On a Conjecture of R. F. Scott (1881)

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To Alston S. Householder on his seventy-fifth birthday.

Submitted by Emeric Deutsch

ABSTRACT

A formula is given for the permanent of a general Cauchy matrix $((x_i - y_j)^{-1})$. In a special case, where the x_i and the y_j are the distinct nth roots of 1 and -1 respectively, a formula for the permanent, conjectured by R. F. Scott, is proved by computing the eigenvalues of related circulants.

1. INTRODUCTION

An n-square complex matrix $A=(a_{pq})$ is called a *Cauchy matrix* if

$$a_{pq}=\frac{1}{x_p-y_q},$$

$p,q=1,\dots,n$, for some $2n$ numbers $x_1,\dots,x_n$, $y_1,\dots,y_n$.

Nearly a century ago R. F. Scott [7] gave, without proof, the following formula for the permanent of an $n\times n$ Cauchy matrix A in which $x_1,\dots,x_n$ and $y_1,\dots,y_n$ are the distinct roots of $x^n=1$ and $y^n=-1$, respectively. He asserted that

$$\operatorname{per}(A)=\begin{cases} n[1\times3\times5\times\cdots\times(n-2)]^2/2^n & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases} \tag{1}$$

(See Conjecture 7 in [5].)

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In this paper I prove that Scott's equality is essentially correct (except for a sign). This is accomplished by actually evaluating the eigenvalues of two circulants related to the matrix A, computing their determinants, and using a theorem of Borchardt [1] to obtain the permanent of A. Clearly the value of the permanent of A is invariant under any permutation of the x_p and of the y_q. On the other hand, the determinant of A and the eigenvalues of the two related circulants depend on the order of the x_p and of the y_q. We shall assume, therefore, that the subscripts are arranged in increasing order of the amplitudes of the x_p and the y_q, in the interval $[0,2\pi)$; that is,

$$x_p = \theta^{2p-2},$$

$p=1,\dots,n$, and

$$y_q = \theta^{2q-1},$$

$q=1,\dots,n$, where $\theta = e^{i\pi/n}$. The $n\times n$ Cauchy matrix whose (p,q) entry is

$$\frac{1}{\theta^{2p-2}-\theta^{2q-1}}$$

is called the $n\times n$ *Scott matrix*. We shall prove Scott's formula for this matrix (Theorem 4).

The paper also contains a formula for the permanent of a general Cauchy matrix (Theorem 1).

2. RESULTS

Let $P_{2k,n}$ be the set of sequences of $2k$ distinct integers $(\omega_1,\dots,\omega_{2k})$ satisfying

$$\begin{aligned} &1 \leqslant \omega_t \leqslant n, && t=1,\dots,2k;\\ &\omega_{2t-1} < \omega_{2t}, && t=1,\dots,k;\\ &\omega_{2t-1} < \omega_{2t+1}, && t=1,\dots,k-1. \end{aligned}$$

THEOREM 1. *Let $A=((x_i-y_j)^{-1})$ be an $n\times n$ Cauchy matrix. Then*

$$\operatorname{per}(A) = (-1)^{n(n-1)/2}$$

$$\cdot\left[\prod_{i=1}^{n} h_i + \sum_{k=1}^{[n/2]} \sum_{\omega\in P_{2k,n}} \frac{h_{\omega'_1}\quad h_{\omega'_2}\cdots h_{\omega'_{n-2k}}}{(x_{\omega_1}-x_{\omega_2})^2(x_{\omega_3}-x_{\omega_4})^2\cdots(x_{\omega_{2k-1}}-x_{\omega_{2k}})^2}\right], \tag{2}$$

where

$$h_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x_i - x_j} - \sum_{j=1}^{n} \frac{1}{x_i - y_j}, \qquad i = i, \ldots, n,$$

ω denotes the sequence $(\omega_1, \ldots, \omega_{2k})$, and $(\omega'_1, \ldots, \omega'_{n-2k})$ is the sequence complementary to ω in $(1, \ldots, n)$.

Let A_n be the $n \times n$ Scott matrix, and let $D = \mathrm{diag}(1, \theta^2, \theta^4, \ldots, \theta^{2n-2})$, where $\theta = e^{i\pi/n}$. Then $C_n = DA_n$ is the circulant with the first row

$$\left[\frac{1}{1-\theta} \quad \frac{1}{1-\theta^3} \quad \frac{1}{1-\theta^5} \quad \cdots \quad \frac{1}{1-\theta^{2n-1}} \right].$$

Clearly

$$\det(C_n) = (-1)^{n-1} \det(A_n).$$

The eigenvalues of C_n are

$$\lambda_t = \sum_{k=1}^{n} \frac{\theta^{2(t-1)(k-1)}}{1 - \theta^{2k-1}}, \tag{3}$$

$t = 1, \ldots, n$ (see, e.g., [4, s. 4.9]).

