card}\big(J^{(k)}\big) = \begin{cases} 2q+2 & \text{for } \ell - r + 1 \leqslant k \leqslant r \\ 2q+1 & \text{otherwise} \end{cases} \quad \text{when } r > \frac{\ell}{2},$$
$$\mathrm{card}\big(J^{(k)}\big) = 2q+1 \quad \text{for any } k \in \{1, \dots, \ell\} \text{ when } r = \frac{\ell}{2}. \tag{18}$$

4.2. *The even case $\ell = 2p$*

For any $k = 1, \dots, p$, set $s_k = \mathrm{card}(I^{(k)})$, $r_k = \mathrm{card}(I^{(k)}) + \mathrm{card}(I^{(\ell-k+1)})$ and define $X^{(k)}$ as the increasing reordering of $\bar{I}^{(k)} \cup I^{(\ell-k+1)}$. Set

$$X^{(k)} = \big(i_1^{(k)}, \dots, i_{r_k}^{(k)}\big). \tag{19}$$

1. If there exists $k \in \{1, \dots, p\}$ such that $\mathrm{card}(X^{(k)}) \neq \mathrm{card}(J^{(k)})$ then $\varepsilon(\mu) = 0$.
2. Otherwise we have $\mathrm{card}(J^{(\ell-k+1)}) = \mathrm{card}(J^{(k)}) = r_k$ since $(J^{(k)})^* = J^{(l-k+1)}$. Let w_0 be the unique element of W such that $\widetilde{w}_0$ maps $X^{(k)}$ to $J^{(l-k+1)}$ for any $k = 1, \dots, p$. Define $\alpha_k = \frac{1}{\ell}(\max J^{(k)} - k)$. For any $k = 1, \dots, p$, consider $\mu^{(k)} \in \widetilde{\mathcal{P}}_{r_k}$ defined by

$$\mu^{(k)} = \left(\mathrm{sign}(i) \frac{\mu_{|i|} + |i| + \mathrm{sign}(i)k - \frac{1+\mathrm{sign}(i)}{2}}{\ell} \;\Big|\; i \in X^{(k)} \right) - (1, \dots, r_k) + (\alpha_k + 1, \dots, \alpha_k + 1). \tag{20}$$

Remark. With q and r as in (18), we can write for any $k = 1, \dots, p$

$$\alpha_k = \begin{cases} q-1 & \text{for } r+1 \leqslant k \leqslant p \\ q & \text{for } k \leqslant r \end{cases} \quad \text{when } r < \frac{\ell}{2},$$
$$\alpha_k = q \quad \text{for any } k \in \{1, \dots, p\} \text{ when } r \geqslant \frac{\ell}{2}.$$

Note also that in step 2, $r_k \in \{2q, 2q+1, 2q+2\}$ according to (18).

We have then with the above notation:

$$\varepsilon(\mu) = \varepsilon(w_0), \qquad G_{\ell,\mu} = GL_{r_1} \times \dots \times GL_{r_p} \quad \text{and} \quad \gamma_{\ell,\mu} = \big(\mu^{(1)}, \dots, \mu^{(p)}\big) \in P^+_{G_{\ell,\mu}}.$$

Example 4.2.1. Put $n = 6$, $\ell = 2$ (thus $p = 1$) and consider $\mu = (2, 5, 5, 6, 7, 9)$. Then $\rho_6 = (1, 2, 3, 4, 5, 6)$ and $\mu + \rho_6 = (3, 7, 8, 10, 12, 15)$. Hence $I^{(2)} = (3, 4, 5)$ and $I^{(1)} = (1, 2, 6)$. Moreover

$J^{(1)} = (\bar{5}, \bar{3}, \bar{1}, 1, 3, 5)$ and $J^{(2)} = (\bar{4}, \bar{2}, 0, 2, 4, 6)$. Then $\widetilde{w}_0$ sends $X^{(1)} = (\bar{6}, \bar{2}, \bar{1}, 3, 4, 5)$ on $J^{(2)}$. This gives

$$\widetilde{w}_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{4} & \bar{5} & \bar{3} & \bar{1} & \bar{2} & 0 & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}$$

by using (16). Hence

$$w_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{5} & \bar{6} & \bar{4} & \bar{2} & \bar{3} & \bar{1} & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}.$$

We have $\varepsilon(\mu) = 1$, $\alpha_1 = 2$ and $\gamma_{\ell,\mu} = (\mu^{(1)})$ where

$$\mu^{(1)} = (-7, -3, -1, 4, 5, 6) - (1, 2, 3, 4, 5, 6) + (3, 3, 3, 3, 3, 3) = (-5, -2, -1, 3, 3, 3).$$

Observe that $G_{\ell,\mu} \simeq GL_6$.

