The integral kernel in the Kuznetsov Type sum formula for $SU(n + 1, 1)$

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1. INTRODUCTION

Let $G$ be a real, connected semisimple Lie group of real rank one, and let $\Gamma$ be a discrete subgroup of finite covolume of $G$. In [MW] Miatello and Wallach have given a generalization of the Kuznetsov sum formula ([K]) for this kind of groups. The sum formula relates spectral data concerning automorphic forms to geometric data concerning the intersection of a discrete subgroup with the big cell in the Bruhat decomposition. The $\tau$-function is the kernel for the integral transformation relating test functions on the spectral side to those on the geometric side. To apply the sum formula, it is necessary to understand this integral transformation well. In the classical case, this integral transformation can be described in terms of classical Bessel functions (see [K], [GW], [MW], Appendix), but in the general case the determination of the Kloosterman term in the Kuznetsov formula is more complicated. In [MW], Theorem 1.9, we see that these Bessel functions are reinterpreted as the $\tau$-function. This $\tau$-function can be computed using the $T(\nu)$-transform ([GW], [MW1]) which transforms conical vectors to Whittaker vectors on the spherical principal series.

In this paper we determine quite explicitly the $\tau$-function in the special case when the group $G$ is locally isomorphic to $SU(n + 1, 1)$, $n \geq 1$. The determination of the $\tau$-function involves solving complicated recurrence relations, and has not been given explicitly before except for the cases of $SO(n + 1, 1)$ and $SU(2, 1)$ (see [MW], Appendix).

Let $G = NAK$ be an Iwasawa decomposition of $G$ and let $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{f}$ be
the corresponding decomposition at the Lie algebra level. Let $M$ be the centralizer of $A$ in $K$ and let $\chi$ be a nontrivial unitary character on $N$. One necessary ingredient in those computations is the knowledge of generators for the $M_\chi$-invariants in the universal Lie algebra $U(\mathfrak{u})$ for those groups. At present, the space of $M_\chi$-invariants is known for general rank one Lie groups (see [MV]). In this paper, we shall use these results to obtain an explicit expression for the $\tau$-function in the case of $SU(n + 1, 1), n \geq 1$.

An outline of the paper is as follows. In §2 we introduce notations and recall some known facts on Whittaker vectors. Also the definition of the $\tau$-function is given. In §3 we carry out the computations needed to give explicit formulas for the coefficients of the Whittaker vector. The case $n = 1$ is treated separately. From these computations we obtain recurrence relations relating the coefficients. These relations, together with the assumption that the first coefficient is equal to 1 determine completely the Whittaker vector. In §4 we apply the results in Section 3 to the explicit determination of the $\tau$-function for $SU(n + 1, 1)$, again treating separately the case $n = 1$.

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2. PRELIMINARIES

Let $I_n$ be the identity $n \times n$ matrix and let

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

We identify $SU(n + 1, 1)$ with the real rank 1 Lie algebra:

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(n + 2, \mathbb{C}) \mid XJ + JX^t = 0, \quad trX = 0 \}$$

and denote by $G = SU(n + 1, 1)$ the connected Lie subgroup of $Gl(n + 2, \mathbb{C})$ with Lie algebra $\mathfrak{g}$. A Cartan involution of $\mathfrak{g}$ is given by $\theta(X) = -\bar{X}'$.

This involution induces the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. We take $\alpha$ the maximal abelian subalgebra in $\mathfrak{p}$ given by $\alpha = \mathbb{R}H$, where