THEOREM 2. *Let C_n be the $n \times n$ circulant whose first row is*

$$\left[\frac{1}{1-\theta} \quad \frac{1}{1-\theta^3} \quad \frac{1}{1-\theta^5} \quad \cdots \quad \frac{1}{1-\theta^{2n-1}} \right],$$

$\theta = e^{i\pi/n}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of C_n ordered as in (3). Then

$$\lambda_1 = \frac{n}{2},$$

and

$$\theta^{t-1} \lambda_t = -\frac{n}{2},$$

$t = 2, \ldots, n$.

COROLLARY 1. *If A_n is the $n \times n$ Scott matrix, then*

$$\det(A_n) = (-i)^{n-1}\left(\frac{n}{2}\right)^n.$$

Let $B^{(2)}$ denote the Hadamard product of matrix $B = (b_{pq})$ with itself—that is, the matrix whose (p,q) entry is b_{pq}^2. We shall relate the permanent of A_n to the determinant of C_n via the determinant of the circulant $C_n^{(2)} = D^2 A_n^{(2)}$. Note that

$$\det(C_n^{(2)}) = \det(A_n^{(2)}).$$

The first row of $C_n^{(2)}$ is

$$\left[\frac{1}{(1-\theta)^2} \quad \frac{1}{(1-\theta^3)^2} \quad \frac{1}{(1-\theta^5)^2} \quad \cdots \quad \frac{1}{(1-\theta^{2n-1})^2}\right],$$

and therefore the eigenvalues of $C_n^{(2)}$ are

$$\mu_t = \sum_{k=1}^{n} \frac{\theta^{2(t-1)(k-1)}}{(1-\theta^{2k-1})^2}, \tag{4}$$

$t = 1, \ldots, n$.

THEOREM 3. *Let $C_n^{(2)}$ be the circulant whose first row is*

$$\left[\frac{1}{(1-\theta)^2} \quad \frac{1}{(1-\theta^3)^2} \quad \frac{1}{(1-\theta^5)^2} \quad \cdots \quad \frac{1}{(1-\theta^{2n-1})^2}\right],$$

$\theta = e^{i\pi/n}$, *and let $\mu_1, \ldots, \mu_n$ be its eigenvalues ordered as in* (4). *Then*

$$\mu_1 = -\tfrac{1}{4}n(n-2)$$

and

$$\theta^{t-1}\mu_t = -\tfrac{1}{4}n(n-2t+4),$$

$t = 2, \ldots, n$.

COROLLARY 2. *If $C_n^{(2)}$ is the matrix in Theorem 3, then*

$$\mu_1 = \theta^2 \mu_3,$$

and

$$\theta^{t-1}\mu_t = -\theta^{n-t+3}\mu_{n-t+4},$$

$t = 4,\dots,n$.

COROLLARY 3. *If $C_n^{(2)}$ is the matrix in Theorem 3, then*

$$\det\left(C_n^{(2)}\right) = \begin{cases} n^{n+1}[1\times3\times5\times\cdots\times(n-2)]^2/4^n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

We conclude with the main result of the paper, the amended version of Scott's formula (1).

THEOREM 4. *If A_n is the $n\times n$ Scott matrix, then*

$$\operatorname{per}(A_n) = \begin{cases} (-1)^{(n-1)/2} n[1\times3\times5\times\cdots\times(n-2)]^2/2^n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{5}$$

3. PROOFS

The main tool in proving both Theorem 1 and Theorem 4 is the following classical result of Borchardt [1].

BORCHARDT'S THEOREM. *If A is a Cauchy matrix, then*

$$\operatorname{per}(A)\det(A) = \det(A^{(2)}), \tag{6}$$

where $A^{(2)}$ denotes the matrix whose entries are the squares of the corresponding entries in A.

For a proof of Borchardt's theorem see [5, Sec. 1.3].

We shall also use the following formula due to Cauchy [2], for the determinant of an $n\times n$ Cauchy matrix A:

$$\det(A)=(-1)^{n(n-1)/2}\frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)}{\prod_{i,j}(x_i-y_j)}. \tag{7}$$

The proof of (7) is straightforward. Note that if the x_i and the y_j are distinct, then $\det(A)\neq 0$.