4.3. *The odd case $\ell = 2p + 1$*

In addition to the sets $X^{(k)}$, $k = 1, \ldots, p$, defined in (19), we have also to consider $I^{(p+1)}$. Set $r_{p+1} = \operatorname{card}(I^{(p+1)})$ and write $I^{(p+1)} = (i_1^{(p+1)}, \ldots, i_{r_{p+1}}^{(p+1)})$. Observe that $(J^{(p+1)})^* = J^{(p+1)}$. Let $X^{(p+1)}$ be the increasing reordering of $\bar{I}^{(p+1)} \cup I^{(p+1)}$.

1. If $\operatorname{card}(I^{(p+1)}) \neq \frac{1}{2} \operatorname{card}(J^{(p+1)})$ or if there exists $k \in \{1, \ldots, p\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varepsilon(\mu) = 0$.
2. Otherwise, we have $\operatorname{card}(J^{(p+1)}) = 2\operatorname{card}(I^{(p+1)}) = 2r_{p+1}$. Let w_0 be the unique element of W mapping $X^{(k)}$ to $J^{(l-k+1)}$ for any $k = 1, \ldots, p$ and $X^{(p+1)}$ to $J^{(p+1)}$. Define

$$\mu^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \,\Big|\, i \in I^{(p+1)} \right) - (1, \ldots, r_{p+1}) \in \mathcal{P}_{r_{p+1}}$$

and for any $k = 1, \ldots, p$, $\mu^{(k)}$ as in the even case. Set $\mathcal{I} = \{I^{(p+1)}, X^{(1)}, \ldots, X^{(p)}\}$. We have then with the above notation

$$\varepsilon(\mu) = \varepsilon(w_0), \qquad G_{\ell,\mu} = GL_{r_1} \times \cdots \times GL_{r_p} \times SO_{2r_{p+1}+1} \quad \text{and}$$
$$\gamma_{\ell,\mu} = \left(\mu^{(p+1)}, \mu^{(1)}, \ldots, \mu^{(p)}\right) \in P^+_{G_{\ell,\mu}}.$$

Remark. With q and r as in (18), we have when $\varepsilon(\mu) \neq 0$ is satisfied $r_{p+1} = \frac{1}{2} \operatorname{card}(J^{(p+1)}) = q$ if $r \leqslant p$ and $r_{p+1} = q + 1$ when $r \geqslant p + 1$.

Example 4.3.1. Put $n = 6$, $\ell = 3$ (thus $p = 1$) and consider $\mu = (1, 5, 5, 6, 7, 9)$. We have $\mu + \rho_6 = (2, 7, 8, 10, 12, 15)$. Thus $I^{(1)} = (2, 4)$, $I^{(2)} = (1, 3)$, $I_3 = (5, 6)$. This gives $X^{(1)} = (\bar{4}, \bar{2}, 5, 6)$ and $X^{(2)} = (\bar{3}, \bar{1}, 1, 3)$. Moreover, $J^{(1)} = (\bar{5}, \bar{2}, 1, 4)$, $J^{(2)} = (\bar{4}, \bar{1}, 2, 5)$ and $J^{(3)} = (\bar{3}, 0, 3, 6)$. In particular $\alpha_1 = 1$. Then

$$\mu^{(1)} = \left(-\frac{10-1}{3} - 1 + 2, -\frac{7-1}{3} - 2 + 2, \frac{12}{3} - 3 + 2, \frac{15}{3} - 4 + 2 \right) = (-2, -2, 3, 3)$$

and $\mu^{(2)} = (\frac{2+1}{3} - 1, \frac{8+1}{3} - 2) = (0, 1)$. By using (16), one obtains

$$\widetilde{w}_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{5} & \bar{2} & \bar{3} & \bar{4} & 0 & \bar{1} & 2 & 1 & 5 & 4 & 3 & 6 \end{pmatrix}.$$

Hence

$$w_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{6} & \bar{3} & \bar{4} & \bar{5} & \bar{1} & \bar{2} & 2 & 1 & 5 & 4 & 3 & 6 \end{pmatrix}$$

and $\varepsilon(\mu) = 1$. We have $G_{\ell,\mu} \simeq GL_4 \times SO_5$.

