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Fredholmness and smooth dependence for linear time-periodic hyperbolic systems

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ABSTRACT

This paper concerns $n \times n$ linear one-dimensional hyperbolic systems of the type

$$\partial_t u_j + a_j(x)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x)u_k = f_j(x,t), \quad j = 1, \dots, n,$$

with periodicity conditions in time and reflection boundary conditions in space. We state conditions on the data a_j and b_{jk} and the reflection coefficients such that the system is Fredholm solvable. Moreover, we state conditions on the data such that for any right-hand side there exists exactly one solution, that the solution survives under small perturbations of the data, and that the corresponding data-to-solution map is smooth with respect to appropriate function space norms. In particular, those conditions imply that no small denominator effects occur.
We show that perturbations of the coefficients a_j lead to essentially different results than perturbations of the coefficients b_{jk}, in general. Our results cover cases of non-strictly hyperbolic systems as well as systems with discontinuous coefficients a_j and b_{jk}, but they are new even in the case of strict hyperbolicity and of smooth coefficients.

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1. Introduction

1.1. Problem and main results

This paper concerns linear inhomogeneous hyperbolic systems of first-order PDEs in one space dimension of the type

$$\partial_t u_j + a_j(x)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x)u_k = f_j(x,t), \quad j=1,\dots,n,\ x\in(0,1) \tag{1.1}$$

with time-periodicity conditions

$$u_j(x,t+2\pi) = u_j(x,t), \quad j=1,\dots,n,\ x\in[0,1] \tag{1.2}$$

and reflection boundary conditions

$$\begin{aligned} u_j(0,t) &= \sum_{k=m+1}^{n} r_{jk}^0 u_k(0,t), \quad j=1,\dots,m, \\ u_j(1,t) &= \sum_{k=1}^{m} r_{jk}^1 u_k(1,t), \quad j=m+1,\dots,n. \end{aligned} \tag{1.3}$$

Here $1 \leqslant m < n$ are fixed natural numbers, r_{jk}^0 and r_{jk}^1 are real numbers, and the right-hand sides $f_j: [0,1]\times\mathbb{R}\to\mathbb{R}$ are supposed to be 2π-periodic with respect to t.

Roughly speaking, we will prove results of the following type:

First, we will state sufficient conditions on the data a_j, b_{jk}, r_{jk}^0, and r_{jk}^1 such that the system (1.1)–(1.3) has a Fredholm type solution behavior, i.e. that it is solvable if and only if the right-hand side is orthogonal to all solutions to the corresponding homogeneous adjoint system

$$\begin{aligned} -\partial_t u_j - \partial_x\big(a_j(x)u_j\big) + \sum_{k=1}^{n} b_{kj}(x)u_k &= 0, \quad j=1,\dots,n,\ x\in(0,1), \\ u_j(x,t+2\pi) &= u_j(x,t), \quad j=1,\dots,n,\ x\in[0,1], \\ a_j(0)u_j(0,t) &= -\sum_{k=1}^{m} r_{kj}^0 a_k(0)u_k(0,t), \quad j=m+1,\dots,n, \\ a_j(1)u_j(1,t) &= -\sum_{k=m+1}^{n} r_{kj}^1 a_k(1)u_k(1,t), \quad j=1,\dots,m. \end{aligned} \tag{1.4}$$

And second, we will state sufficient conditions on the data a_j, b_{jk}, r_{jk}^0, and r_{jk}^1 such that the system (1.1)–(1.3) is uniquely solvable for any right-hand side, that this unique solvability property survives under small perturbations of the data, and that the corresponding data-to-solution maps are smooth with respect to appropriate function space norms. For example, under those sufficient conditions the following are true:

(I) If $\partial_t^j f \in L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$ for $j=0,1$, then the map $b\mapsto u$ is C^∞-smooth from an open set in $L^\infty((0,1);\mathbb{M}_n)$ into $L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$.

(II) If $\partial_t^j f \in L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$ for $j=0,1,2$, then the map $b\mapsto u$ is C^∞-smooth from an open set in $L^\infty((0,1);\mathbb{M}_n)$ into $C([0,1]\times[0,2\pi];\mathbb{R}^n)$.

(III) If $\partial_t^j f \in L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$ for $j=0,1,\dots,k$ with $k\geqslant 2$, then the map $a\mapsto u$ is C^{k-1}-smooth (resp. C^{k-2}-smooth) from an open subset of $BV((0,1);\mathbb{M}_n)$ into $L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$. (resp. $C([0,1]\times[0,2\pi];\mathbb{R}^n)$).

Here and in what follows we denote by

$$a := \operatorname{diag}(a_1,\dots,a_n),\qquad b := [b_{jk}]_{j,k=1}^n,\qquad f := (f_1,\dots,f_n),\quad \text{and}\quad u := (u_1,\dots,u_n)$$

the diagonal matrix of the coefficient functions a_j, the matrix of the coefficient functions b_{jk}, and the vectors of the right-hand sides f_j and the solutions u_j, respectively, and $\mathbb{M}_n$ is the space of all real $n\times n$ matrices.

In order to formulate our results more precisely, let us introduce the following function spaces: For $\gamma\geqslant 0$ we denote by W^γ the vector space of all locally integrable functions $f:[0,1]\times\mathbb{R}\to\mathbb{R}^n$ such that $f(x,t)=f(x,t+2\pi)$ for almost all $x\in(0,1)$ and $t\in\mathbb{R}$ and that

$$\|f\|_{W^\gamma}^2 := \sum_{s\in\mathbb{Z}}\left(1+s^2\right)^\gamma \int_0^1 \left\| \int_0^{2\pi} f(x,t)e^{-ist}\,dt\right\|^2 dx < \infty. \tag{1.5}$$

Here and in what follows $\|\cdot\|$ is the Hermitian norm in $\mathbb{C}^n$. It is well known (see, e.g., [2], [19, Chapter 5.10], and [21, Chapter 2.4]) that W^γ is a Banach space with the norm (1.5). In fact, it is the anisotropic Sobolev space of all measurable functions $u:[0,1]\times\mathbb{R}\to\mathbb{R}^n$ such that $u(x,t)=u(x,t+2\pi)$ for almost all $x\in(0,1)$ and $t\in\mathbb{R}$ and that the distributional partial derivatives of u with respect to t up to the order γ are locally quadratically integrable.

Further, for $\gamma\geqslant 1$ and $a\in L^\infty((0,1);\mathbb{M}_n)$ with $\operatorname{ess\,inf}|a_j|>0$ for all $j=1,\dots,n$ we will work with the function spaces

$$U^\gamma(a) := \left\{u\in W^\gamma\colon\ \partial_x u\in W^0,\ \partial_t u + a\partial_x u\in W^\gamma\right\}$$

endowed with the norms

$$\|u\|_{U^\gamma(a)}^2 := \|u\|_{W^\gamma}^2 + \|\partial_t u + a\partial_x u\|_{W^\gamma}^2.$$

Remark that the space $U^\gamma(a)$ depends on a. In particular, it is larger than the space of all $u\in W^\gamma$ such that $\partial_t u\in W^\gamma$ and $\partial_x u\in W^\gamma$ (which does not depend on a). For $u\in U^\gamma(a)$ there exist traces $u(0,\cdot), u(1,\cdot)\in L^2_{loc}(\mathbb{R};\mathbb{R}^n)$ (see Section 2), and, hence, it makes sense to consider the closed subspaces in $U^\gamma(a)$

$$V^\gamma(a,r) := \left\{u\in U^\gamma(a)\colon\ (1.3)\text{ is fulfilled}\right\},$$
$$\tilde{V}^\gamma(a,r) := \left\{u\in U^\gamma(a)\colon\ (1.4)\text{ is fulfilled}\right\}.$$

Here we use the notation

$$r := \left(r^0, r^1\right)\quad \text{with}\ r^0 := \left[r^0_{jk}\right]_{j=1,\,k=m+1}^{m\quad\ n},\ r^1 := \left[r^1_{jk}\right]_{j=m+1,\,k=1}^{n\quad\ m}$$

for the matrices of the reflection coefficients r^0_{jk} and r^1_{jk}. Further, we denote by

$$b^0 := \operatorname{diag}(b_{11}, b_{22},\dots,b_{nn})\quad\text{and}\quad b^1 := b - b^0$$

the diagonal and the off-diagonal parts of the coefficient matrix b, respectively.

Further, we introduce operators $\mathcal{A}(a,b^0)\in\mathcal{L}(V^\gamma(a,r);W^\gamma)$, $\tilde{\mathcal{A}}(a,b^0)\in\mathcal{L}(\tilde{V}^\gamma(a,r);W^\gamma)$ and $\mathcal{B}(b^1), \tilde{\mathcal{B}}(b^1)\in\mathcal{L}(W^\gamma)$ by

$$
\begin{aligned}
\mathcal{A}(a,b^0)u &:= \partial_t u + a\partial_x u + b^0 u,\\
\tilde{\mathcal{A}}(a,b^0)u &:= -\partial_t u - \partial_x(au) + b^0 u,\\
\mathcal{B}(b^1)u &:= b^1 u,\\
\tilde{\mathcal{B}}(b^1)u &:= (b^1)^T u.
\end{aligned}
$$

Remark that the operators $\mathcal{A}(a,b^0)$, $\mathcal{B}(b^1)$, and $\tilde{\mathcal{B}}(b^1)$ are well defined for $a_j, b_{jk} \in L^\infty(0,1)$, while $\tilde{\mathcal{A}}(a,b^0)$ is well defined under additional regularity assumptions with respect to the coefficients a_j, for example, for $a_j \in C^{0,1}([0,1])$. Obviously, the operator equation

$$\mathcal{A}(a,b^0)u + \mathcal{B}(b^1)u = f \tag{1.6}$$

is an abstract representation of the periodic-Dirichlet problem (1.1)–(1.3).

Finally, for $s \in \mathbb{Z}$ we introduce the following complex $(n-m)\times(n-m)$ matrices

$$R_s(a,b^0,r) := \left[\sum_{l=1}^m e^{is(\alpha_j(1)-\alpha_l(1))+\beta_j(1)-\beta_l(1)} r^1_{jl} r^0_{lk}\right]_{j,k=m+1}^n, \tag{1.7}$$

where

$$\alpha_j(x) := \int_0^x \frac{1}{a_j(y)}\,dy, \qquad \beta_j(x) := \int_0^x \frac{b_{jj}(y)}{a_j(y)}\,dy. \tag{1.8}$$

Our first result concerns an isomorphism property of $\mathcal{A}(a,b^0)$:

Theorem 1.1. *For all $c > 0$ there exists $C > 0$ such that the following is true: If*

$$a_j, b_{jj} \in L^\infty(0,1) \quad \text{and} \quad \operatorname{ess\,inf}|a_j| \geqslant c \quad \text{for all } j = 1,\dots,n, \tag{1.9}$$

$$\sum_{j=1}^n \|b_{jj}\|_\infty + \sum_{j=1}^m \sum_{k=m+1}^n |r^0_{jk}| + \sum_{j=m+1}^n \sum_{k=1}^m |r^1_{jk}| \leqslant \frac{1}{c}, \tag{1.10}$$

and

$$\left|\det\left(I - R_s(a,b^0,r)\right)\right| \geqslant c \quad \text{for all } s \in \mathbb{Z}, \tag{1.11}$$

then for all $\gamma \geqslant 1$ the operator $\mathcal{A}(a,b^0)$ is an isomorphism from $V^\gamma(a,r)$ onto W^γ and

$$\left\|\mathcal{A}(a,b^0)^{-1}\right\|_{\mathcal{L}(W^\gamma; V^\gamma(a,r))} \leqslant C.$$

Our second result concerns the Fredholm solvability of (1.6):