Proof of Theorem 1. We prove the theorem using Borchardt's theorem. The determinant of $A^{(2)}$ is evaluated by a method suggested by Borchardt [1]:

$$\det(A^{(2)})=(-1)^n\frac{\partial^n}{\partial x_1\,\partial x_2\cdots\partial x_n}\det(A). \tag{8}$$

The formula (8) is obvious. We apply it to the expression in (7). We prove by induction that if

$$F=\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)$$

and

$$G=\prod_{i,j}(x_i-y_j),$$

where $x_1,\dots,x_n$, $y_1,\dots,y_n$ are variables, then for any t, $1\leqslant t\leqslant n$,

$$\frac{\partial^t}{\partial x_1\cdots\partial x_t}\frac{F}{G}=\frac{F}{G}\left[\prod_{i=1}^{t}h_i+\sum_{k=1}^{[t/2]}\sum_{\omega\in P_{2k,t}}\frac{\prod_{i=1}^{t-2k}h_{\omega_i'}}{\prod_{i=1}^{k}(x_{\omega_{2i-1}}-x_{\omega_{2i}})^2}\right]. \tag{9}$$

If $t=1$, then

$$\frac{\partial}{\partial x_1}\frac{F}{G}=\frac{1}{G^2}\left(G\frac{\partial F}{\partial x_1}-\frac{\partial G}{\partial x_1}\right)$$

$$=\frac{1}{G^2}\left[\left(GF\sum_{j=2}^{n}\frac{1}{x_1-x_j}\right)-FG\sum_{j=1}^{n}\frac{1}{x_1-y_j}\right]$$

$$=\frac{F}{G}h_1,$$

which is (9) with $t=1$. In fact,

$$\frac{\partial}{\partial x_s}\frac{F}{G}=\frac{F}{G}h_s$$

for any s, $1\leqslant s\leqslant n$. Now assume that $t\geqslant 2$ and that (9) holds for $t-1$, i.e., that

$$\frac{\partial^{t-1}}{\partial x_1\cdots\partial x_{t-1}}\frac{F}{G}=\frac{F}{G}\left[\prod_{i=1}^{t-1}h_i+\sum_{k=1}^{[(t-1)/2]}\sum_{\omega\in P_{2k,t-1}}\frac{\prod_{i=1}^{t-2k-1}h_{\omega_i'}}{\prod_{i=1}^{k}(x_{\omega_{2i-1}}-x_{\omega_{2i}})^2}\right].$$

Then

$$\frac{\partial^{t}}{\partial x_1\cdots\partial x_{t}}\frac{F}{G}=\frac{\partial}{\partial x_t}\frac{\partial^{t-1}}{\partial x_1\cdots\partial x_{t-1}}\frac{F}{G}$$

$$=\frac{F}{G}h_t\left[\prod_{i=1}^{t-1}h_i+\sum_{k=1}^{[(t-1)/2]}\sum_{\omega\in P_{2k,t-1}}\frac{\prod_{i=1}^{t-2k-1}h_{\omega_i'}}{\prod_{i=1}^{k}(x_{\omega_{2i-1}}-x_{\omega_{2i}})^2}\right]$$

$$+\frac{F}{G}\left[\sum_{s=1}^{t-1}\frac{1}{(x_s-x_t)^2}\prod_{\substack{i=1\\ i\neq s}}^{t-1}h_i+\sum_{k=1}^{[t/2]-1}\sum_{\omega\in P_{2k,t-1}}\sum_{s=1}^{t-2k-1}\frac{1}{(x_{\omega_s'}-x_t)^2}\frac{\prod_{\substack{i=1\\ i\neq s}}^{t-2k-1}h_{\omega_i'}}{\prod_{i=1}^{k}(x_{\omega_{2i-1}}-x_{\omega_{2i}})^2}\right],$$

which clearly equals the right-hand side of (9). We now set $t=n$ in (9), obtaining, via (8), a formula for $\det(A^{(2)})$, which we use together with (7) in Borchardt's theorem to obtain (2). ■

Theorem 4 can be derived, in principle, from Theorem 1 by specifying

$$x_t=\theta^{2t-2},$$

and

$$y_t=\theta^{2t-1},$$

$t=1,\dots,n$, where $\theta=e^{i\pi/n}$. Unfortunately this method is not practicable, due to the complexity of the resulting expression. We proceed, therefore, as follows. We first obtain the eigenvalues of C_n (Theorem 2) and of $C_n^{(2)}$ (Theorem 3); we then compute the determinants of A_n (Corollary 1) and of $A_n^{(2)}$ (Corollary 3), and use Borchardt's theorem to deduce Theorem 4.

We require the following key auxiliary result.