4.4. The stabilization phenomenon

We begin this paragraph with further remarks:

Remarks. (i) Suppose $\varepsilon(\mu)\neq 0$. In the even case, we have $G_{\ell,\mu}=GL_{r_1}\times\cdots\times GL_{r_p}$. In n is odd and $n\geqslant p+1$, $G_{\ell,\mu}$ is not a direct product of linear groups since $G_{\ell,\mu}=\cdots\times SO_{2r_{p+1}+1}$ due to remark in Section 4.3.

(ii) When $\ell=2$, we have always $\mathrm{card}(X^{(1)})=n=\mathrm{card}(J^{(2)})$. Hence $\varepsilon(\mu)\neq 0$ for all partitions μ. Observe that it does not mean that the expansion (15) is infinite. In fact all but a finite number of the branching coefficients $[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ vanishes in this situation. Indeed, we have $G_{\ell,\mu}\simeq GL_n$ in this case. Thus the coefficient of $s_\mu^{\mathfrak{so}_{2n+1}}$ is not equal to zero if and only if $V^{\mathfrak{gl}_n}(\gamma_{\ell,\mu})$ is isomorphic to an irreducible component of the restriction of $V^{\mathfrak{so}_{2n+1}}(\lambda)$ to GL_n. Since the number of such components is finite and the map $\mu\mapsto\gamma_{\ell,\mu}$ is injective, the expansion (15) is finite.

(iii) We have seen that the non-zero parts of the ℓ-quotient μ/ℓ does not depend on the number of zero parts in μ (see Proposition 3.2.2). This notably implies the stability of the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{gl}}$. The situation is more subtle for the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$. Indeed, the dominant weights $\gamma_{\ell,\mu}$ given by the previous algorithm do *not* stabilize in general when the number of zero parts in μ increases. Let us consider for example $\mu=(1,5,5,6,9)$ and $\ell=2$. By adding parts 0 to μ, we obtain successively for the dominant weights

$$(-1,2,4,4,5),\quad(-5,-4,-4,-2,-2,1),\quad(-1,2,2,2,4,4,5),\quad\text{etc.}\tag{21}$$

This is not incompatible with Proposition 3.3.2 which asserts that $\psi_\ell(s_\mu^{\mathfrak{so}_{2n+1}})$ stabilizes in large rank. In fact, this only means that, when no assumption is made on the size of n, there can exist non-zero coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}_{2n+1}}$ in the decomposition

$$\psi_\ell\big(s_\lambda^{\mathfrak{so}_{2n+1}}\big)=\sum_{\mu\in\mathcal{P}_n}a_{\lambda,\mu}^{\ell,\mathfrak{so}_{2n+1}}s_\mu^{\mathfrak{so}_{2n+1}}$$

such that $a_{\lambda,\mu}^{\ell,\mathfrak{so}}=0$. In the rest of this paragraph, we are going to see that the dominant weights μ for which $\gamma_{\ell,\mu}$ do not stabilize are such that $a_{\lambda,\mu}^{\ell,\mathfrak{so}}=0$, that is their contribution to $\psi_\ell(s_\lambda^{\mathfrak{so}_{2n+1}})$ vanishes in large rank. Moreover, we are going to characterize precisely these weights.

Suppose first $\ell=2p$ is even. Consider $\mu\in\mathcal{P}_m$ such that $\varepsilon(\mu)\neq 0$. Set $\gamma_{\ell,\mu}=(\mu^{(1)},\dots,\mu^{(p)})$. Write ν for the partition of $\mathcal{P}_{m+\ell}$ obtained by adding ℓ parts 0 in μ. For any $k\in\{1,\dots,p\}$, set $\mu^{(k)}=(\mu_-^{(k)},\mu_+^{(k)})$ where $\mu_-^{(k)}$ (respectively $\mu_+^{(k)}$) is the sequence formed by the s_k leftmost (respectively r_k-s_k rightmost) components of $\mu^{(k)}$ (see Section 4.2 for the notation).