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

Let $K$ and $A$ be the connected Lie subgroups of $G$ corresponding to $\mathfrak{h}$ and $\alpha$, respectively. Let $M$ be the centralizer of $A$ in $K$, and let $\mathfrak{m}$ be the corresponding Lie algebra of $M$. If $\alpha \in \alpha^*$ is such that $\alpha(H) = 1$, then let $\mathfrak{n}_\alpha$ and $\mathfrak{n}_{2\alpha}$ be the root spaces associated to $\alpha$ and $2\alpha$, respectively. We have: $\mathfrak{n}_\alpha = \{ X(x) \mid x \in \mathbb{C}^\alpha \}$ and $\mathfrak{n}_{2\alpha} = \mathbb{R}Z(i)$, where
$$X(x) = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & -x' \\ 0 & 0 & 0 \end{pmatrix}, \quad Z(i) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Also we have

$$m = \left\{ M(A) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & A & \vdots \\ 0 & \cdots & a \end{pmatrix} \mid A \in M_n(\mathbb{C}), A + \bar{A}' = 0, \ 2a + tr(A) = 0 \right\}.$$  

If $n = n_\alpha \oplus n_\alpha$, then $g$ has the Iwasawa decomposition $g = n \oplus \alpha \oplus f$. Let $G = NAK$ the corresponding Iwasawa decomposition at the group level. If $\bar{\theta} = \theta n$, the $g = n \oplus \alpha \oplus m \oplus \bar{\theta}$.  

Let $B(X, Y) = \frac{1}{2} tr(XY)$, $(X, Y) = -B(X, \theta Y)$, $X, Y \in g$. Then $B$ is $g$-invariant and $B(H, H) = 1$.  

If $e_1, \ldots, e_n$ denotes the canonical basis in $\mathbb{R}^n$, we set $X_j = X(e_j)$ and $X'_j = X'(ie_j)$, $Y_j = -\theta X_j$, $Y'_j = \theta X'_j$, $Z = Z(i)$, $Z' = -\theta Z$. Then $\{X_j, X'_k, Z \mid 1 \leq j, k \leq n\}$ is an orthonormal basis of $n$ with respect to $(\ , \ )$. Note that $[X_i, X'_j] = 2 \delta_{ij}Z$.  

If $\chi$ is a character of $N$, then there exists $X_\chi \in n_\alpha$ such that $d\chi(X) = i(X, X_\chi) = -iB(X, \theta X_\chi)$, for $X \in n$. Set $M_\chi = \{ m \in M \mid Ad(u)X_\chi = X_\chi \}$. As $M$ acts transitively on the unit sphere of $n_\alpha$ (cf. [MV, Introduction]) there is $u_0 \in M$ such that $Ad(u_0)X_\chi = cX_1$, $c \in \mathbb{R}^+$. So $M_\chi = u_0M_1u_0^{-1}$, where $M_1 = \{ u \in M \mid Ad(u)X_1 = X_1 \}$.  

Now consider de Verma module $M(-\nu) = U(\mathfrak{g}) \otimes_{U(p)} C_{-\nu - \rho}$, where $p = m \oplus \alpha \oplus n$ and $C_{-\nu - \rho}$ denotes the $p$-module $C$ with $m \oplus \alpha$ acting by $0$ and $\alpha$ acting by $-\nu - \rho$, $\nu \in \alpha^*$. Let $M(-\nu)[\bar{\theta}]$ denote the $\bar{\theta}$-completion of $M(-\nu)$ (see [GW], §2). If $J = (j_1, j_2, \ldots, j_m) \in \mathbb{N}^m$, $(\mathbb{N} = \{0, 1, 2, \ldots\})$, $m = 2n + 1$, and $Y(J) = Y^{j_1}_1 \ldots Y^{j_2}_n Z^{j_m}$, then by Poincaré-Birkhoff-Witt theorem, the set $\{ Y(J) \mid J \in \mathbb{N}^m \}$ constitutes a basis of $U(\bar{\theta})$. Hence every element in $M(-\nu)[\bar{\theta}]$ has an expansion of the type $\sum a_j Y(J) \otimes 1$, $a_j \in C$. A $\chi$-Whittaker vector is an element $v_\chi(-\nu)$ in $M(-\nu)[\bar{\theta}]$ that satisfies the equation