Theorem 1.2. *Suppose that conditions (1.9) and (1.11) are fulfilled for some $c > 0$. Suppose also that*

$$\left.\begin{array}{l}\text{for all } j \neq k \text{ there is } c_{jk} \in BV(0,1) \text{ such that}\\ a_k(x)b_{jk}(x) = c_{jk}(x)\big(a_j(x) - a_k(x)\big) \text{ for a.a. } x \in [0,1].\end{array}\right\} \tag{1.12}$$

Then the following are true:

(i) *The operator* $\mathcal{A}(a,b^0)+\mathcal{B}(b^1)$ *is a Fredholm operator with index zero from* $V^\gamma(a,r)$ *into* W^γ *for all* $\gamma \geqslant 1$*, and*

$$\ker\bigl(\mathcal{A}\bigl(a,b^0\bigr)+\mathcal{B}\bigl(b^1\bigr)\bigr) := \bigl\{u \in V^\gamma(a,r)\colon \bigl(\mathcal{A}\bigl(a,b^0\bigr)+\mathcal{B}\bigl(b^1\bigr)\bigr)u = 0\bigr\}$$

does not depend on γ*.*

(ii) *Suppose* $a \in C^{0,1}([0,1];\mathbb{M}_n)$*. Then*

$$\begin{aligned}&\bigl\{\bigl(\mathcal{A}\bigl(a,b^0\bigr)+\mathcal{B}\bigl(b^1\bigr)\bigr)u\colon\ u \in V^\gamma(a,r)\bigr\}\\ &\quad= \bigl\{f \in W^\gamma\colon\ \langle f,u\rangle_{L^2} = 0 \text{ for all } u \in \ker\bigl(\tilde{\mathcal{A}}\bigl(a,b^0\bigr)+\tilde{\mathcal{B}}\bigl(b^1\bigr)\bigr)\bigr\},\end{aligned}$$

where

$$\ker\bigl(\tilde{\mathcal{A}}\bigl(a,b^0\bigr)+\tilde{\mathcal{B}}\bigl(b^1\bigr)\bigr) := \bigl\{u \in \tilde{V}^\gamma(a,r)\colon \bigl(\tilde{\mathcal{A}}\bigl(a,b^0\bigr)+\tilde{\mathcal{B}}\bigl(b^1\bigr)\bigr)u = 0\bigr\}$$

does not depend on γ*.*

Here we write

$$\langle f,u\rangle_{L^2} := \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \bigl\langle f(x,t),u(x,t)\bigr\rangle\,dx\,dt \tag{1.13}$$

for the usual scalar product in the Hilbert space $L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$, and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$ (as well as the Hermitian scalar product in $\mathbb{C}^n$).

The main tools of the proofs of Theorems 1.1 and 1.2 are separation of variables (cf. (3.3)–(3.4)), integral representation of the solutions of the corresponding boundary value problems of the ODE systems (cf. (3.10)), and an abstract criterion for Fredholmness (cf. Lemma 4.1). In the special case $m=1$, $n=2$, $a_1(x)=1$, and $a_2(x)=-1$ Theorem 1.2 was proved in [9].

Our last results concern the solution behavior of (1.6) under small perturbations of the data a and b and under arbitrary perturbations of f. In order to describe this we use the following notation for the corresponding open balls (for $\varepsilon>0$):

$$\begin{aligned}A_\varepsilon(a) &:= \Bigl\{\tilde{a} \in BV\bigl((0,1);\mathbb{M}_n\bigr)\colon \tilde{a} = \operatorname{diag}(\tilde{a}_1,\dots,\tilde{a}_n),\ \max_{1\leqslant j\leqslant n}\|\tilde{a}_j - a_j\|_\infty < \varepsilon\Bigr\},\\ B_\varepsilon^\infty(b) &:= \Bigl\{\tilde{b} \in L^\infty\bigl((0,1);\mathbb{M}_n\bigr)\colon \max_{1\leqslant j,k\leqslant n}\|\tilde{b}_{jk} - b_{jk}\|_\infty < \varepsilon\Bigr\},\\ B_\varepsilon(b) &:= \bigl\{\tilde{b} \in B_\varepsilon^\infty(b)\colon \tilde{b}_{jk} \in BV(0,1) \text{ for all } 1 \leqslant j \neq k \leqslant n\bigr\}.\end{aligned}$$

The set $B_\varepsilon^\infty(b)$ is open in the Banach space $L^\infty((0,1);\mathbb{M}_n)$. The sets $A_\varepsilon(a)$ and $B_\varepsilon(b)$ will be considered as open sets in the (not complete) normed vector spaces $\{\tilde{a} \in BV((0,1);\mathbb{M}_n)\colon \tilde{a} = \operatorname{diag}(\tilde{a}_1,\dots,\tilde{a}_n)\}$ and $\{\tilde{b} \in L^\infty((0,1);\mathbb{M}_n)\colon \tilde{b}_{jk} \in BV(0,1) \text{ for all } 1\leqslant j\neq k\leqslant n\}$, equipped with the corresponding L^∞-norms.

The solution behavior of (1.6) under small perturbations of b and f follows directly from Theorem 1.2 and the Implicit Function Theorem, because the map

$$b \in L^\infty\bigl((0,1);\mathbb{M}_n\bigr) \mapsto \bigl(\mathcal{A}\bigl(a,b^0\bigr),\mathcal{B}\bigl(b^1\bigr)\bigr) \in \mathcal{L}\bigl(V^\gamma(a,r);W^\gamma\bigr)\times\mathcal{L}\bigl(W^\gamma\bigr) \tag{1.14}$$

is affine and continuous:

Corollary 1.3. *Suppose* (1.9), (1.11) *for some* $c > 0$, (1.12), *and*

$$\dim \ker\big(\mathcal{A}\big(a, b^0\big) + \mathcal{B}\big(b^1\big)\big) = 0. \tag{1.15}$$

Then there exists $\varepsilon > 0$ *such that for all* $\gamma \geqslant 1$, $\tilde{b} \in B_\varepsilon^\infty(b)$, *and* $f \in W^\gamma$ *there exists exactly one* $u \in V^\gamma(a, r)$ *with* $\mathcal{A}(a, \tilde{b}^0)u + \mathcal{B}(\tilde{b}^1)u = f$. *Moreover, the map*

$$(\tilde{b}, f) \in B_\varepsilon^\infty(b) \times W^\gamma \mapsto u \in V^\gamma(a, r)$$

is C^∞*-smooth.*

In particular, Corollary 1.3 implies assertion (I) above, and, because of the continuous embedding $V^\gamma(a, r) \hookrightarrow C([0, 1] \times [0, 2\pi]; \mathbb{R}^n)$ for $\gamma > 3/2$ (see Lemma 2.2(iii)), also assertion (II).

The solution behavior of (1.6) under small perturbations of a and r seems to be more complicated. Under those perturbations the function spaces $V^\gamma(a, r)$ change, in general. This makes them inappropriate. On the other hand, we don't know any Fredholmness results for the operator $\mathcal{A}(a, b^0) + \mathcal{B}(b^1)$ besides that which is described in Theorem 1.2 and, hence, which is related to the choice of the function spaces $V^\gamma(a, r)$ and W^γ.

Theorem 1.4. *Suppose* (1.15) *and*

$$a_j \in BV(0, 1), \qquad b_{jj} \in L^\infty(0, 1), \quad \text{and} \quad \inf |a_j| > 0 \quad \text{for all } j = 1, \dots, n, \tag{1.16}$$

$$b_{jk} \in BV(0, 1) \quad \text{and} \quad \inf |a_j - a_k| > 0 \quad \text{for all } 1 \leqslant j \neq k \leqslant n \tag{1.17}$$

and

$$\sum_{j,k=m+1}^{n} \sum_{l=1}^{m} e^{2(\beta_j(1) - \beta_l(1))} \big|r_{jl}^1 r_{lk}^0\big|^2 < 1. \tag{1.18}$$

Then there exists $\varepsilon > 0$ *such that for all* $\gamma \geqslant 2$, $\tilde{a} \in A_\varepsilon(a)$, $\tilde{b} \in B_\varepsilon(b)$, *and* $f \in W^\gamma$ *there exists exactly one* $u \in V^\gamma(a, r)$ *with* $\mathcal{A}(\tilde{a}, \tilde{b}^0)u + \mathcal{B}(\tilde{b}^1)u = f$. *Moreover, the map*

$$(\tilde{a}, \tilde{b}, f) \in A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \mapsto u \in W^{\gamma-k-1} \cap C\big([0, 1] \times [0, 2\pi]; \mathbb{R}^n\big) \tag{1.19}$$

is C^k*-smooth for all nonnegative integers* $k \leqslant \gamma - 1$.

In particular, for $k = \gamma - 1$ (resp. $k = \gamma - 2$) we get assertion (III) above.

The present paper has been motivated mainly by two reasons:

The first reason is that the Fredholm property of the linearization is a key for many local investigations for nonlinear equations, such as small periodic forcing of stationary solutions to nonlinear autonomous problems (see, e.g. [18]) or Hopf bifurcation (see, e.g. [7,10]). In particular, those techniques are well established for nonlinear ODEs and nonlinear parabolic PDEs, but almost nothing is known if those techniques work for nonlinear dissipative hyperbolic PDEs.

The second reason are applications to semiconductor laser dynamics [12,16,17]. Phenomena like Hopf bifurcation (describing the appearance of selfpulsations of lasers) and periodic forcing of stationary solutions (describing the modulation of stationary laser states by time periodic electric pumping) are essential for many applications of semiconductor laser devices in communication systems (see, e.g., [17]).

Remark that our smoothness assumptions concerning a_j, b_{jk}, and $f_j(\cdot, t)$ are quite weak. This is important for the applications to laser dynamics. But it turns out that any stronger smoothness assumption with respect to the space variable x would not essentially improve our results and would not simplify the proofs.

Boundary value problems for hyperbolic systems of the type (1.1), (1.3) are also used for modeling of correlated random walks (see, e.g. [3,4,6,14]).

Our paper is organized as follows: In Section 1.2 we comment about sufficient conditions for the key assumptions (1.11), (1.12), (1.15), and (1.18) and about the question if those conditions as well as the assertions of Theorems 1.1 and 1.2 are stable under small perturbations of the data. In Section 2 we introduce the main properties of the function spaces, used in this paper. In Section 3 we prove Theorem 1.1, in Sections 4 and 5 we prove Theorem 1.2, and, finally, in Section 6 we prove Theorem 1.4.

1.2. Some comments

Remark 1.5 *(About small denominators).* In Section 3 we show the following: If one considers system (1.1)–(1.3) with vanishing nondiagonal coefficients, i.e. with $b_{jk} = 0$ for $j \neq k$, and if one makes a Fourier series ansatz for the solution, one ends up with linear algebraic systems for the vector valued Fourier coefficients. The system for the Fourier coefficient of order s is uniquely solvable if and only if $\det(I - R_s(a, b^0, r)) \neq 0$. In this case $\det(I - R_s(a, b^0, r))$ appears in the denominator of the formula for the Fourier coefficient. The condition (1.11) implies that the denominators are uniformly bounded from below, thereby ensuring the convergence of the Fourier series. Using classical terminology, one can say that (1.11) allows us to avoid small denominators.

Remark 1.6 *(About the case $m = 1, n = 2$).* In the case $m = 1$, $n = 2$ the matrix $R_s(a, b^0, r)$ is the complex number

$$R_s\big(a, b^0, r\big) = e^{is(\alpha_2(1)-\alpha_1(1))+\beta_2(1)-\beta_1(1)} r_{21}^1 r_{12}^0.$$

Hence, in this case condition (1.11) is equivalent to

$$e^{\beta_2(1)-\beta_1(1)} r_{21}^1 r_{12}^0 \neq 1.$$

This fact was proved in our paper [9]. For the cases $n - m > 1$ we don't know any s-independent equivalent of condition (1.11).