Lemma 4.4.1. *We have $\varepsilon(\nu)=\varepsilon(\mu)$. Moreover if we set $\gamma_{\ell,\nu}=(\nu^{(1)},\dots,\nu^{(p)})$, we obtain*

$$\nu^{(k)}=\big(\mu_-^{(k)},\alpha_k+1-s_k,\alpha_k+1-s_k,\mu_+^{(k)}\big)$$

for any $k\in\{1,\dots,p\}$, that is $\nu^{(k)}$ is obtained by inserting in $\mu^{(k)}$ two components equal to α_k+1-s_k. In particular

$$|\gamma_{\ell,\nu}|=|\gamma_{\ell,\mu}|+2\sum_{k=1}^{p}|\alpha_k+1-s_k|.\tag{22}$$

Proof. Let us slightly abuse the notation by writing $I^{(k)}(\mu)$, $J^{(k)}(\mu)$, $I^{(k)}(\nu)$, $J^{(k)}(\nu)$, $k=1,\dots,\ell$, and $X^{(k)}(\mu)$, $X^{(k)}(\nu)$, $k=1,\dots,p$, for the sequences defined from μ and ν by applying the procedure of Section 4.2. We define $\alpha_k(\mu)$ and $\alpha_k(\nu)$, $k=1,\dots,p$, similarly. We have $I^{(k)}(\nu)=\{k\}\cup(I^{(k)}(\mu)+\ell)$ for $k=1,\dots,\ell$. We have also for $k=1,\dots,\ell-1$, $J^{(k)}(\nu)=\{k\}\cup\{k-\ell\}\cup{}^tJ^{(k)}(\nu)$ where ${}^tJ^{(k)}(\nu)$ is obtained by adding ℓ to the positive integers of $J^{(k)}(\nu)$ and $-\ell$ to the integers less or equal to 0.

Similarly, $J^{(\ell)}(\nu)=\{\ell\}\cup\{0\}\cup{}^{t}J^{(\ell)}(\nu)$. In all cases, this implies that $\operatorname{card}(J^{(k)}(\nu))=\operatorname{card}(J^{(k)}(\mu))+2$ and $\alpha_k(\nu)=\alpha_k(\mu)+1$ for any $k\in\{1,\dots,\ell\}$. Thus $\operatorname{card}(X^{(k)}(\nu))=\operatorname{card}(J^{(k)}(\nu))$ for any $k\in\{1,\dots,p\}$ and we have $\varepsilon(\nu)=\varepsilon(\mu)$. So it makes sense to consider $\gamma_{\ell,\nu}=(\nu^{(1)},\dots,\nu^{(p)})$. It then follows by a direct application of the formulas (20) that

$$\nu^{(k)}=\left(\mu_-^{(k)},\alpha_k+1-s_k,\alpha_k+1-s_k,\mu_+^{(k)}\right)$$

and thus $|\gamma_{\ell,\nu}|=|\gamma_{\ell,\mu}|+2\sum_{k=1}^{p}|\alpha_k+1-s_k|$. □

When $\ell=2p+1$ is odd and $\gamma_{\ell,\mu}=(\mu^{(1)},\dots,\mu^{(p)},\mu^{(p+1)})$, we can define ν similarly. Then, one proves that $\varepsilon(\nu)=\varepsilon(\mu)$. We have $\gamma_{\ell,\nu}=(\nu^{(1)},\dots,\nu^{(p)},\nu^{(p+1)})$ with

$$\nu^{(k)}=\left(\mu_-^{(k)},\alpha_k+1-s_k,\alpha_k+1-s_k,\mu_+^{(k)}\right)\quad\text{for any } k=1,\dots,p \tag{23}$$

and $\nu^{(p+1)}=(0,\mu^{(p+1)})$. Hence (22) still holds. With the notation of Sections 4.2 and 4.3, we obtain the following stabilization theorem:

Theorem 4.4.2. *Consider μ a partition such that $\varepsilon(\mu)\neq 0$. Let ℓ be a positive integer. Then for any partition λ*

1. *$a_{\lambda,\mu}^{\ell,\mathfrak{so}}\neq 0$ only if $s_k=\alpha_k+1$ for any $k=1,\dots,p$.*
2. *In this case we have $a_{\lambda,\mu}^{\ell,\mathfrak{so}}=\varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ and this coefficient does not depend on the number of parts 0 in λ and μ.*

Proof. Suppose there exists $k\in\{1,\dots,p\}$ such that $s_k\neq\alpha_k+1$. Write $\mu(a)$ for the partition obtained by adding $a\ell$ components 0 to μ. By (22), we have then $|\gamma_{\ell,\mu(a)}|\geqslant|\gamma_{\ell,\mu}|+2a$. Thus, for a sufficiently large, one has $|\gamma_{\ell,\mu(a)}|>|\lambda|$. For such a, we will obtain $[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]=0$. Hence $\varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ does not coincide with a non-zero coefficient $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$. When $s_k=\alpha_k+1$ for any $k=1,\dots,p$, we obtain from (23) that $a_{\lambda,\mu}^{\ell,\mathfrak{so}}=a_{(0^\ell,\lambda),(0^\ell,\mu)}^{\ell,\mathfrak{so}}$. Since the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$ must stabilize by Lemma 3.3.1, this implies that $a_{\lambda,\mu}^{\ell,\mathfrak{so}}=a_{(0^a,\lambda),(0^a,\mu)}^{\ell,\mathfrak{so}}$ for any nonnegative integer a. □

Remarks. (i) There exist efficient recursive procedures to compute the branching coefficients $[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ (see [2]). By the previous theorem, they permit to derive the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$.

(ii) One can check that condition 1 of Theorem 4.4.2 is satisfied in Example 4.2.1 but fails in (21) where $s_1=1$ and $\alpha_1=2$.

(iii) Assume that $\alpha_k+1=s_k$ for all k and put $\overline{\mu}^{(k)}=(\frac{\mu_i+i+(\ell-k)}{\ell}\mid i\in I^{(k)})-(1,\dots,s_k)$. We have then for $1\leqslant k\leqslant p$, $\mu^{(k)}=[\overline{\mu}^{(k)},\overline{\mu}^{(\ell-k+1)}]$ in the notation $[\gamma^-,\gamma^+]$ of (1) and $\mu^{(p+1)}=\overline{\mu}^{(p+1)}$ if ℓ is odd. The effect of enlarging n by one and considering $\nu=(0,\mu)$ will be $\overline{\nu}^{(1)}=(0,\overline{\mu}^{(\ell)})$, $\overline{\nu}^{(k)}=\overline{\mu}^{(k-1)}$ for $2\leqslant k\leqslant\ell$ as in the type A case in terms of the $\overline{\mu}^{(k)}$, but in terms of the $\mu^{(k)}$. This gives

$$\gamma_{\ell,\mu}=\left(\left[\overline{\mu}^{(1)},\overline{\mu}^{(\ell)}\right],\left[\overline{\mu}^{(2)},\overline{\mu}^{(\ell-1)}\right],\dots,\left[\overline{\mu}^{(p)},\overline{\mu}^{(\ell-p+1)}\right],\overline{\mu}^{(p+1)}\right),$$
$$\gamma_{\ell,\nu}=\left(\left[\left(0,\overline{\mu}^{(\ell)}\right),\overline{\mu}^{(\ell-1)}\right],\left[\overline{\mu}^{(1)},\overline{\mu}^{(\ell-2)}\right],\dots,\left[\overline{\mu}^{(p-1)},\overline{\mu}^{(\ell-p)}\right],\overline{\mu}^{(p)}\right)$$

where the rightmost component appears only when ℓ is odd. By using the decomposition of the branching coefficients $\gamma_{\ell,\mu}=[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ and $\gamma_{\ell,\nu}=[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_{\ell,\nu}}(\gamma_{\ell,\nu})]$ in terms of the Littlewood–Richardson coefficients obtained in Theorems A1 (1) and (3) of [5], one can then prove the equality $\gamma_{\ell,\mu}=\gamma_{\ell,\nu}$. This notably means that the use of Lemma 3.3.1 in the proof of Theorem 4.4.2 is not properly needed. The stabilization phenomenon emerges in fact naturally from the algorithms of Sections 4.2 and 4.3.