$$(2.1) \quad X.v = d\chi(X)v \quad \forall X \in n.$$  

Such a vector has an expression of the form

$$(2.2) \quad v_\chi(-\nu) = \sum_{I \in \mathbb{N}^m} a_I(\chi, \nu) Y(I) \otimes 1$$

where the coefficients $a_I(\chi, \nu) \in C$ are expressions depending on $\chi$ and $\nu$. There is a unique such Whittaker vector with $a_0(\chi, \nu) = 1$ (see [BM] §6, Lemma 11, for instance). Since $m \cdot v_\chi(-\nu) = v_\chi(-\nu)$ for $m \in M_\chi$, we have that $Y(I)$ must be a polynomial in the $M_\chi$-invariants of $U(\bar{\theta})$.  

Let $\chi$ be such that

$$(2.3) \quad d\chi(X_1) = \lambda, \quad d\chi(X_i) = d\chi(X'_j) = 0 \quad i > 1, \ j \geq 1$$

$\lambda \in i\mathbb{R}$. For this choice of $\chi$, we shall use the notation $u(\lambda, -\nu)$ for the $\chi$-
Whittaker vector. Then \( u(\lambda, -\nu) \) must be an element in \( \mathcal{U}(\mathfrak{n})^{M_1} \). We note that \( M_1 \cong SU(n - 1) \) is the subgroup of matrices in \( M \) of the form

\[
u_1(b, B) = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & b \end{pmatrix},
\]

\( B \in SU(n - 1) \). Hence \( u(\lambda, -\nu) = \sum Y_i f_i \), where \( f_i \) is an \( M_1 \)-invariant polynomial in \( Y_2, \ldots, Y_n, Y'_1, \ldots, Y'_n \) and \( Z' \). Then, by \([MV]\), \( f_i \in \mathbb{C}[Y'_1, Z', q_1] \), where \( q_1 = \sum_{i=2}^{n} Y_i^2 + Y'_i^2 \).

Let \( E_{i,j} \) be the matrix in \( gl(n + 2, \mathbb{C}) \) where the entry \( (i, j) \) is equal to \( 1 \), and all the other entries are zero. In order to have simpler formulas, it is convenient to change the basis of \( \mathfrak{n}_\mathbb{C} \) to

\[
\{ V_1, V_2, Y_2, \ldots, Y_n, Y'_2, \ldots, Y'_n, T \},
\]

where \( V_1 = E_{2,1}, V_2 = E_{n+2,2}, T = E_{n+2,1} \). Note that \( Y_1 = V_1 - V_2, Y_1' = -iV_1 - iV_2 \), and \( Z' = iT \). Let \( q \) be the element in \( \mathcal{U}(\mathfrak{n})^{M_1} \),

\[
q = \sum_{i=1}^{n} Y_i^2 + Y_i'^2.
\]

Now it is clear that we may write

\[
u(\lambda, -\nu) = \sum_{j,k,l} a_{j,k,l}(\lambda, \nu) V_1^j V_2^k T^l \otimes 1 \quad n = 1
\]

\[
u(\lambda, -\nu) = \sum_{j,k,l,m} a_{j,k,l,m}(\lambda, \nu) V_1^j V_2^k T^l q^m \otimes 1 \quad n > 1
\]

Now we use the \( T(\nu) \)-transform \([GW], [MW1]\) to compute the \( \tau \)-function. We see in \([MW]\) that this \( \tau \)-function is the main ingredient in the Bessel transform in the Kloosterman term of the Kuznetsov type Sum Formula and also it appears in the Fourier coefficients of the Poincaré series. To get the \( \tau \)-function one has to consider two parabolic \( \Gamma \)-percuspidal subgroups \( P \) and \( P' \). Then \( P = \text{NAM} \) and \( P' = \text{N}'A'M' \). Let \( \chi \) and \( \chi_1 \) be nontrivial unitary characters on \( N \) and \( N' \) respectively. If \( W(A) = \{ 1, s \} \) is the Weyl group of \( (P, A) \) then we take \( s^* \) a representative of \( s \) in \( K \). Then the \( \tau \)-function is given by the following formula (see \([MW]\) Proposition 1.2):

\[
\tau(\chi_1, \chi, ua, \nu) = \sum_{I \in \mathbb{N}^m} a_I(\chi, \nu) d\chi_1(\text{Ad}(ua s^*)) Y(I)^T \otimes 1
\]

\( u \in M, a \in A \) and where the coefficients \( a_I \) are given by the formula (2.2). Here, \( Y \rightarrow Y^T \) is the automorphism in the universal enveloping algebra given by \( X \rightarrow -X \) if \( X \in \mathfrak{g} \).