Remark 1.7 *(About a sufficient condition for (1.11)).* Let us formulate, for general m and n, a sufficient condition for (1.11), in which the parameter s does not appear. Condition (1.11) is satisfied iff for all $s \in \mathbb{Z}$ the matrix $I - R_s(a, b^0, r)$ is invertible and the operator norm $\|(I - R_s(a, b^0, r))^{-1}\|$ is bounded uniformly in $s \in \mathbb{Z}$. For that it is sufficient to have

$$\big\|R_s\big(a, b^0, r\big)\big\| \leqslant \text{const} < 1 \quad \text{for all } s \in \mathbb{Z}. \tag{1.20}$$

Here we can use any operator norm in $\mathbb{M}_{n-m}$, corresponding to any norm in $\mathbb{R}^{n-m}$. If we take the Euclidean norm in $\mathbb{R}^{n-m}$, then the corresponding operator norm in $\mathbb{M}_{n-m}$ can be estimated by the Euclidean norm in $\mathbb{M}_{n-m}$. In other words: (1.20) and, hence, (1.11) are satisfied if, for example, condition (1.18) is satisfied. This can be interpreted as a kind of control on small denominators via parameters a, b^0, and r.

Example 1.8 *(About a correlated random walk model).* In the case $m = 1$, $n = 2$ the sufficient for (1.11) condition (1.20) reads as

$$\big|r_{21}^1 r_{12}^0\big| \exp \int_0^1 \left(\frac{b_{22}(x)}{a_2(x)} - \frac{b_{11}(x)}{a_1(x)}\right) dx < 1. \tag{1.21}$$

Consider the following correlated random walk model for chemotaxis (chemosensitive movement, see [5,20]), consisting of the hyperbolic system

$$\left.\begin{aligned} \partial_t u^+ + \partial_x\big(a^+(x)u^+\big) &= -\mu^+(x)u^+ + \mu^-(x)u^-, \\ \partial_t u^- - \partial_x\big(a^-(x)u^-\big) &= -\mu^-(x)u^- + \mu^+(x)u^+, \end{aligned}\right\} \quad x \in (0,1)$$

with "natural" boundary conditions

$$a^+(x)u^+(x,t) = a^-(x)u^-(x,t), \quad x = 0, 1.$$

Translating the new notation to the old one, we get

$$a_1 = a^+, \qquad a_2 = -a^-, \qquad b_{11} = \mu^+ + \partial_x a^+, \qquad b_{22} = \mu^- - \partial_x a^-$$

and

$$r_{12}^0 = \frac{a^-(0)}{a^+(0)}, \qquad r_{21}^1 = \frac{a^+(1)}{a^-(1)}.$$

Therefore

$$\exp\int_0^1 \frac{b_{11}(x)}{a_1(x)}\,dx = \exp\int_0^1 \frac{\mu^+(x) + \partial_x a^+(x)}{a^+(x)}\,dx = \frac{a^+(1)}{a^+(0)} \exp\int_0^1 \frac{\mu^+(x)}{a^+(x)}\,dx$$

and analogously

$$\exp\int_0^1 \frac{b_{22}(x)}{a_2(x)}\,dx = \frac{a^-(0)}{a^-(1)} \exp\int_0^1 \frac{\mu^+(x)}{a^+(x)}\,dx.$$

Hence, condition (1.21) is

$$\int_0^1 \left(\frac{\mu^+(x)}{a^+(x)} + \frac{\mu^-(x)}{a^-(x)}\right) dx > 0. \tag{1.22}$$

Remark 1.9 *(About small perturbations of the data in (1.11) and (1.18)).* Let us comment about the behavior of the assumption (1.11) and its sufficient condition (1.18) under small perturbations of the data.

If condition (1.18) is satisfied for given data, then it remains to be satisfied under sufficiently small perturbations of the coefficients r_{jk}^0, r_{jk}^1 and under sufficiently small (in $L^\infty(0,1)$) perturbations of the coefficient functions a_j and b_{jk}.

If condition (1.11) is satisfied, then it remains to be satisfied under sufficiently small perturbations of r_{jk}^0, r_{jk}^1, and b_{jk}, but not under small perturbations of a_j, in general. In other words, (1.11) is not sufficient for (1.18). It may happen that there exist arbitrarily small perturbations of a_j that destroy the validity of (1.11):

For example, consider the case $m = 1$, $n = 2$, $a_1(x) = \alpha$, $a_2(x) = -\alpha$, $b_{jk}(x) = 0$ for $j, k = 1, 2$, $r_{1,2}^0 = 1$, $r_{2,1}^1 = -1$. Then (1.11) reads as

$$\left|1 + e^{\frac{2is}{\alpha}}\right| \geqslant \text{const} > 0 \quad \text{for all } s \in \mathbb{Z}. \tag{1.23}$$

This is satisfied iff

$$\alpha = \frac{2l+1}{k\pi} \quad \text{with } k \in \mathbb{Z} \text{ and } l \in \mathbb{N}. \tag{1.24}$$

In this case the set of all values α such that condition (1.11) is satisfied, is dense in $\mathbb{R}$, but the set of all values α such that (1.11) is not satisfied, is dense too.

Remark 1.10 *(About Fredholmness of $\mathcal{A}(a, b^0) + \mathcal{B}(b^1)$ under small perturbations of the data).* Let us comment about the behavior of the conclusions of Theorem 1.2, mainly the Fredholmness of the operator $\mathcal{A}(a, b^0) + \mathcal{B}(b^1)$, under small perturbations of the data.

Suppose that for given data a and b the assumptions of Theorem 1.2 are satisfied. Then, under sufficiently small perturbations of b_{jk} in $L^\infty(0, 1)$, independently whether (1.12) remains to be true or not, the Fredholmness of $\mathcal{A}(a, b^0)$ survives because the map (1.14) is continuous and because the set of index zero Fredholm operators between two fixed Banach spaces is open.

But if a_j, r^0_{jk}, or r^1_{jk} are perturbed, then the function space $V^\gamma(a, r)$ is changed, in general, and it may happen that there exist arbitrarily small perturbations that destroy the Fredholmness:

For example, consider again the case $m = 1$, $n = 2$, $a_1(x) = \alpha$, $a_2(x) = -\alpha$, $f(x) = 0$, $b_{jk}(x) = 0$ for $j, k = 1, 2$, $r^0_{1,2} = 1$, $r^1_{2,1} = -1$. Then (1.11) reads as (1.23) which is equivalent to (1.24). Hence, by Theorem 1.2, if (1.24) is true, then $\mathcal{A}(a, b^0)$ is Fredholm. Condition (1.24) and, hence, condition (1.23) is not satisfied, for example, if

$$\alpha = \frac{2q}{(2p+1)\pi} \quad \text{with } p, q \in \mathbb{N}, \tag{1.25}$$

and in this case $\mathcal{A}(a, b^0)$ is not Fredholm because $\dim\ker \mathcal{A}(a, b^0) = \infty$: Indeed, we have $(u_1, u_2) \in \ker \mathcal{A}(a, b^0)$ iff

$$\partial_t u_1 + \alpha \partial_x u_1 = \partial_t u_2 - \alpha \partial_x u_2 = 0, \quad x \in [0, 1], \ t \in \mathbb{R}, \tag{1.26}$$

$$u_j(x, t + 2\pi) = u_j(x, t), \quad j = 1, 2, \ x \in [0, 1], \ t \in \mathbb{R}, \tag{1.27}$$

$$u_1(0, t) = u_2(0, t), \qquad u_2(1, t) = -u_1(1, t), \quad t \in \mathbb{R}. \tag{1.28}$$

The solutions of (1.26) are of the type $u_1(x, t) = U_1(t - \frac{x}{\alpha})$ and $u_2(x, t) = U_2(t + \frac{x}{\alpha})$. They satisfy (1.27) iff the functions U_1 and U_2 are 2π-periodic. From the boundary condition in $x = 0$ follows $U_1 = U_2$, and, hence, the boundary condition in $x = 1$ reads as

$$U_1\left(t - \frac{1}{\alpha}\right) = -U_1\left(t + \frac{1}{\alpha}\right). \tag{1.29}$$

Choosing $U_1(y) = \sin(ry)$, $r \in \mathbb{Z}$, and using (1.25), condition (1.29) transforms into

$$\sin\left(r\left(t - \frac{2p+1}{2q}\pi\right)\right) = -\sin\left(r\left(t + \frac{2p+1}{2q}\pi\right)\right).$$

This is fulfilled, for example, for $r = (2k + 1)q$ and any choice of $k \in \mathbb{Z}$, i.e. we found infinitely many linearly independent solutions to (1.26)–(1.28).

The set of all values α of the type (1.25) is dense in $[0, \infty)$. Hence, we get: In this case the set of all $\alpha > 0$ such that $\mathcal{A}(a, b^0) + \mathcal{B}(b^1)$ is Fredholm, is dense in $[0, \infty)$, but the set of all $\alpha > 0$ such that $\mathcal{A}(a, b^0) + \mathcal{B}(b^1)$ is not Fredholm, is dense in $[0, \infty)$ too.

Remark 1.11 *(About assumptions (1.12) and (1.17)).* Obviously, the condition (1.12) is not necessary for the conclusions of Theorem 1.2 because the conclusions of Theorem 1.2 survive under small (in $L^\infty(0,1)$) perturbations of the coefficients b_{jk}, but the assumption (1.12) does not, in general.

The following example shows that Theorem 1.2 is not true, in general, if all its assumptions are fulfilled with the exception of (1.12): Take $m=1$, $n=2$, $a_1(x)=a_2(x)=1$, $b_{11}(x)=b_{12}(x)=b_{22}(x)=f_1(x,t)=f_2(x,t)=0$, $b_{21}=b=\text{const}$. Then (1.1)–(1.3) looks like

$$\partial_t u_1+\partial_x u_1=\partial_t u_2+\partial_x u_2+bu_1=0,$$
$$u_1(x,t+2\pi)-u_1(x,t)=u_2(x,t+2\pi)-u_2(x,t)=0,$$
$$u_1(0,t)-r_{12}^0u_2(0,t)=u_2(1,t)-r_{21}^1u_1(1,t)=0.$$

If $r_{12}^0r_{21}^1<1$ and $b\neq 0$, then all assumptions of Theorem 1.2 are fulfilled with the exception of (1.12). If, moreover,

$$b=\frac{r_{12}^0r_{21}^1-1}{r_{12}^0},$$

then

$$u_1(x,t)=\sin l(t-x),\qquad u_2(x,t)=b\left(\frac{1}{1-r_{12}^0r_{21}^1}-x\right)\sin l(t-x),\quad l\in\mathbb{N},$$

are infinitely many linearly independent solutions. Hence, the conclusion of Theorem 1.2 is not true.

Finally, let us remark that, surprisingly, the assumption (1.12) is used also in quite another circumstances, for proving the spectrum-determined growth condition in L^p-spaces [1,13,15] and in C-spaces [11] for semiflows generated by hyperbolic systems of the type (1.1), (1.3).