4.5. Coefficients $a^{\ell}_{\lambda,\mu}$ and restriction to Levi subgroups

By combining the results of Sections 3 and 4 we derive the following theorem which expresses $a^{\ell,\mathfrak{so}}_{\lambda,\mu}$ and $a^{\ell,\mathfrak{sp}}_{\lambda,\mu}$ as branching coefficients corresponding to restrictions to Levi subgroups.

Theorem 4.5.1. *Consider $\lambda \in \mathcal{P}_m$ and ℓ a positive integer. Let $\mathfrak{g}$ be a symplectic or orthogonal Lie algebra with rank n. Then we have*:

1. $a^{\ell,\mathfrak{so}}_{\lambda,\mu} = \varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ *for any $n \geqslant \ell l(\lambda)$ where $\varepsilon(\mu)$, $\mathfrak{g}_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ are determined by the algorithms of Section 4,*
2. $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = \varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{g}_{\ell,\mu'}}(\gamma_{\ell,\mu'})]$ *for any $n \geqslant \ell l(\lambda')$ if $|\lambda|$ is even, and*
3. $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = (-1)^{\ell-1}\varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{g}_{\ell,\mu'}}(\gamma_{\ell,\mu'})]$ *for any $n \geqslant \ell l(\lambda')$ if $|\lambda|$ is odd.*

Proof. Assertion 1 follows from Proposition 3.3.2 and Theorem 4.1.1. By remark following Lemma 3.3.3, one has $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = a^{\ell,\mathfrak{so}}_{\lambda',\mu'}$ if $|\lambda|$ is even and $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = (-1)^{\ell-1}a^{\ell,\mathfrak{so}}_{\lambda',\mu'}$ otherwise which proves assertion 2. □

In the sequel, we will assume for simplicity $\ell \geqslant 2$ and $n \geqslant \max(\ell l(\lambda), \ell l(\lambda'))$.

5. Splitting $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ into its symmetric and antisymmetric parts

5.1. Decomposition of the plethysms $p_2 \circ s^{\mathfrak{g}}_{\lambda}$

Consider a partition $\lambda \in \mathcal{P}_m$. According to Theorem 4.5.1, we have with the notation of Sections 3.2 and 3.3

$$p_2 \circ s^{\mathfrak{gl}_n}_{\lambda} = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\mu^{(1)})} s^{\mathfrak{gl}_n}_{\mu},$$

$$p_2 \circ s^{\mathfrak{so}}_{\lambda} = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu)\big[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)\big] s^{\mathfrak{so}}_{\mu} \quad \text{and} \quad p_2 \circ s^{\mathfrak{sp}}_{\lambda} = (-1)^{|\lambda|} p_2 \circ s^{\mathfrak{so}}_{\lambda'} \tag{24}$$

for any $n \geqslant 2|\lambda|$. Here we have written for short γ_μ for $\gamma_{2,\mu}$ and $V^{\mathfrak{gl}_n}(\gamma_\mu)$ instead of $V^{\mathfrak{g}_{2,\mu}}(\gamma_\mu)$ (see Remark (ii) of Section 4.4). Since $n \geqslant m$ and $\gamma_\mu = (\gamma^-, \gamma^+)$ belongs to $\widetilde{\mathcal{P}}_n$ we have the following decomposition (see [2]):

$$\big[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)\big] = \sum_{\delta,\xi \in \mathcal{P}_n} c^{\lambda}_{\delta,\xi} c^{\xi}_{\gamma^-,\gamma^+}. \tag{25}$$