The case \( \chi = \chi_1 \) has a special interest because it gives information on the discrete spectrum of the Laplacian on \( L^2(\Gamma \backslash G/K) \) (cf. \([MW1]\)).
3. AN EXPLICIT FORMULA FOR THE WHITTAKER VECTOR

The aim of this section is to look for a Whittaker vector \( u(\lambda, \nu + \rho) \) in the \( \bar{\Pi} \)-completion of the Verma module \( M(\nu + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}_\nu} \mathbb{C}_\nu \), where \( \mathfrak{m} \otimes \mathfrak{n} \) acts by 0 and \( H \) acts by \( \nu(H) \), \( \nu \in \mathfrak{a}^* \). We shall write \( \nu \) instead of \( \nu(H) \).

**Lemma 3.1.** Let \( X_1, X_1', V_1, V_2, T \) and \( H \) be as above. Let \( \Lambda = \frac{1}{2} [H + iM(2iE_{11})] \) and \( \Pi = \frac{1}{2} [-H + iM(2iE_{11})] \). Then, the following commutation relations hold for \( j, k, l \geq 1 \):

\[
\begin{align*}
[X_1, V_1'] &= jV_1^{j-1}(\Lambda - j + 1) \\
[X_1, V_2^k] &= kV_2^{k-1}(\Pi + k - 1) \\
[X_1, T'] &= -l(V_1 + V_2)T'^{-1} \\
[X_1', V_1'] &= jV_1^{-j-1}(i\Lambda - i(j - 1)) \\
[X_1', V_2^k] &= kV_2^{-k-1}(-i\Pi - i(k - 1)) \\
[X_1', T'] &= il(V_1 - V_2)T'^{-1}
\end{align*}
\]

Also we have: \( [\Lambda, V_2] = V_2, [\Lambda, T] = -T \) and \( [\Pi, T] = T \).

**Proof.** The formulas are proved by the usual \( \mathfrak{sl}(2, \mathbb{C}) \)-technique. They are based on the identities: \( [X_1, V_1] = \Lambda, [X_1, V_2] = \Pi, [X_1', V_1] = i\Lambda, [X_1', V_2] = -i\Pi \) and the fact that \( [V_1, V_2] = T \).

Thus, it follows from Lemma 1 that:

\[
X_1V_1^jV_2^kT'^l \otimes 1 = \frac{(X_1, V_1')V_2^kT'^l \otimes 1 + V_1^j(X_1, V_2^k)V_2^kT'^l \otimes 1 + V_1^jV_2^k(X_1, T'^l) \otimes 1}{-jV_1^{j-1}(\Lambda - j + 1)V_2^kT'^l \otimes 1 + kV_1^jV_2^{k-1}(\Pi + k - 1)T'^l \otimes 1} \\
= jV_1^{j-1}V_2^kT'(k - l - j + 1 + \Lambda) \otimes 1 + kV_1^jV_2^{k-1}T'(l + k - 1 + \Pi) \otimes 1 \\
- lV_1^jV_2^kV_1T'^{-l} \otimes 1 - lV_1^jV_2^{k+1}T'^{-l} \otimes 1
\]