Remark 1.12 *(About sufficient conditions for (1.15)).* Similarly to [8], one can provide a wide range of sufficient conditions for (1.15). We here concentrate on the physically relevant case

$$\sum_{j=1}^{m}\sum_{k=m+1}^{n}\left|r_{jk}^0\right|^2\leqslant 1\quad\text{and}\quad\sum_{j=m+1}^{n}\sum_{k=1}^{m}\left|r_{jk}^1\right|^2\leqslant 1. \tag{1.30}$$

If (1.1)–(1.3) with $f=0$ is satisfied, then

$$0=\int_0^{2\pi}\left(u_j^2(1,t)-u_j^2(0,t)\right)dt+2\int_0^{2\pi}\int_0^1 a_j^{-1}(x)\left(b_{jj}(x)u_j^2+\sum_{k\neq j}b_{jk}(x)u_ju_k\right)dx\,dt. \tag{1.31}$$

Using the reflection boundary conditions, summing up separately the first m equations of (1.31) and the rest $n-m$ equations of (1.31), and subtracting the second resulting equality from the first one, we get

$$\int_0^{2\pi}\left(\sum_{j=1}^{m}u_j^2(1,t)-\sum_{j=m+1}^{n}\left(\sum_{k=1}^{m}r_{jk}^1u_k(1,t)\right)^2+\sum_{j=m+1}^{n}u_j^2(0,t)-\sum_{j=1}^{m}\left(\sum_{k=m+1}^{n}r_{jk}^0u_k(0,t)\right)^2\right)dt$$

$$+2\sum_{j=1}^{m}\int_0^{2\pi}\int_0^1 \frac{1}{a_j(x)}\left(b_{jj}(x)u_j^2+\sum_{k\neq j}b_{jk}(x)u_ju_k\right)dx\,dt$$
$$-2\sum_{j=m+1}^{n}\int_0^{2\pi}\int_0^1 \frac{1}{a_j(x)}\left(b_{jj}(x)u_j^2+\sum_{k\neq j}b_{jk}(x)u_ju_k\right)dx\,dt=0. \tag{1.32}$$

Applying Hölder's inequality and assumption (1.30), we derive that

$$\int_0^{2\pi}\left(\sum_{j=1}^{m}u_j^2(1,t)-\sum_{j=m+1}^{n}\left(\sum_{k=1}^{m}r_{jk}^1u_k(1,t)\right)^2\right)dt\geqslant\left(1-\sum_{j=m+1}^{n}\sum_{k=1}^{m}\left|r_{jk}^1\right|^2\right)\int_0^{2\pi}\sum_{j=1}^{m}u_j(1,t)^2\,dt\geqslant 0.$$

A similar estimate is true for the second boundary summand in (1.32) as well. Set

$$c_{jk}(x):=\frac{b_{jk}(x)}{a_j(x)}\quad\text{for } 1\leqslant j\leqslant m\quad\text{and}\quad c_{jk}(x):=-\frac{b_{jk}(x)}{a_j(x)}\quad\text{for } m+1\leqslant j\leqslant n.$$

Then (1.30) together with

$$\sum_{j,k=1}^{n}c_{jk}(x)\xi_j\xi_k\geqslant C\sum_{j=1}^{n}|\xi_j|^2\quad\text{for all }\xi\in\mathbb{R}^n\text{ and a.a. }x\in(0,1), \tag{1.33}$$

where the constant $C>0$ does not depend on ξ and x, is sufficient for (1.15). It is easily seen that estimate (1.33) is true if, for instance,

$$\operatorname{ess\,inf}\left\{\frac{b_{jj}}{a_j}-\sum_{k\neq j}\left(\left|\frac{b_{jk}}{a_j}\right|+\left|\frac{b_{jk}}{a_k}\right|\right)\right\}>0\quad\text{for all } j=1,\dots,m,$$
$$\operatorname{ess\,inf}\left\{-\frac{b_{jj}}{a_j}-\sum_{k\neq j}\left(\left|\frac{b_{jk}}{a_j}\right|+\left|\frac{b_{jk}}{a_k}\right|\right)\right\}>0\quad\text{for all } j=m+1,\dots,n.$$

Summarizing, we get: In order the main conditions (1.15) and (1.18) to be satisfied, it is sufficient that (1.30) is fulfilled as well as

$$\operatorname{ess\,inf} a_j>0\quad\text{for } j=1,\dots,m,$$
$$\operatorname{ess\,sup} a_j<0\quad\text{for } j=m+1,\dots,n,$$
$$\operatorname{ess\,inf} b_{jj}>0\quad\text{for } j=1,\dots,n,$$
$$\operatorname{ess\,sup}|b_{jk}|\approx 0\quad\text{for } 1\leqslant j\neq k\leqslant n.$$

2. Some properties of the used function spaces

In this section we formulate some properties of the function spaces W^γ, $V^\gamma(a,r)$, and $U^\gamma(a)$ introduced in Section 1. For each $u\in W^\gamma$ we have

$$u(x,t)=\sum_{s\in\mathbb{Z}} u^s(x)e^{ist} \quad \text{with } u^s(x):=\frac{1}{2\pi}\int_0^{2\pi} u(x,t)e^{-ist}\,dt, \tag{2.1}$$

where $u^s \in L^2((0,1);\mathbb{C}^n)$, and the series in (2.1) converges to u in the complexification of W^γ. And vice versa: For any sequence $(u^s)_{s\in\mathbb{Z}}$ with

$$u^s \in L^2\big((0,1);\mathbb{C}^n\big), \qquad u^{-s}=\overline{u^s}, \qquad \sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma \big\|u^s\big\|^2_{L^2((0,1);\mathbb{C}^n)}<\infty \tag{2.2}$$

there exists exactly one $u \in W^\gamma$ with (2.1). In what follows, we will identify functions $u \in W^\gamma$ and sequences $(u^s)_{s\in\mathbb{Z}}$ with (2.2) by means of (2.1), and we will keep for the functions and the sequences the notations u and $(u^s)_{s\in\mathbb{Z}}$, respectively.

The following lemma gives a compactness criterion in W^γ (see [9, Lemma 6]):

Lemma 2.1. *A set $M \subset W^\gamma$ is precompact in W^γ if and only if the following two conditions are satisfied*:

(i) *There exists $C>0$ such that for all $u \in M$ it holds*

$$\sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma \int_0^1 \big\|u^s(x)\big\|^2\,dx \leqslant C.$$

(ii) *For all $\varepsilon>0$ there exists $\delta>0$ such that for all $\xi,\tau \in (-\delta,\delta)$ and all $u \in M$ it holds*

$$\sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma \int_0^1 \big\|u^s(x+\xi)e^{is\tau}-u^s(x)\big\|^2\,dx<\varepsilon,$$

where $u^s(x+\xi):=0$ for $x+\xi \notin [0,1]$.

Concerning the spaces $U^\gamma(a)$ we have the following result:

Lemma 2.2.

(i) *The space $U^\gamma(a)$ is complete.*
(ii) *If $\gamma \geqslant 1$, then for any $x \in [0,1]$ there exists a continuous trace map $u \in U^\gamma(a) \mapsto u(x,\cdot) \in L^2((0,2\pi);\mathbb{R}^n)$.*
(iii) *If $\gamma>3/2$, then $U^\gamma(a)$ is continuously embedded into $C([0,1]\times[0,2\pi];\mathbb{R}^n)$.*

Proof. (i) Let $(u^k)_{k\in\mathbb{N}}$ be a fundamental sequence in $U^\gamma(a)$. Then $(u^k)_{k\in\mathbb{N}}$ and $(\partial_t u^k + a\partial_x u^k)_{k\in\mathbb{N}}$ are fundamental sequences in W^γ. This implies that $(\partial_t u^k)_{k\in\mathbb{N}}$ and, hence, $(a\partial_x u^k)_{k\in\mathbb{N}}$ are fundamental sequences in $W^{\gamma-1}$. On the account of $a_j \in L^\infty(0,1)$ and $\operatorname{ess\,inf}|a_j|>0$ for all $j=1,\dots,n$, the latter entails that $(\partial_x u^k)_{k\in\mathbb{N}}$ is a fundamental sequence in $W^{\gamma-1}$ as well. Because W^γ is complete for any γ, there exist $u \in W^\gamma$ and $v,w \in W^{\gamma-1}$ such that

$$u^k \to u \quad \text{in } W^\gamma, \qquad \partial_t u^k \to v \quad \text{in } W^{\gamma-1}, \qquad \partial_x u^k \to w \quad \text{in } W^{\gamma-1} \text{ as } k\to\infty.$$

It is obvious that $\partial_t u = v$ and $\partial_x u = w$ in the sense of the generalized derivatives: Take a smooth function $\varphi:(0,1)\times(0,2\pi)\to\mathbb{R}^n$ with compact support. Then

$$\int_0^{2\pi}\int_0^1 \langle u, \partial_t\varphi\rangle\,dx\,dt = \lim_{k\to\infty}\int_0^{2\pi}\int_0^1 \langle u^k, \partial_t\varphi\rangle\,dx\,dt = -\lim_{k\to\infty}\int_0^{2\pi}\int_0^1 \langle \partial_t u^k, \varphi\rangle\,dx\,dt = -\int_0^{2\pi}\int_0^1 \langle v, \varphi\rangle\,dx\,dt,$$

and similarly for $\partial_x u$ and w. Hence $\partial_t u + a\partial_x u = v + aw$ in $W^{\gamma-1}$. Since $(\partial_t u^k + a\partial_x u^k)_{k\in\mathbb{N}}$ is fundamental in W^γ, then $\partial_t u + a\partial_x u = v + aw$ in W^γ as desired.

Properties (ii) and (iii) can be proved similarly to [9, Lemma 8 and Remark 9]. □

Now, let us consider the dual spaces $(W^\gamma)^*$.

Obviously, for any $\gamma \geqslant 0$ the spaces W^γ are densely and continuously embedded into the Hilbert space $L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$. Hence, there is a canonical dense continuous embedding

$$L^2\big((0,1)\times(0,2\pi);\mathbb{R}^n\big) \hookrightarrow \big(W^\gamma\big)^* : [u,v]_{W^\gamma} = \langle u, v\rangle_{L^2} \tag{2.3}$$

for all $u \in L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$ and $v \in W^\gamma$. Here $[\cdot,\cdot]_{W^\gamma} : (W^\gamma)^* \times W^\gamma \to \mathbb{R}$ is the dual pairing, and $\langle\cdot,\cdot\rangle_{L^2}$ is the scalar product introduced in (1.13).

Let us denote

$$e_s(t) := e^{ist} \quad \text{for } s \in \mathbb{Z} \text{ and } t \in \mathbb{R}. \tag{2.4}$$

If a sequence $(\varphi^s)_{s\in\mathbb{Z}}$ with $\varphi^s \in L^2((0,1);\mathbb{C}^n)$ is given, then the pointwise products $\varphi^s e_s$ belong to $L^2((0,1)\times(0,2\pi);\mathbb{C}^n)$. Hence, they belong to the complexification of $(W^\gamma)^*$ (by means of the complexified version of (2.3)), and it makes sense to ask if the series

$$\sum_{s\in\mathbb{Z}} \varphi^s e_s \tag{2.5}$$

converges in the complexification of $(W^\gamma)^*$. Moreover, we have (see [9, Lemma 10])

Lemma 2.3.

(i) *For any $\varphi \in (W^\gamma)^*$ there exists a sequence $(\varphi^s)_{s\in\mathbb{Z}}$ with*

$$\varphi^s \in L^2\big((0,1);\mathbb{C}^n\big), \qquad \varphi^{-s} = \overline{\varphi^s}, \qquad \sum_{s\in\mathbb{Z}}\big(1+s^2\big)^{-\gamma}\big\|\varphi^s(x)\big\|^2_{L^2((0,1);\mathbb{C}^n)} < \infty, \tag{2.6}$$

and the series (2.5) *converges to φ in the complexification of $(W^\gamma)^*$. Moreover, it holds*

$$\int_0^1 \big\langle \varphi^s(x), u(x)\big\rangle\,dx = [\varphi, ue_{-s}]_{W^\gamma} \quad \text{for all } s \in \mathbb{Z} \text{ and } u \in L^2\big((0,1);\mathbb{R}^n\big). \tag{2.7}$$

(ii) *For any sequence $(\varphi^s)_{s\in\mathbb{Z}}$ with* (2.6) *the series* (2.5) *converges in the complexification of $(W^\gamma)^*$ to some $\varphi \in (W^\gamma)^*$, and* (2.7) *is satisfied.*

3. Isomorphism property (proof of Theorem 1.1)

Let $\gamma \geqslant 1$ and $f \in W^\gamma$ be arbitrarily fixed. We have $f(x,t) = \sum_{s\in\mathbb{Z}} f^s(x)e^{ist}$ with

$$f^s \in L^2\big((0,1);\mathbb{C}^n\big), \qquad \sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma \int_0^1 \big\|f^s(x)\big\|^2\,dx < \infty. \tag{3.1}$$

We have to show that, if (1.9), (1.10), and (1.11) hold, then there exists exactly one $u \in V^\gamma(a,r)$ with

$$\mathcal{A}\big(a,b^0\big)u = f \quad \text{and} \quad \|u\|_{V^\gamma(a,r)} \leqslant C\|f\|_{W^\gamma},$$

where the constant C does not depend on γ, a, b^0, u, and f, but only on the constant c, which was introduced in the assumptions of Theorem 1.1. But

$$\|u\|_{V^\gamma(a,r)} = \|u\|_{W^\gamma} + \|\partial_t u + a\partial_x u\|_{W^\gamma} = \|u\|_{W^\gamma} + \big\|f - b^0 u\big\|_{W^\gamma},$$

hence we have to show that there exists exactly one $u \in V^\gamma(a,r)$ with

$$\mathcal{A}\big(a,b^0\big)u = f \quad \text{and} \quad \|u\|_{W^\gamma} \leqslant C\|f\|_{W^\gamma} \tag{3.2}$$

with a constant C, which does not depend on γ, a, b^0, u, and f, but only on c.