LEMMA. *Let t and n be integers, $0\leqslant t\leqslant n$. Then*

$$\sum_{k=1}^{n}\frac{\cos[(2k-1)t\pi/n]}{1-\cos[(2k-1)\pi/n]}=\tfrac{1}{2}n(n-2t). \tag{10}$$

Proof of the lemma. Use induction on n. If $t=0$, the left-hand side of (10) is

$$\sum_{k=1}^{n}\frac{1}{1-\cos[(2k-1)\pi/n]}=\frac{1}{2}\sum_{k=1}^{n}\csc^2\frac{(2k-1)\pi}{2n}$$

$$=\tfrac{1}{2}n^2.$$

(For the last equality see [3, Series Nos. 441 and 442].) If $t=1$, the series is

$$\sum_{k=1}^{n}\frac{\cos[(2k-1)\pi/n]}{1-\cos[(2k-1)\pi/n]}=-n+\sum_{k=1}^{n}\frac{1}{1-\cos[(2k-1)\pi/n]}$$

$$=-n+\tfrac{1}{2}n^2$$

$$=\tfrac{1}{2}n(n-2).$$

Now assume that $t \geqslant 2$. Then for any φ,

$$\frac{\cos t\varphi}{1-\cos\varphi} = \frac{2\cos(t-1)\varphi\cdot\cos\varphi - \cos(t-2)\varphi}{1-\cos\varphi}$$

$$= \frac{2\cos(t-1)\varphi}{1-\cos\varphi} - 2\cos(t-1)\varphi - \frac{\cos(t-2)\varphi}{1-\cos\varphi}. \qquad (11)$$

If we set $\varphi = (2k-1)\pi/n$, we get

$$\sum_{k=1}^{n} \frac{\cos[(2k-1)t\pi/n]}{1-\cos[(2k-1)\pi/n]} = 2\sum_{k=1}^{n} \frac{\cos[(2k-1)(t-1)\pi/n]}{1-\cos[(2k-1)\pi/n]}$$

$$-2\sum_{k=1}^{n}\cos[(2k-1)(t-1)\pi/n] - \sum_{k=1}^{n}\frac{\cos[(2k-1)(t-2)\pi/n]}{1-\cos[(2k-1)\pi/n]}.$$

But (see [3, Series No. 420])

$$2\sum_{k=1}^{n}\cos\frac{(2k-1)(t-1)\pi}{n} = \sin\frac{2n(t-1)\pi}{n}\csc\frac{(t-1)\pi}{n}$$

$$=0.$$

Hence, using the induction hypothesis,

$$\sum\frac{\cos[(2k-1)t\pi/n]}{1-\cos[(2k-1)\pi/n]} = 2\times\tfrac{1}{2}n(n-2t+2) - \tfrac{1}{2}n(n-2t+4)$$

$$= \tfrac{1}{2}n(n-2t). \qquad \blacksquare$$

Proof of Theorem 2. The eigenvalues of the circulant C_n are [4, s. 49]

$$\lambda_t = \sum_{k=1}^{n}\frac{\theta^{2(t-1)(k-1)}}{1-\theta^{2k-1}},$$

$t=1,\dots,n$, where $\theta = e^{i\pi/n}$. Hence

$$\theta^{t-1}\lambda_t = \sum_{k=1}^{n}\frac{\theta^{(t-1)(2k-1)}}{1-\theta^{2k-1}},$$

$t=1,\ldots,n$. But θ^{2k-1} and $\theta^{2n-2k+1}$ are conjugate for every k, and it follows that

$$\theta^{t-1}\lambda_t=\sum_{k=1}^{n}\operatorname{Re}\frac{\theta^{(t-1)(2k-1)}}{1-\theta^{2k-1}}$$

$$=\sum_{k=1}^{n}\frac{\cos[(t-1)\varphi_k](1-\cos\varphi_k)-\sin[(t-1)\varphi_k]\sin\varphi_k}{2(1-\cos\varphi_k)},$$

where $\varphi_k=(2k-1)\pi/n$. Therefore,

$$\lambda_1=\frac{n}{2},$$

and for $t\geqslant 2$,

$$\theta^{t-1}\lambda_t=\sum_{k=1}^{n}\frac{\cos[(t-1)\varphi_k]-\cos[(t-2)\varphi_k]}{2(1-\cos\varphi_k)}$$

$$=\tfrac{1}{4}n(n-2t+2)-\tfrac{1}{4}n(n-2t+4)$$

$$=-\frac{n}{2},$$

by the lemma. ■

Theorem 2 implies that

$$\det(C_n)=\prod_{t=1}^{n}\lambda_t$$

$$=\theta^{-n(n-1)/2}\prod_{t=1}^{n}\theta^{t-1}\lambda_t$$

$$=(-i)^{n-1}\frac{n}{2}\left(-\frac{n}{2}\right)^{n-1}$$

$$=i^{n-1}\left(\frac{n}{2}\right)^{n}.$$

This proves Corollary 1, since $\det(A_n)=(-1)^{n-1}\det(C_n)$.