Using that \( [V_1, V_2^k] = -kV_2^{k-1}T \) and the fact that \( \Lambda \) and \( \Pi \) act by \( \nu/2 \) and \(-\nu/2\) respectively, the last expression is equal to

\[
jV_1^{j-1}V_2^kT'(k - l - j + 1 + \nu/2) \otimes 1 + kV_1^jV_2^{k-1}T'(k - 1 - \nu/2) \otimes 1 \\
- lV_1^jV_2^{k+1}V_2T'^{-l-1} \otimes 1 - lV_1^jV_2^{k+1}T'^{-l-1} \otimes 1
\]

Now, formula (2.1) implies that:

\[
\sum_{j, k, l \geq 0} a_{j, k, l}(\lambda, \nu)X_1V_1^jV_2^kT'^l \otimes 1 = \lambda \sum_{j, k, l \geq 0} a_{j, k, l}(\lambda, \nu)V_1^jV_2^kT'^l \otimes 1
\]

\[
\sum_{j, k, l \geq 0} a_{j, k, l}(\lambda, \nu)X_1'V_1^jV_2^kT'^l \otimes 1 = 0
\]

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From formulas (3.2) and (3.3) we obtain the following recurrence relation for the coefficients $a_{j,k,l} = a_{j,k,l}(\nu, \lambda)$:

$$\lambda a_{j,k,l} = (j + 1)(k - j - (\nu/2))a_{j+1,k,l} + (k + 1)(\nu/2)a_{j,k+1,l} - (l + 1)a_{j-1,k,l+1} - (l + 1)a_{j,k-1,l+1}$$

(3.5)

Also, we have

$$X^j_1 V_1^j V_2^k T^l \otimes 1 =$$

$$= jV_1^j (i(\lambda - j + 1 + k - l)) \otimes 1 +$$

$$+ V_1^j k V_2^{k-1} T^l (\Pi_2 - i(k - 1 + l)) \otimes 1 +$$

$$+ il V_1^j V_2^k V_1^l T^{-1} \otimes 1 - il V_1^j V_2^k T^{-1} \otimes 1$$

(3.6)

This computation together with the formula (3.4) gives a second recurrence relation for the coefficients $a_{j,k,l}$:

$$(j + 1)(-j + k - l + (\nu/2))a_{j+1,k,l} + (k + 1)(-k + (\nu/2))a_{j,k+1,l} + (l + 1)a_{j-1,k,l+1} - (l + 1)a_{j,k-1,l+1} = 0$$

(3.7)

Finally, from the identities (3.5) and (3.7) we get the following relations:

$$(3.8) \hspace{1cm} \lambda a_{j,k,l} = (j + 1)(\nu - 2j + 2k - 2l)a_{j+1,k,l} - 2(l + 1)a_{j,k-1,l+1}$$

$$(3.9) \hspace{1cm} \lambda a_{j,k,l} = (k + 1)(-\nu + 2k)a_{j,k+1,l} - 2(l + 1)a_{j-1,k,l+1}$$

Proposition 3.2. Let $\chi$ be as in (2.3). If $u(\lambda, \nu + \rho)$ denotes the canonical $\chi$-Whittaker vector in $M(\nu + \rho)$ for $SU(2,1)$, then the coefficients $a_{j,k,l}(\nu, \lambda)$ of $u(\lambda, \nu + \rho)$ are given by the formula:

$$a_{j,k,l}(\nu, \lambda) = \frac{\chi^{j + k + 2l} \prod_{i=k+l}^{j+2l} (-1)^{k+i} (\nu + i)}{2^j j! k! l! \prod_{i=0}^{j-1} (\nu + 2i) \prod_{i=0}^{k-1} (\nu - 2i) \prod_{i=0}^{j+l-1} (\nu + 1 - i)}$$

(3.10)

$$a_{0,0,0}(\nu, \lambda) = 1$$

Proof. We shall prove the formula using induction in $n$, where $n = j + k + 2l$. We see from (3.8) and (3.9) that $\lambda a_{0,0,0} = \nu a_{1,0,0} = -\nu a_{0,1,0}$, so $a_{j,k,l}$ satisfies (3.10) if $j + k + 2l = 1$.