Writing u as series according to (2.1) and (2.2), it is easy to see that (3.2) is satisfied if for all $s \in \mathbb{Z}$ we have $u^s \in H^1((0,1);\mathbb{C}^n)$ and

$$a_j(x)\frac{d}{dx}u_j^s(x) + \big(is + b_{jj}(x)\big)u_j^s(x) = f_j^s(x), \quad j = 1,\ldots,n, \tag{3.3}$$

$$\left.\begin{aligned} u_j^s(0) &= \sum_{k=m+1}^{n} r_{jk}^0 u_k^s(0), \quad j = 1,\ldots,m, \\ u_j^s(1) &= \sum_{k=1}^{m} r_{jk}^1 u_k^s(1), \quad j = m+1,\ldots,n, \end{aligned}\right\} \tag{3.4}$$

$$\sum_{s\in\mathbb{Z}} \big(1+s^2\big)^\gamma \int_0^1 \big|u_j^s(x)\big|^2\,dx \leqslant C\|f\|_{W^\gamma}, \quad j = 1,\ldots,n. \tag{3.5}$$

And vice versa: If (3.2) is satisfied, then we have $u^s \in H^1((0,1);\mathbb{C}^n)$ and (3.3)–(3.5). Indeed, take a smooth test function $\varphi : (0,1) \to \mathbb{R}$ with compact support. Then we have

$$\begin{aligned} \int_0^1 \frac{f_j^s(x)\varphi(x)}{a_j(x)}\,dx &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{(\partial_t u_j(x,t) + a_j(x)\partial_x u_j(x,t) + b_{jj}(x)u_j(x,t))\varphi(x)e^{-ist}}{a_j(x)}\,dt\,dx \\ &= \int_0^1 \left(-u_j^s(x)\varphi'(x) + \frac{(is + b_{jj}(x))u_j^s(x)\varphi(x)}{a_j(x)}\right)dx. \end{aligned}$$

This implies $u_j^s \in H^1((0,1);\mathbb{C})$ and (3.3). After that it follows easily that also the boundary conditions (3.4) are fulfilled.

Now we are going to show that there exists exactly one tuple of sequences $(u_j^s)_{s\in\mathbb{Z}}$, $j = 1,\ldots,n$, with $u_j^s \in H^1((0,1);\mathbb{C})$ satisfying (3.3)–(3.5).

By means of the variation of constants formula, (3.3) is fulfilled if and only if

$$u_j^s(x) = e^{-is\alpha_j(x)-\beta_j(x)}\left(u_j^s(0) + \int_0^x e^{is\alpha_j(y)+\beta_j(y)}\frac{f_j^s(y)}{a_j(y)}\,dy\right), \tag{3.6}$$

where the functions α_j and β_j are defined in (1.8). The boundary conditions (3.4) are satisfied if and only if

$$u_j^s(0) = \sum_{k=m+1}^{n} r_{jk}^0 u_k^s(0), \quad j = 1, \dots, m, \tag{3.7}$$

and

$$e^{-is\alpha_j(1)-\beta_j(1)}\left(u_j^s(0) + \int_0^1 e^{is\alpha_j(y)+\beta_j(y)} \frac{f_j^s(y)}{a_j(y)}\,dy\right)$$
$$= \sum_{k=1}^{m} r_{jk}^1 e^{-is\alpha_k(1)-\beta_k(1)}\left(u_k^s(0) + \int_0^1 e^{is\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)}\,dy\right), \quad j = m+1, \dots, n.$$

This is equivalent to (3.7),

$$e^{-is\alpha_j(1)-\beta_j(1)} u_j^s(0) - \sum_{k=1}^{m} \sum_{p=m+1}^{n} e^{-is\alpha_k(1)-\beta_k(1)} r_{jk}^1 r_{kp}^0 u_p^s(0)$$
$$= -e^{-is\alpha_j(1)-\beta_j(1)} \int_0^1 e^{is\alpha_j(y)+\beta_j(y)} \frac{f_j^s(y)}{a_j(y)}\,dy$$
$$+ \sum_{k=1}^{m} e^{-is\alpha_k(1)-\beta_k(1)} r_{jk}^1 \int_0^1 e^{is\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)}\,dy, \quad j = m+1, \dots, n. \tag{3.8}$$

The system (3.8) has a unique solution $(u_{m+1}^s(0), \dots, u_n^s(0))$ if and only if its coefficient matrix $I - R_s(a, b^0, r)$ (where $R_s(a, b^0, r)$ is introduced in (1.7)) is regular. If, moreover, assumptions (1.10)–(1.11) are satisfied, then there exist coefficients c_{jk}^s and a constant C such that

$$u_j^s(0) = \sum_{k=1}^{n} c_{jk}^s e^{-is\alpha_k(1)-\beta_k(1)} \int_0^1 e^{is\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)}\,dy, \quad j = m+1, \dots, n, \tag{3.9}$$

and $|c_{jk}^s| \leqslant C$ uniformly with respect to a, b^0, r, and $s \in \mathbb{Z}$ with (1.10)–(1.11). Hence, for each $s \in \mathbb{Z}$ the boundary value problem (3.3)–(3.4) is uniquely solvable, and we have the integral representation (3.6) of the solution, where $u_j^s(0)$ for $1 \leqslant j \leqslant m$ is given by (3.7) and for $m+1 \leqslant j \leqslant n$ by (3.9). Putting this together, we get

$$u_j^s(x) = e^{-is\alpha_j(x)-\beta_j(x)}\left(\int_0^x e^{is\alpha_j(y)+\beta_j(y)} \frac{f_j^s(y)}{a_j(y)}\,dy\right.$$
$$\left.+ \sum_{k=1}^{n} d_{jk}^s e^{-is\alpha_k(1)-\beta_k(1)} \int_0^1 e^{is\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)}\,dy\right), \quad j = 1, \dots, n, \tag{3.10}$$

with certain coefficients d_{jk}^s such that there exists a constant C (depending neither on f nor on a, b^0, r, and $s \in \mathbb{Z}$ satisfying (1.10)–(1.11)) with

$$\left|d_{jk}^s\right| \leqslant C. \tag{3.11}$$

In addition, (3.10) and (3.11) imply that there exists a constant C (depending neither on f nor on a, b^0, r, and $s \in \mathbb{Z}$ satisfying (1.10)–(1.11)) such that

$$\left|u_j^s(x)\right| \leqslant C \int_0^1 \left\| f^s(x) \right\| dx. \tag{3.12}$$

The estimate (3.5) now follows from (3.1).

4. Fredholmness property (proof of Theorem 1.2)

In Sections 4 and 5 we suppose the data a, b, and r to be fixed and to satisfy (1.11)–(1.12). Hence we will omit the arguments in the operators and the spaces:

$$\mathcal{A} := \mathcal{A}\bigl(a, b^0\bigr), \qquad \mathcal{B} := \mathcal{B}\bigl(b^1\bigr), \qquad \tilde{\mathcal{A}} := \tilde{\mathcal{A}}\bigl(a, b^0\bigr), \qquad \tilde{\mathcal{B}} := \tilde{\mathcal{B}}\bigl(b^1\bigr),$$
$$V^\gamma := V^\gamma(a, r), \qquad \tilde{V}^\gamma := \tilde{V}^\gamma(a, r).$$

In this section we prove that $\mathcal{A} + \mathcal{B}$ is Fredholm from V^γ into W^γ, which is part of the assertions of Theorem 1.2.

Obviously, $\mathcal{A} + \mathcal{B}$ is Fredholm from V^γ into W^γ if and only if $I + \mathcal{B}\mathcal{A}^{-1}$ is Fredholm from W^γ into W^γ. Here I is the identity in W^γ.

We will prove that $I + \mathcal{B}\mathcal{A}^{-1}$ is Fredholm from W^γ into W^γ using the following abstract criterion for Fredholmness (see, e.g., [9, Lemma 11] and [22, Proposition 5.7.1]):

Lemma 4.1. *Let W be a Banach space, I the identity in W, and $\mathcal{C} \in \mathcal{L}(W)$ such that $\mathcal{C}^2$ is compact. Then $I + \mathcal{C}$ is Fredholm.*

In order to use Lemma 4.1 with $W := W^\gamma$ and $\mathcal{C} := \mathcal{B}\mathcal{A}^{-1}$ we have to show that $(\mathcal{B}\mathcal{A}^{-1})^2$ is compact from W^γ into W^γ. For this purpose we will use Lemma 2.1.

Condition (i) of Lemma 2.1 is satisfied because $\mathcal{B}\mathcal{A}^{-1}$ is a bounded operator from W^γ into W^γ.

It remains to check condition (ii) of Lemma 2.1. For this purpose we will use the integral representation (3.10) of $\mathcal{A}^{-1}$:

Take a bounded set $N \subset W^\gamma$ and $f \in N$. Denote $u := \mathcal{A}^{-1} f$ and $\tilde{u} := (\mathcal{B}\mathcal{A}^{-1})^2 f$. Then

$$\tilde{u}_j^s(x) = \sum_{k \neq j} b_{jk}(x) e^{-is\alpha_k(x) - \beta_k(x)} \left(\int_0^x e^{is\alpha_k(y) + \beta_k(y)} a_k^{-1}(y) \sum_{l \neq k} b_{kl}(y) u_l^s(y)\, dy \right.$$
$$\left. + \sum_{l=1}^n d_{kl}^s e^{-is\alpha_l(1) - \beta_l(1)} \int_0^1 e^{is\alpha_l(y) + \beta_l(y)} a_l^{-1}(y) \sum_{r \neq l} b_{lr}(y) u_r^s(y)\, dy \right).$$

Therefore $\tilde{u}_j^s(x + \xi) e^{is\tau} - \tilde{u}_j^s(x) = P_j^s(x, \xi, \tau) + Q_j^s(x, \xi, \tau) + R_j^s(x, \xi)$ with

$$P_j^s(x, \xi, \tau) := \sum_{j \neq k \neq l} \int_x^{x+\xi} e^{is(-\alpha_k(x+\xi) + \tau + \alpha_k(y)) - \beta_k(x+\xi) + \beta_k(y)} \times a_k^{-1}(y) b_{jk}(x + \xi) b_{kl}(y) u_l^s(y)\, dy,$$

$$Q_j^s(x,\xi,\tau) := \sum_{k\neq j} b_{jk}(x+\xi)e^{-\beta_k(x+\xi)}\big(e^{is(-\alpha_k(x+\xi)+\tau)} - e^{-is\alpha_k(x)}\big)S_k^s(x),$$

$$R_j^s(x,\xi) := \sum_{k\neq j}\big(b_{jk}(x+\xi)e^{-\beta_k(x+\xi)} - b_{jk}(x)e^{-\beta_k(x)}\big)S_k^s(x)$$

and

$$S_k^s(x) := \int_0^x e^{is\alpha_k(y)+\beta_k(y)}a_k^{-1}(y)\sum_{l\neq k} b_{kl}(y)u_l^s(y)\,dy + \sum_{l=1}^n d_{kl}^s e^{-is\alpha_l(1)-\beta_l(1)}\int_0^1 e^{is\alpha_l(y)+\beta_l(y)}a_l^{-1}(y)\sum_{r\neq l} b_{lr}(y)u_r^s(y)\,dy. \tag{4.1}$$

We have to show that

$$\sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma\int_0^1\big(\big|P_j^s(x,\xi,\tau)\big|^2 + \big|Q_j^s(x,\xi,\tau)\big|^2 + \big|R_j^s(x,\xi)\big|^2\big)\,dx \to 0$$

for $|\xi|+|\tau|\to 0$ uniformly with respect to $f\in N$.