5.2. Symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$

Consider $\lambda \in \mathcal{P}_m$. By Propositions 3.2.3 and 3.3.2, for any rank $n \geqslant 2|\lambda|$ the plethysms $h_2 \circ s^{\mathfrak{g}}_{\lambda}$ and $e_2 \circ s^{\mathfrak{g}}_{\lambda}$ stabilize. Set

$$h_2 \circ s^{\mathfrak{g}}_{\lambda} = \sum_{\mu \in \mathcal{P}_n} m^{\mathfrak{G},+}_{\lambda,\mu} s^{\mathfrak{g}}_{\mu} \quad \text{and} \quad e_2 \circ s^{\mathfrak{g}}_{\lambda} = \sum_{\mu \in \mathcal{P}_n} m^{\mathfrak{G},-}_{\lambda,\mu} s^{\mathfrak{g}}_{\mu}$$

where $\mathfrak{G} = \mathfrak{gl}, \mathfrak{so}$ or $\mathfrak{sp}$ respectively when $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_N$ or $\mathfrak{sp}_{2n}$. Recall that $h_2 \circ s^{\mathfrak{g}}_{\lambda}$ and $e_2 \circ s^{\mathfrak{g}}_{\lambda}$ are the characters of $S^2(V^{\mathfrak{g}}(\lambda))$ and $\Lambda^2(V^{\mathfrak{g}}(\lambda))$. By using (10) and Theorem 4.5.1, we obtain for any rank $n \geqslant 2|\lambda|$

$$m^{\mathfrak{gl},\pm}_{\lambda,\mu} = \frac{1}{2}\big(c^{\mu}_{\lambda,\lambda}, \pm\varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\mu^{(1)})}\big),$$

$$m^{\mathfrak{so},\pm}_{\lambda,\mu} = \frac{1}{2}\big(d^{\mu}_{\lambda,\lambda}, \pm\varepsilon(\mu)\big[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)\big]\big),$$

$$m^{\mathfrak{sp},\pm}_{\lambda,\mu} = \frac{1}{2}\big(d^{\mu'}_{\lambda',\lambda'}, \pm(-1)^{|\lambda|}\varepsilon(\mu')\big[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{gl}_n}(\gamma_{\mu'})\big]\big)$$

where the coefficients $d^{\mu}_{\lambda,\lambda}$ are the multiplicities appearing in Proposition 2.2.2. Observe that the values of $\varepsilon(\mu)$ appearing in (10) and Theorem 4.5.1 may not coincide. Now these multiplicities can be expressed in terms of the Littlewood coefficients [1]. Namely we have $d^{\mu}_{\lambda,\lambda}=\sum_{\delta,\xi,\eta}c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta}$. In particular we recover the equality $d^{\mu'}_{\lambda',\lambda'}=d^{\mu}_{\lambda,\lambda}$ since $c^{\gamma}_{\delta,\eta}=c^{\gamma'}_{\delta',\eta'}$ for any partitions δ, η and γ. By using (25), this thus permits to express the multiplicities appearing in the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ in terms of the Littlewood–Richardson coefficients. Note that formulas for computing the plethysms $h_2\circ s^{\mathfrak{g}}_{\lambda}$ and $e_2\circ s^{\mathfrak{g}}_{\lambda}$ were introduced without a complete proof by Littlewood in [9].

Proposition 5.2.1. *With the above notation we have for any rank $n\geqslant 2|\lambda|$*

$$m^{\mathfrak{gl},\pm}_{\lambda,\mu}=\frac{1}{2}\left(c^{\mu}_{\lambda,\lambda}\pm\varepsilon(\mu)c^{\lambda}_{(\mu^{(0)},\mu^{(1)})}\right),$$

$$m^{\mathfrak{so},\pm}_{\lambda,\mu}=\frac{1}{2}\left(\sum_{\delta,\xi,\eta\in\mathcal{P}_n}c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta}\pm\varepsilon(\mu)\sum_{\delta,\xi\in\mathcal{P}_n}c^{\lambda}_{\delta,\xi}c^{\xi}_{\gamma^-,\gamma^+}\right),$$

$$m^{\mathfrak{sp},\pm}_{\lambda,\mu}=\frac{1}{2}\left(\sum_{\delta,\xi,\eta\in\mathcal{P}_n}c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta}\pm(-1)^{|\lambda|}\varepsilon(\mu')\sum_{\delta,\xi\in\mathcal{P}_n}c^{\lambda'}_{\delta,\xi}c^{\xi}_{\kappa^-,\kappa^+}\right)$$

where $\gamma_{\mu}=(\gamma^-,\gamma^+)$ and $\gamma_{\mu'}=(\kappa^-,\kappa^+)$.

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