If $n = j + k + 2l$ is greater than 1 then we have to consider several cases:
(a) \( j = 0, k \neq 0 \), (b) \( j \neq 0, k = 0 \), (c) \( j = k = 0 \) and (d) \( j \neq 0, k \neq 0 \). In case (a) we may apply formula (9) putting \( j = 0 \) and using \( k - 1 \) instead of \( k \) so to obtain

\[
\lambda a_{0,k-1,l} - k \cdot (-\nu + 2k - 2)a_{0,k,l}.
\]

The case (b) is similar but using formula (3.8) with \( k = 0 \) and using \( j - 1 \) instead of \( j \).

In the case of (c) we apply the formulas (3.8) with \( j = 0, k = 1 \) and \( l - 1 \) instead of \( l \) and (3.9) with \( j = 1, k = 0 \) and \( l - 1 \) instead of \( l \). Then we get:

\[
(3.11) \quad \lambda a_{0,1,l-1} = (\nu + 2 - 2l + 2)a_{1,1,l-1} - 2la_{0,0,l},
\]

\[
(3.12) \quad \lambda a_{1,0,l-1} = (-\nu)a_{1,1,l-1} - 2la_{0,0,l}.
\]

Then \( a_{1,1,l-1} \) is determined by \( a_{0,1,l-1} \) and \( a_{1,0,l-1} \):

\[
\lambda a_{0,1,l-1} - \lambda a_{1,0,l-1} = 2(\nu - l + 2)a_{1,1,l-1},
\]

more precisely:

\[
2(\nu - l + 2)a_{1,1,l-1} = \lambda \left[ \frac{\lambda^{2l-1}(-1)^l}{2^{l-1}(l-1)! \prod_{0}^{l-2} (\nu - 2i) \prod_{0}^{l-2} (\nu + 1 - i)} \right. \\
- \frac{\lambda^{2l-1}(-1)^l(-1)^l(1)}{2^{l-1}(l-1)! \prod_{0}^{l-1} (\nu - 2i) \prod_{0}^{l-2} (\nu + 1 - i)} a_{0,0,0} \\
\left. \frac{\lambda^{2l}(1)^l(\nu + 1 - l)}{2^l(l-1)! \prod_{0}^{l-2} (\nu - 2i) \prod_{0}^{l-1} (\nu + 1 - i)} a_{0,0,0} \right].
\]

Now we replace the value of \( a_{1,1,l-1} \) in formula (3.12) and we get a formula for \( a_{0,0,l} \):

\[
2la_{0,0,l} = -\nu a_{1,1,l-1} - \lambda a_{1,0,l-1}
\]

\[
= \frac{\lambda^{2l}(-1)^l}{2^{l-1}(l-1)! \prod_{0}^{l-2} (\nu - 2i) \prod_{0}^{l-1} (\nu + 1 - i)} \times [-(-\nu + l) + (\nu + 1 - i) + 1]a_{0,0,0}
\]

so

\[
a_{0,0,l} = \frac{\lambda^{2l}(-1)^l}{2^l l! \prod_{0}^{l-1} (\nu - 2i) \prod_{0}^{l-1} (\nu + 1 - i)} a_{0,0,0}.
\]

(d) If \( j \neq 0 \) and \( k \neq 0 \) then we use formula (3.8) putting \( j - 1 \) instead of \( j \) and putting \( k - 1 \) in formula (3.9) instead of \( k \). Substracting one formula from the other we get a formula for \( a_{j,k,l} \) in terms of \( a_{j-1,k,l} \) and \( a_{j,k-1,l} \). □

Now let us consider the case \( n > 1 \).
Lemma 3.3. Let \( X_i, Y_i, Y'_i, V_1, V_2, T \) and \( q \) be as in the preliminaries. Then the following identities hold: for \( i \geq 2 \),

\[
[X_i, V'_i] = jV_i^{j-1}M_i \quad \text{where} \quad M_i = \frac{1}{2}(M(E_{ii} - E_{il}) + iM(E_{ii} + iE_{il}))
\]

\[
[X_i, V_2^k] = kV_2^{k-1}(M_i)'
\]

\[
[X_i, T'] = -iT^{i-1}(Y_i')
\]

\[
[M_i, V_2^k] = (k/2)(-Y_i + iY_i')V_2^{k-1}
\]