Because of $\mathcal{A}u = f$ we have (3.12). This implies

$$\sum_{s\in\mathbb{Z}}\big(1+s^2\big)^\gamma\int_0^1\big|P_j^s(x,\xi,\tau)\big|^2\,dx \leqslant C\xi^2\|f\|_{W^\gamma}^2, \tag{4.2}$$

where the constant C does not depend on j, ξ, τ, and f. Hence, the left-hand side of (4.2) tends to zero for $|\xi|\to 0$ uniformly with respect to $f\in N$.

In order to estimate $Q_j^s(x,\xi,\tau)$ and $R_j^s(x,\xi)$, let us first estimate $S_j^s(x)$. Again we use $\mathcal{A}u = f$. From (3.3) it follows

$$\frac{d}{dy}\big(e^{is\alpha_l(y)}u_l^s(y)\big) = e^{is\alpha_l(y)}\frac{f_l^s(y) - b_{ll}(y)u_l^s(y)}{a_l(y)}.$$

Using this, we get

$$\begin{aligned} is\frac{a_l(y)-a_k(y)}{a_k(y)a_l(y)}e^{is\alpha_k(y)}u_l^s(y) &= e^{is(\alpha_k(y)-\alpha_l(y))}\frac{d}{dy}\big(e^{is\alpha_l(y)}u_l^s(y)\big) - \frac{d}{dy}\big(e^{is\alpha_k(y)}u_l^s(y)\big)\\ &= e^{is\alpha_k(y)}\frac{f_l^s(y)-b_{ll}(y)u_l^s(y)}{a_l(y)} - \frac{d}{dy}\big(e^{is\alpha_k(y)}u_l^s(y)\big). \end{aligned}$$

Therefore

$$\begin{aligned} e^{is\alpha_k(y)+\beta_k(y)}\frac{b_{kl}(y)}{a_k(y)}u_l^s(y) &= \frac{e^{\beta_k(y)}}{is}\frac{a_l(y)b_{kl}(y)}{a_l(y)-a_k(y)}\\ &\quad\times\left(e^{is\alpha_k(y)}\frac{f_l^s(y)-b_{ll}(y)u_l^s(y)}{a_l(y)} - \frac{d}{dy}\big(e^{is\alpha_k(y)}u_l^s(y)\big)\right). \end{aligned} \tag{4.3}$$

Moreover, because of assumption (1.12), for all $k \neq l$ the function

$$y \in [0,1] \mapsto e^{\beta_k(y)} \frac{a_l(y) b_{kl}(y)}{a_l(y) - a_k(y)}$$

is in $BV(0,1)$. Hence,

$$\left| \int_0^x e^{\beta_k(y)} \frac{a_l(y) b_{kl}(y)}{a_l(y) - a_k(y)} \frac{d}{dy} \left(e^{is\alpha_k(y)} u_l^s(y) \right) dy \right| \leqslant C \left\| u_l^s \right\|_\infty, \tag{4.4}$$

the constant C being independent of x, k, l, s, and u. Therefore, (3.12) and (4.3) imply

$$\left| \int_0^x e^{is\alpha_k(y)+\beta_k(y)} \frac{b_{kl}(y)}{a_k(y)} u_l^s(y)\, dy \right| \leqslant \frac{C}{1+|s|} \int_0^1 \left\| f^s(y) \right\| dy \tag{4.5}$$

for some constant C being independent of x, k, l, s, and f. Similar estimates are true for all other integrals in (4.1). As a consequence,

$$\left| S_j^s(x) \right| \leqslant \frac{C}{1+|s|} \int_0^1 \left\| f^s(y) \right\| dy, \tag{4.6}$$

where C does not depend on x, j, s, and f. This gives

$$\int_0^1 \left| Q_j^s(x,\xi,\tau) \right|^2 dx \leqslant \frac{C}{1+|s|} \max_{k=1,\dots,n} \left| e^{is(-\alpha_k(x+\xi)+\tau)} - e^{-is\alpha_k(x)} \right|^2 \int_0^1 \left\| f^s(y) \right\|^2 dy,$$

where C does not depend on x, ξ, τ, j, s, and f. But assumption (1.9) and notation (1.8) imply that

$$\left| e^{is(-\alpha_k(x+\xi)+\tau)} - e^{-is\alpha_k(x)} \right| \leqslant Cs\left(|\xi| + |\tau| \right),$$

hence

$$\sum_{s\in\mathbb{Z}} \left(1+s^2\right)^\gamma \int_0^1 \left| Q_j^s(x,\xi,\tau) \right|^2 dx \leqslant C\left(\xi^2+\tau^2\right) \|f\|_{W^\gamma}^2, \tag{4.7}$$

where the constants, again, do not depend on j, k, ξ, τ, and f. Hence, the left-hand side of (4.7) tends to zero for $|\xi| + |\tau| \to 0$ uniformly with respect to $f \in N$.

Finally, (4.6) gives

$$\begin{aligned} &\sum_{s\in\mathbb{Z}} \left(1+s^2\right)^\gamma \int_0^1 \left| R_j^s(x,\xi) \right|^2 dx \\ &\quad \leqslant C \max_{k=1,\dots,n} \int_0^1 \left| b_{jk}(x+\xi) e^{-\beta_k(x+\xi)} - b_{jk}(x) e^{-\beta_k(x)} \right|^2 dx \, \|f\|_{W^\gamma}^2, \end{aligned} \tag{4.8}$$

where the constant C does not depend on j, ξ, τ, and f. Hence, the left-hand side of (4.8) tends to zero for $|\xi| \to 0$ uniformly with respect to $f \in N$ because of the continuity in the mean of the functions $x \mapsto b_{jk}(x)e^{-\beta_k(x)}$.

5. Fredholm alternative (still proof of Theorem 1.2)

To finish the proof of the assertion (i) of Theorem 1.2, it remains to show that the index of the operator $I + \mathcal{B}\mathcal{A}^{-1}$ is zero. This is a straightforward consequence of Lemma 4.1 and a homotopy argument: Since $(\mathcal{B}\mathcal{A}^{-1})^2 \in \mathcal{L}(W^\gamma)$ is a compact operator, the operators $(s\mathcal{B}\mathcal{A}^{-1})^2 \in \mathcal{L}(W^\gamma)$ are compact for any $s \in \mathbb{R}$ as well. By Lemma 4.1, the operators $I + s\mathcal{B}\mathcal{A}^{-1}$ are Fredholm. Furthermore, they depend continuously on s. Since I has index zero, the homotopy argument gives the same property for the operator $I + s\mathcal{B}\mathcal{A}^{-1}$ for any $s \in \mathbb{R}$, in particular, for $s = 1$. Assertion (i) is thereby proved.

Summarizing, we proved the Fredholm alternative for $\mathcal{A} + \mathcal{B} \in \mathcal{L}(V^\gamma, W^\gamma)$. Hence, we have

$$\left.\begin{aligned} &\dim\ker(\mathcal{A}+\mathcal{B}) = \dim\ker(\mathcal{A}+\mathcal{B})^* < \infty, \\ &\operatorname{im}(\mathcal{A}+\mathcal{B}) = \left\{f \in W^\gamma\colon [\varphi, f]_{W^\gamma} = 0 \text{ for all } \varphi \in \ker(\mathcal{A}+\mathcal{B})^*\right\}. \end{aligned}\right\} \tag{5.1}$$

Here $(\mathcal{A}+\mathcal{B})^*$ is the dual operator to $\mathcal{A}+\mathcal{B}$, i.e. a linear bounded operator from $(W^\gamma)^*$ into $(V^\gamma)^*$, and $[\cdot,\cdot]_{W^\gamma} : (W^\gamma)^* \times W^\gamma \to \mathbb{R}$ is the dual pairing in W^γ.

To prove assertion (ii) of Theorem 1.2, we have to prove something slightly different, namely, that

$$\operatorname{im}(\mathcal{A}+\mathcal{B}) = \left\{f \in W^\gamma\colon \langle f, u\rangle_{L^2} = 0 \text{ for all } u \in \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})\right\}$$

and that $\ker(\mathcal{A}+\mathcal{B})$ and $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$ do not depend on γ. Here $\langle\cdot,\cdot\rangle_{L^2}$ is the scalar product in $W^0 = L^2((0,1)\times(0,2\pi);\mathbb{R}^n)$ introduced in (1.13).

Directly from the definitions of the operators $\mathcal{A}$, $\tilde{\mathcal{A}}$, $\mathcal{B}$, and $\tilde{\mathcal{B}}$ it follows

$$\left\langle(\mathcal{A}+\mathcal{B})u, \tilde{u}\right\rangle_{L^2} = \left\langle u, (\tilde{\mathcal{A}}+\tilde{\mathcal{B}})\tilde{u}\right\rangle_{L^2} \quad \text{for all } u \in V^\gamma \text{ and } \tilde{u} \in \tilde{V}^\gamma. \tag{5.2}$$

Using the continuous dense embedding (cf. (2.3)) $\tilde{V}^\gamma \hookrightarrow W^\gamma \hookrightarrow W^0 \hookrightarrow (W^\gamma)^*$, it makes sense to compare the subspaces $\ker(\mathcal{A}+\mathcal{B})^*$ of $(W^\gamma)^*$ and $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$ of $\tilde{V}^\gamma$:

Lemma 5.1. $\ker(\mathcal{A}+\mathcal{B})^* = \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$.

Proof. Because of (2.3) and (5.2), we have for all $u \in V^\gamma$ and $\tilde{u} \in \tilde{V}^\gamma$ that

$$\left\langle(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})\tilde{u}, u\right\rangle_{L^2} = \left\langle\tilde{u}, (\mathcal{A}+\mathcal{B})u\right\rangle_{L^2} = \left[\tilde{u}, (\mathcal{A}+\mathcal{B})u\right]_{W^\gamma} = \left[(\mathcal{A}+\mathcal{B})^*\tilde{u}, u\right]_{W^\gamma}.$$

This implies $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}}) \subseteq \ker(\mathcal{A}+\mathcal{B})^*$.