Proof of Theorem 3. We have

$$\theta^{t-1}\mu_t = \sum_{k=1}^{n} \frac{\theta^{(2k-1)(t-1)}}{(1-\theta^{2k-1})^2}$$

$$= \sum_{k=1}^{n} \operatorname{Re} \frac{\theta^{(2k-1)(t-1)}}{(1-\theta^{2k-1})^2}$$

$$= \sum_{k=1}^{n} \operatorname{Re} \frac{\{\cos[(t-1)\varphi_k] + i\sin[(t-1)\varphi_k]\}(1-\cos\varphi_k + i\sin\varphi_k)^2}{4(1-\cos\varphi_k)^2}$$

$$= \sum_{k=1}^{n} \frac{\{\cos[(t-1)\varphi_k] - \cos[(t-1)\varphi_k]\cos\varphi_k - 2\sin[(t-1)\varphi_k]\sin\varphi_k - \cos[(t-1)\varphi_k](1+\cos\varphi_k)\}}{4(1-\cos\varphi_k)}$$

$$= -\sum_{k=1}^{n} \frac{\cos[(t-2)\varphi_k]}{2(1-\cos\varphi_k)},$$

where, as before, $\varphi_k = (2k-1)\pi/n$. Hence by the lemma,

$$\mu_1 = -\frac{1}{2}\sum_{n=1}^{n} \frac{\cos[-(2k-1)\pi/n]}{1-\cos[(2k-1)\pi/n]}$$

$$= -\tfrac{1}{4}n(n-2),$$

and for $t \geq 2$,

$$\theta^{t-1}\mu_t = -\frac{1}{2}\sum_{k=1}^{n} \frac{\cos[(t-2)(2k-1)\pi/n]}{1-\cos[(2k-1)\pi/n]}$$

$$= -\tfrac{1}{4}n(n-2t+4). \qquad \blacksquare$$

Corollary 2 is an immediate consequence of Theorem 3. To prove Corollary 3 we note that if n is odd, then by Corollary 2,

$$\prod_{t=1}^{n} \theta^{t-1}\mu_t = \mu_1^2 \cdot \theta\mu_2 \cdot \prod_{t=4}^{(n+3)/2} \left[-\left(\theta^{t-1}\mu_t\right)^2 \right]$$

$$= (-1)^{(n-1)/2} \frac{[n(n-2)]^2 n^2 \left(\prod_{t=4}^{(n+3)/2} [n(n-2t+4)]^2 \right)}{4^n}$$

$$= (-1)^{(n-1)/2} \frac{n^{n+1}((n-2)(n-4)\times \cdots \times 3 \times 1)^2}{4^n}.$$

Hence if n is odd,

$$\det\left(C_n^{(2)}\right) = \prod_{t=1}^{n} \mu_t$$

$$= \theta^{-n(n-1)/2} \prod_{t=1}^{n} \theta^{t-1}\mu_t$$

$$= \frac{n^{n+1}[1\times 3\times 5\times \cdots \times (n-2)]^2}{4^n},$$

since $\theta^{-n(n-1)/2} = (-1)^{(n-1)/2}$ for an odd n.

If n is even, then by Theorem 3,

$$\mu_{(n+4)/2} = 0,$$

and therefore

$$\det\left(C_n^{(2)}\right) = 0.$$

This concludes the proof of Corollary 3.

Proof of Theorem 4. Recall that

$$\det(A_n) = (-1)^{n-1}\det(C_n)$$

and

$$\det(A_n^{(2)}) = \det(C_n^{(2)}).$$

Hence by Borchardt's theorem,

$$\operatorname{per}(A_n) = \frac{\det(A_n^{(2)})}{\det(A_n)}$$

$$= (-1)^{n-1}\frac{\det(C_n^{(2)})}{\det(C_n)}. \tag{12}$$

If n is even, then (12) with Corollary 3 gives

$$\operatorname{per}(A) = 0.$$

If n is odd, then (12) with Corollaries 1 and 3 yields

$$\operatorname{per}(A) = \frac{(-1)^{n-1}n^{n+1}[1\times3\times5\times\cdots\times(n-2)]^2/4^n}{(-i)^{n-1}n^n/2^n}$$

$$= (-1)^{(n-1)/2}\frac{n[1\times3\times5\cdots\times(n-2)]^2}{2^n}. \qquad \blacksquare$$

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