For \( i \geq 1 \)

\[
[X_i, q^m] = 2 \sum_{s=0}^{m-1} \epsilon_s 4^s U_{i,s} T^s q^{m-1-s}
\]

(3.13)

\[
\sum_{j=s}^{m-1} \frac{j}{s} (n + H - 2(m - 1 - j)) \mod U(g)^m
\]

where \( \epsilon_s U_{i,s} = Y_i \) if \( s \) is even, and \( \epsilon_s U_{i,s} = -iY_i' \) if \( s \) is odd.

Proof. The first identities follow as in Lemma 3.1. We will prove the formula (3.13) by using induction on \( m \). We need the formulas:

\[
[[X_i, Y_j], Y_k] = -\delta_{ij} Y_k + \delta_{jk} Y_i - \delta_{ik} Y_j
\]

\[
[[X_i, Y'_j], Y'_k] = \delta_{ij} Y_k + \delta_{jk} Y_i + \delta_{ik} Y_j
\]

\[
[X_i, Y_j] = \delta_{ij} H + E_{ij} - E_{ji}
\]

\[
[X_i, Y'_j] = M(i(E_{ij} + E_{ji}))
\]

Then

\[
[X_i, q] = \sum_{j=1}^{n} \left( [[X_i, Y_j^2] + [X_i, Y'_j^2]] \right)
\]

\[
= \sum_{j=1}^{n} \left( [[[X_i, Y_j], Y_j] + [[[X_i, Y'_j], Y'_j] + 2Y_j[X_i, Y_j] + 2Y'_j[X_i, Y'_j]] + 2Y_i H \mod U(g)^m \right)
\]

\[
= \sum_{j=1}^{n} (2\delta_{ij} Y_j + Y_i + 2\delta_{ij} Y_j + Y_i) \mod U(g)^m
\]

\[
= 2Y_i(n + H) \mod U(g)^m
\]

Now, if \( m \geq 1 \) and since \([H, q^m] = (-2m)q^m \) and since \( q \) is \( m \) invariant we have:

\[
[X_i, q^{m+1}] = [X_i, q^m]q + q^m[X_i, q]
\]

(3.15)

\[
= 2 \sum_{s=0}^{m-1} \epsilon_s 4^s U_{i,s} T^s q^{m-1-s} \sum_{j=s}^{m-1} \frac{j}{s} (n + H - 2(m - j))
\]

\[
+ 2q^m Y_i(n + H) \mod U(g)^m
\]

Now we compute the bracket \([q^m, Y_i] \). \( \square \)

Lemma 3.4. Let \( \epsilon_s \) and \( U_{i,s} \) as in Lemma 3.3. Then, for \( i \geq 1 \),
\[
\begin{align*}
[Y_i, q'] &= \sum_{s=1}^{j} -\varepsilon_s 4^s \binom{j}{s} U_{i,s} T^s q^{j-s} \\
[Y_i', q'] &= i \sum_{s=1}^{j} -\varepsilon_s -1 4^s \binom{j}{s} U_{i,s-1} T^s q^{j-s}
\end{align*}
\]

**Proof.** (of Lemma 4): First we see that \([Y_i, q] = 4i Y_i'T\) and \([Y_i', q] = -4i Y_i T\).

By arguing by induction we see that for \(j \geq 1\):

\[
\begin{align*}
[Y_i, q^{j+1}] &= \sum_{s=1}^{j} -\varepsilon_s 4^s \binom{j}{s} U_{i,s} T^s q^{j+1-s} + 4iq^j Y_i'T \\
&= \sum_{s=1}^{j} -\varepsilon_s 4^s \binom{j}{s} U_{i,s} T^s q^{j+1-s} + 4iY_i'Tq^j + 4i([q^j, Y_i'])T \\
&= \sum_{s=1}^{j} -\varepsilon_s 4^s \binom{j}{s} U_{i,s} T^s q^{j+1-s} + 4iY'_Tq^j \\
(3.16)
\end{align*}
\]