Now, take an arbitrary $\varphi \in \ker(\mathcal{A}+\mathcal{B})^*$ and show that $\varphi \in \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$. By Lemma 2.3, we have (using notation (2.4)) $\varphi = \sum_{s\in\mathbb{Z}} \varphi^s e_s$ with

$$\varphi^s \in L^2\big((0,1);\mathbb{C}^n\big), \qquad \sum_{s\in\mathbb{Z}} \big(1+s^2\big)^{-\gamma} \left\|\varphi^s(x)\right\|^2_{L^2((0,1);\mathbb{C}^n)} < \infty.$$

It follows that for all $u \in V^\gamma$

$$0 = \left[(\mathcal{A}+\mathcal{B})^*\varphi, u\right]_{W^\gamma} = \left[\varphi, (\mathcal{A}+\mathcal{B})u\right]_{W^\gamma} = \sum_{s\in\mathbb{Z}} \int_0^1 \left\langle \varphi^s, a(x)\frac{d}{dx}u^{-s} - isu^{-s} + b(x)^T u^{-s}\right\rangle dx.$$

Therefore

$$\int_0^1 \left\langle \varphi^s, a(x)\frac{d}{dx}u^{-s} - isu^{-s} + b(x)^T u^{-s}\right\rangle dx = 0$$

for all $u^s \in H^1((0,1);\mathbb{C}^n)$ with (3.4). Since $a \in C^{0,1}([0,1];M_n)$, by a standard argument, we conclude that $\varphi^s \in H^1((0,1);\mathbb{C}^n)$ and that it satisfies the differential equation

$$-a(x)\frac{d}{dx}\varphi^s + \left(-is + b(x)^T - \frac{d}{dx}a(x)\right)\varphi^s = 0 \tag{5.3}$$

and the boundary conditions

$$\left.\begin{aligned} a_j(0)\varphi_j^s(0) &= -\sum_{k=1}^{m} r_{kj}^0 a_k(0)\varphi_k^s(0), \quad m+1 \leqslant j \leqslant n,\\ a_j(1)\varphi_j^s(1) &= -\sum_{k=m+1}^{n} r_{kj}^1 a_k(1)\varphi_k^s(1), \quad 1 \leqslant j \leqslant m. \end{aligned}\right\} \tag{5.4}$$

In other words: the functions $\varphi^s(x)e^{ist}$ belong to $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$ and, hence, to $\ker(\mathcal{A}+\mathcal{B})^*$. But they are linearly independent, and $\dim\ker(\mathcal{A}+\mathcal{B})^* < \infty$, hence there is $s_0 \in \mathbb{N}$ such that $\varphi^s = 0$ for $|s| > s_0$. Therefore, $\varphi \in \tilde{V}^\gamma$ for all $\gamma \geqslant 1$ and $(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})\varphi = 0$ as desired. $\square$

As it follows from the proof of Lemma 5.1,

$$\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}}) = \left\{\sum_{|s|\leqslant s_0}\varphi^s e_s \;\middle|\; \varphi^s \text{ solves (5.3), (5.4)}\right\}$$

does not depend on γ. By (5.1), $\ker(\mathcal{A}+\mathcal{B})$ does not depend on γ as well. Claim (ii) of Theorem 1.2 follows.

6. C^k-smoothness of the data-to-solution map (proof of Theorem 1.4)

In this section we prove Theorem 1.4. Hence, we suppose the assumptions of Theorem 1.4 to be satisfied, i.e. the data a, b, and r, which satisfy (1.15)–(1.18), are given and fixed.

Recall that in Theorem 1.4 the sets $A_\varepsilon(a)$ and $B_\varepsilon(b)$ are the open balls around a and b of radius ε in the (not complete) normed vector spaces

$$\mathbb{A} := \left\{\tilde{a} = \operatorname{diag}(\tilde{a}_1,\ldots,\tilde{a}_n) \in BV\big((0,1);\mathbb{M}_n\big)\right\} \quad \text{with } \|a\|_{\mathbb{A}} := \max_{1\leqslant j\leqslant n}\|\tilde{a}_j\|_\infty$$

and

$$\begin{aligned}\mathbb{B} := \left\{\tilde{b} \in L^\infty\big((0,1);\mathbb{M}_n\big):\ \tilde{b}_{jk} \in BV(0,1) \text{ for all } 1 \leqslant j \neq k \leqslant n\right\}\\ \text{with } \|b\|_{\mathbb{B}} := \max_{1\leqslant j,k\leqslant n}\|\tilde{b}_{jk}\|_\infty, \end{aligned} \tag{6.1}$$

respectively.

Because the assumptions of Theorem 1.4 are satisfied, there exists $\varepsilon > 0$ such that for all $\tilde{a} \in \mathbb{A}$ and $\tilde{b} \in \mathbb{B}$ with

$$\|\tilde{a} - a\|_{\mathbb{A}} + \|\tilde{b} - b\|_{\mathbb{B}} < \varepsilon \tag{6.2}$$

the assumptions of Theorem 1.2 are satisfied. Therefore, for those $\tilde{a}$ and $\tilde{b}$ the operators $\mathcal{A}(\tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$ are Fredholm of index zero from $V^\gamma(\tilde{a}, r)$ into W^γ for all $\gamma \geqslant 2$.

Moreover, all $\tilde{a} = \operatorname{diag}(\tilde{a}_1, \ldots, \tilde{a}_n)$ and $\tilde{b} \in L^\infty((0,1); \mathbb{M}_n)$ with (6.2) fulfill the assumptions of Theorem 1.1. Hence, for those $\tilde{a}$ and $\tilde{b}$ and for all $\gamma \geqslant 1$ the operator $\mathcal{A}(\tilde{a}, \tilde{b}^0)$ is an isomorphism from $V^\gamma(\tilde{a}, r)$ onto W^γ and

$$\left\|\mathcal{A}\big(\tilde{a}, \tilde{b}^0\big)^{-1}\right\|_{\mathcal{L}(W^\gamma; V^\gamma(\tilde{a}, r))} \leqslant C, \tag{6.3}$$

where the constant $C > 0$ does not depend on $\tilde{a}$, $\tilde{b}$, and γ, but only on ε.

For the sake of shortness, write $\mathcal{C} := C([0,1] \times [0, 2\pi]; \mathbb{R}^n)$.

Lemma 6.1. *For all $\gamma \geqslant 2$ the map $(\tilde{a}, \tilde{b}^0) \mapsto \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)^{-1}$ is locally Lipschitz continuous as a map from a subset of $L^\infty((0,1); \mathbb{M}_n) \times L^\infty((0,1); \mathbb{M}_n)$ into $\mathcal{L}(W^\gamma; W^{\gamma-1} \cap \mathcal{C})$.*

Proof. Take $a', a'' \approx a$ and $b', b'' \approx b^0$ in $L^\infty((0,1); \mathbb{M}_n)$, $\gamma \geqslant 2$, $f \in W^\gamma$, $u' \in V^\gamma(a', r)$, and $u'' \in V^\gamma(a'', r)$ such that

$$\mathcal{A}\big(a', b'\big)u' = \mathcal{A}\big(a'', b''\big)u'' = f.$$

Then (6.3) and Lemma 2.2(iii) yield that there exists a constant $c > 0$ such that

$$c\left\|\partial_t u'\right\|_{W^{\gamma-1}} \leqslant c\left\|u'\right\|_{W^\gamma \cap \mathcal{C}} \leqslant \left\|u'\right\|_{V^\gamma(a', r)} \leqslant C\|f\|_{W^\gamma} \tag{6.4}$$

and

$$\begin{aligned} c \min_{1 \leqslant j \leqslant n} \operatorname{ess\,inf}\left|a'_j(x)\right| \left\|\partial_x u'\right\|_{W^{\gamma-1}} &\leqslant c\left\|a' \partial_x u'\right\|_{W^{\gamma-1}} \\ &\leqslant c\left\|\partial_t u' + a' \partial_x u'\right\|_{W^{\gamma-1}} + c\left\|\partial_t u'\right\|_{W^{\gamma-1}} \\ &\leqslant c\left\|u'\right\|_{V^\gamma(a', r)} + C\|f\|_{W^\gamma} \leqslant C(c+1)\|f\|_{W^\gamma}, \end{aligned} \tag{6.5}$$

where the constant $C > 0$ is the same as in (6.3) and is independent of γ, a', a'', b', and b''. Moreover, we have

$$\begin{aligned} \mathcal{A}\big(a'', b''\big)\big(u'' - u'\big) &= \big(\mathcal{A}\big(a'', b''\big) - \mathcal{A}\big(a'', b''\big)\big)u' \\ &\quad + \big(\mathcal{A}\big(a', b''\big) - \mathcal{A}\big(a'', b''\big)\big)u' + \big(\mathcal{A}\big(a', b'\big) - \mathcal{A}\big(a', b''\big)\big)u' \\ &= \big(a' - a''\big)\partial_x u' + \big(b' - b''\big)u'. \end{aligned}$$

This is a well-defined equation in $W^{\gamma-1}$, and (6.3), (6.4), and (6.5) yield

$$\left\|u'' - u'\right\|_{W^{\gamma-1} \cap \mathcal{C}} \leqslant C\Big(\max_{1 \leqslant j \leqslant n}\left\|a''_j - a'_j\right\|_\infty + \max_{1 \leqslant j \leqslant n}\left\|b''_{jj} - b'_{jj}\right\|_\infty\Big)\|f\|_{W^\gamma},$$

with a new constant C being independent of γ, a', a'', b', b'', again. □

Lemma 6.2. *There exists* $\varepsilon > 0$ *with the following property*:

For each $\gamma \geqslant 2$ *there exists* $C > 0$ *such that for all* $\tilde{a} \in A_\varepsilon(a)$, *and* $\tilde{b} \in B_\varepsilon(b)$ *the operator* $\mathcal{A}(\tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$ *is an isomorphism from* $V^\gamma(\tilde{a}, r)$ *onto* W^γ *and*

$$\left\| \left(\mathcal{A}\left(\tilde{a}, \tilde{b}^0\right) + \mathcal{B}\left(\tilde{b}^1\right)\right)^{-1} \right\|_{\mathcal{L}(W^\gamma; V^\gamma(\tilde{a}, r))} \leqslant C. \tag{6.6}$$

Proof. As mentioned above, there exists $\varepsilon > 0$ such that for all $\tilde{a} \in A_\varepsilon(a)$, and $\tilde{b} \in B_\varepsilon(b)$ the operator $\mathcal{A}(\tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$ is Fredholm of index zero from $V^\gamma(\tilde{a}, r)$ into W^γ.

Let us show that ε can be chosen so small that for all $\tilde{a} \in A_\varepsilon(a)$, and $\tilde{b} \in B_\varepsilon(b)$ the operator $\mathcal{A}(\tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$ is injective (and, hence, bijective).

Suppose the contrary. Then there exist sequences $\alpha_k \to a$ in $L^\infty((0,1); \mathbb{M}_n)$, $\beta_k \to b$ in $L^\infty((0,1); \mathbb{M}_n)$, and $v_k \in V^\gamma(\alpha_k, r)$ such that

$$\left(\mathcal{A}\left(\alpha_k, \beta_k^0\right) + \mathcal{B}\left(\beta_k^1\right)\right) v_k = 0 \quad \text{and} \quad v_k \neq 0. \tag{6.7}$$

Set $w_k := \mathcal{A}(\alpha_k, \beta_k^0) v_k$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$

$$\left(I + \mathcal{B}\left(\beta_k^1\right) \mathcal{A}\left(\alpha_k, \beta_k^0\right)^{-1}\right) w_k = 0. \tag{6.8}$$

Hence

$$w_k = \left(\mathcal{B}\left(b_k^1\right) \mathcal{A}\left(\alpha_k, \beta_k^0\right)^{-1}\right)^2 w_k,$$

and, consequently,

$$\begin{aligned} z_k &:= \frac{w_k}{\|w_k\|_{W^\gamma}} \\ &= \left(\mathcal{B}\left(b^1\right) \mathcal{A}\left(a, b^0\right)^{-1}\right)^2 z_k + \left(\left(\mathcal{B}\left(\beta_k^1\right) \mathcal{A}\left(\alpha_k, \beta_k^0\right)^{-1}\right)^2 - \left(\mathcal{B}\left(b^1\right) \mathcal{A}\left(a, b^0\right)^{-1}\right)^2\right) z_k. \end{aligned} \tag{6.9}$$

Because the operator $(\mathcal{B}(b^1)\mathcal{A}(a, b^0)^{-1})^2$ is compact from W^γ into W^γ, it is also compact from W^γ into $W^{\gamma-1}$. Hence there exist $z \in W^{\gamma-1}$ and a subsequence z_{k_l} such that $(\mathcal{B}(b^1)\mathcal{A}(a, b^0)^{-1})^2 z_{k_l} \to z$ in $W^{\gamma-1}$. Therefore Lemma 6.1 and (6.9) yield that $z_{k_l} \to z$ in $W^{\gamma-1}$, and Lemma 6.1 and (6.8) yield that

$$\left(I + \mathcal{B}\left(b^1\right) \mathcal{A}\left(a, b^0\right)^{-1}\right) z = 0 \quad \text{in } W^{\gamma-1}.$$

Hence z belongs to $\ker(I + \mathcal{B}(b^1)\mathcal{A}(a, b^0)^{-1})$. Since $\|z\|_{W^\gamma} = 1$, we get a contradiction to assumption (1.15).

It remains to prove (6.6). Suppose the contrary, i.e. that for any choice of ε (6.6) is not true. Then there exist sequences $\alpha_k \to a$ in $L^\infty((0,1); \mathbb{M}_n)$, $\beta_k \to b$ in $L^\infty((0,1); \mathbb{M}_n)$, and $v_k \in V^\gamma(\alpha_k, r)$ such that

$$\left(\mathcal{A}\left(\alpha_k, \beta_k^0\right) + \mathcal{B}\left(\beta_k^1\right)\right) v_k \to 0 \quad \text{in } W^\gamma \text{ as } k \to \infty \quad \text{and} \quad \|v_k\|_{V^\gamma(\alpha_k, r)} = 1. \tag{6.10}$$

Now we proceed as above, replacing (6.7) by (6.10), to get a contradiction. □

Similarly to Lemma 6.1 one can prove

Lemma 6.3. *For all* $\gamma \geqslant 2$ *the map*

$$(\tilde{a}, \tilde{b}) \in A_\varepsilon(a) \times B_\varepsilon(b) \mapsto \left(\mathcal{A}\left(\tilde{a}, \tilde{b}^0\right) + \mathcal{B}\left(\tilde{b}^1\right)\right)^{-1} \in \mathcal{L}\left(W^\gamma; W^{\gamma-1} \cap \mathcal{C}\right)$$

is locally Lipschitz continuous.

Let us introduce the data-to-solution map

$$(\tilde{a},\tilde{b},f)\in A_\varepsilon(a)\times B_\varepsilon(b)\times W^\gamma\mapsto \hat{u}(\tilde{a},\tilde{b},f):=\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)^{-1}f\in V^\gamma(\tilde{a},r). \tag{6.11}$$

Lemma 6.4. *The map $\hat{u}$ is C^1-smooth as a map into $W^{\gamma-2}\cap\mathcal{C}$ for all $\gamma\geqslant 3$.*

Proof. We have to show that all partial derivatives $\partial_a\hat{u}$, $\partial_b\hat{u}$, and $\partial_f\hat{u}$ exist and are continuous.

First, consider $\partial_f\hat{u}$. From the definition (6.11) follows that it exists and that

$$\partial_f\hat{u}(\tilde{a},\tilde{b},f)\bar{f}=\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)^{-1}\bar{f}.$$

The continuity of the map

$$(\tilde{a},\tilde{b},f)\in A_\varepsilon(a)\times B_\varepsilon(b)\times W^\gamma\mapsto\partial_f\hat{u}(\tilde{a},\tilde{b},f)\in\mathcal{L}\big(W^\gamma;W^{\gamma-1}\cap\mathcal{C}\big)$$

follows from Lemma 6.3.

Now, consider $\partial_b\hat{u}$. From Corollary 1.3 follows that it exists, and (6.11) yields

$$\partial_b\hat{u}(\tilde{a},\tilde{b},f)\bar{b}=-\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)^{-1}\times\big(\partial_b\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)\bar{b}+\mathcal{B}'\big(\tilde{b}^1\big)\bar{b}\big)\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)^{-1}f.$$

Moreover, we have

$$\big(\partial_b\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)\bar{b}\big)u=\bar{b}^0u\quad\text{and}\quad\big(\mathcal{B}'\big(\tilde{b}^1\big)\bar{b}\big)u=\bar{b}^1u. \tag{6.12}$$

Hence, $\partial_b\mathcal{A}(\tilde{a},\tilde{b}^0)$ and $\mathcal{B}'(\tilde{b}^1)$ do not depend on $\tilde{a}$, and $\tilde{b}$. Therefore, again Lemma 6.3 yields the continuity of the map $(\tilde{a},\tilde{b},f)\in A_\varepsilon(a)\times B_\varepsilon(b)\times W^\gamma\mapsto\partial_b\hat{u}(\tilde{a},\tilde{b},f)\in\mathcal{L}(\mathbb{B};W^{\gamma-1}\cap\mathcal{C})$.

Further, consider $\partial_a\hat{u}$. If $\partial_a\hat{u}(\tilde{a},\tilde{b},f)$ exists as an element of the space $\mathcal{L}(\mathbb{A};W^{\gamma-2}\cap\mathcal{C})$, then for any $\bar{a}\in BV((0,1),\mathbb{M}_n)$ we have

$$\begin{aligned}\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)\partial_a\hat{u}(\tilde{a},\tilde{b},f)\bar{a}&=-\bar{a}\partial_x\hat{u}(\tilde{a},\tilde{b},f)\\&=\bar{a}\tilde{a}^{-1}\big(\partial_t\hat{u}(\tilde{a},\tilde{b},f)+\tilde{b}^1\hat{u}(\tilde{a},\tilde{b},f)-f\big).\end{aligned} \tag{6.13}$$

The right-hand side belongs to $W^{\gamma-1}\cap\mathcal{C}$, hence this equation determines uniquely the candidate

$$\hat{v}(\tilde{a},\tilde{b},f)\bar{a}:=-\big(\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)^{-1}\big(\partial_a\mathcal{A}\big(\tilde{a},\tilde{b}^0\big)\bar{a}\big)\hat{u}(\tilde{a},\tilde{b},f)$$

for $\partial_a\hat{u}(\tilde{a},\tilde{b},f)$ in $\mathcal{L}(\mathbb{A};W^{\gamma-1}\cap\mathcal{C})$. Moreover, because of Lemma 6.3 the candidate $\hat{v}$ for $\partial_a\hat{u}$ is continuous as a map from $A_\varepsilon(a)\times B_\varepsilon(b)\times W^\gamma$ into the space $\mathcal{L}(\mathbb{A};W^{\gamma-2}\cap\mathcal{C})$.

It remains to prove that $\hat{v}$ is really $\partial_a\hat{u}$. In order to show this, take $a',a''\in A_\varepsilon(a)$, $\tilde{b}\in B_\varepsilon(b)$, $\gamma\geqslant 2$, $f\in W^\gamma$, $u'\in V^\gamma(a',r)$, and $u''\in V^\gamma(a'',r)$ such that

$$\big(\mathcal{A}\big(a',\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)u'=\big(\mathcal{A}\big(a'',\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)u''=f.$$

Then

$$\begin{aligned}\big(\mathcal{A}\big(a'',\tilde{b}^0\big)+\mathcal{B}\big(\tilde{b}^1\big)\big)\big(u''-u'-\hat{v}\big(a',\tilde{b}^0\big)\big(a''-a'\big)\big)&=\big(\mathcal{A}\big(a',\tilde{b}^0\big)-\mathcal{A}\big(a'',\tilde{b}^0\big)\big)\hat{v}\big(a',\tilde{b}^0\big)\big(a''-a'\big)\\&=-\big(a''-a'\big)^2\partial_x\hat{v}\big(a',\tilde{b}^0\big).\end{aligned}$$

Here we used that the map $\mathcal{A}(\cdot, \tilde{b})$ is affine. Hence

$$\left\| \left(\mathcal{A}\left(a'', \tilde{b}^0\right) + \mathcal{B}\left(\tilde{b}^1\right)\right)\left(u'' - u' - \hat{v}\left(a', \tilde{b}^0\right)\left(a'' - a'\right)\right) \right\|_{W^{\gamma-2} \cap \mathcal{C}} = o\left(\left\|a'' - a'\right\|_{\mathbb{A}}\right),$$

and Lemma 6.2 yields

$$\left\| u'' - u' - \hat{v}\left(a', \tilde{b}^0\right)\left(a'' - a'\right) \right\|_{W^{\gamma-2} \cap \mathcal{C}} = o\left(\left\|a'' - a'\right\|_{\mathbb{A}}\right). \qquad \square$$

Lemma 6.5. *The map $\hat{u}$ is C^k-smooth as a map into $W^{\gamma-k-1} \cap \mathcal{C}$ for all $1 \leqslant k \leqslant \gamma - 1$.*

Proof. For $k = 1$ the lemma is true, and the first partial derivatives satisfy

$$\partial_f \hat{u}(\tilde{a}, \tilde{b}, f)\bar{f} = \hat{u}(\tilde{a}, \tilde{b}, \bar{f}), \tag{6.14}$$

$$\partial_b \hat{u}(\tilde{a}, \tilde{b}, f)\bar{b} = -\hat{u}\left(\tilde{a}, \tilde{b}, \bar{\mathcal{B}}\left(\hat{u}(\tilde{a}, \tilde{b}, f), \bar{b}\right)\right), \tag{6.15}$$

$$\partial_a \hat{u}(\tilde{a}, \tilde{b}, f)\bar{a} = -\hat{u}\left(\tilde{a}, \tilde{b}, \bar{\mathcal{A}}\left(\tilde{a}, \tilde{b}, f, \hat{u}(\tilde{a}, \tilde{b}, f)\right)\bar{a}\right). \tag{6.16}$$

Here we denoted by $\bar{\mathcal{A}}(\tilde{a}, \tilde{b}, f, u) : \mathbb{A} \to W^{\gamma-1}$ the linear bounded operator which is defined by (cf. (6.13))

$$\bar{\mathcal{A}}(\tilde{a}, \tilde{b}, f, u)\bar{a} := \bar{a}\tilde{a}^{-1}\left(\partial_t u + \tilde{b}^1 u - f\right),$$

and $\bar{\mathcal{B}} : W^\gamma \times \mathbb{B} \to W^\gamma$ is the bilinear bounded operator which is defined by (cf. (6.12))

$$\bar{\mathcal{B}}(u, b) := bu.$$

Obviously, the map

$$(\tilde{a}, \tilde{b}, f, u) \in \mathbb{R} \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \times W^\gamma \mapsto \bar{\mathcal{A}}(\tilde{a}, \tilde{b}, f, u) \in \mathcal{L}\left(\mathbb{A}; W^{\gamma-1} \cap \mathcal{C}\right)$$

is C^∞-smooth. Hence, (6.14)–(6.16), Lemma 6.4 and the chain rule imply that the data-to-solution map $\hat{u}$ is C^2-smooth, and one gets corresponding formulae for the second partial derivatives by differentiating the identities (6.14)–(6.16). For example, it holds

$$\partial_f^2 \hat{u}(\tilde{a}, \tilde{b}, f) = 0,$$

$$\partial_f \partial_b \hat{u}(\tilde{a}, \tilde{b}, f)(\bar{f}, \bar{b}) = \partial_b \hat{u}(\tilde{a}, \tilde{b}, \bar{f})\bar{b} = -\hat{u}\left(\tilde{a}, \tilde{b}, \bar{\mathcal{B}}\left(\hat{u}(\tilde{a}, \tilde{b}, \bar{f}), \bar{b}\right)\right),$$

$$\partial_f \partial_a \hat{u}(\tilde{a}, \tilde{b}, f)(\bar{f}, \bar{a}) = \partial_a \hat{u}(\tilde{a}, \tilde{b}, \bar{f})\bar{a} = -\hat{u}\left(\tilde{a}, \tilde{b}, \bar{\mathcal{A}}\left(\tilde{a}, \tilde{b}, \bar{f}, \hat{u}(\tilde{a}, \tilde{b}, \bar{f})\right)\bar{a}\right).$$

If $\gamma \geqslant 4$, then those formulae and the chain rule imply that all second partial derivatives of $\hat{u}$ are C^1-smooth, etc. $\square$

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