The proof for \([Y_i', q^j]\) is similar. \(\square\)

We go back to the proof of Lemma 3.3. We have that \([X_i, q^{m+1}]\) is equal, modulo \(\mathcal{U}(g)_{m}\) to:

\[
\begin{align*}
[X_i, q^{m+1}] &= 2\epsilon_0 U_{i,0} q^m \left( \sum_{j=0}^{m} (n + H - 2(m-j)) + (n + H) \right) \\
&\quad + 2 \sum_{m=1}^{s} \epsilon_s 4^s U_{i,s} T^s q^m -s \sum_{m=1}^{j=s} \binom{j}{s} (n + H - 2(m-j)) \\
&\quad + 2 \sum_{m=1}^{s} \epsilon_s 4^s \binom{m}{s} U_{i,s} T^s q^m -s (n + H) \\
&= 2\epsilon_0 U_{i,0} q^m \left( \sum_{j=0}^{m} (n + H - 2(m-j)) \right) \\
&\quad + 2 \sum_{m=1}^{s} \epsilon_s 4^s U_{i,s} T^s q^m -s \sum_{j=s}^{m} \binom{j}{s} (n + H - 2(m-j)) \\
&\quad + 2\epsilon_m 4^m U_{i,m} T^m (n + H) \\
(3.17)
&= 2 \sum_{s=0}^{m} \epsilon_s 4^s U_{i,s} T^s q^m -s \sum_{j=s}^{m} \binom{j}{s} (n + H - 2(m+1-j))
\end{align*}
\]

Then we can write:
\[ [X_i, q^m] = Y_i \left( 2 \sum_{s=0, s \text{ even}}^{m-1} 4^s T^s q^{m-1-s} \sum_{m=1}^{j=s} \left( \frac{j}{s} \right) (n + H - 2(m - 1 - j)) \right) \]

\[ - i Y_i' \left( 2 \sum_{s=0, s \text{ odd}}^{m-1} 4^s T^s q^{m-1-s} \sum_{m=1}^{j=s} \left( \frac{j}{s} \right) (n + H - 2(m - 1 - j)) \right) \]

mod \mathcal{U}(q)_m

We set

\[ \sigma(m - 1, s, \nu) = \sum_{j=s}^{m-1} \left( \frac{j}{s} \right) (n + \nu - 2(m - 1 - j)). \]

From Lemma 3.3 we have that for \( i \geq 2, \)

\[ X_i(V_1 j V_2 k T^l q^m \otimes 1) = j V_1 j^{-1} M_i V_2 k T^l q^m \otimes 1 \]

\[ + V_1 j k V_2 k^{-1} (M_i)^{T^l} q^m \otimes 1 + V_1 j V_2 k (-i l T^{l-1} Y_i') q^m \]

\[ \otimes 1 + V_1 j V_2 k T^l (X_i q^m) \otimes 1 \]

As \([M, T] = [M, q] = 0\) for \( M \in m, \) Lemma 3.3 implies that formula (3.20) is equal to:

\[ \frac{-jk}{2} (Y_i - i Y_i') V_1 j^{-1} V_2 k^{-1} T^l q^m \otimes 1 - i l Y_i' V_1 j V_2 k T^{l-1} q^m \otimes 1 \]

\[ + V_1 j V_2 k T^l (X_i q^m) \otimes 1 \]

Hence

\[ X_i(V_1 j V_2 k T^l q^m \otimes 1) \]

\[ = Y_i \left( \frac{-jk}{2} V_1 j^{-1} V_2 k^{-1} T^l q^m \otimes 1 + 2 \sum_{s=0, s \text{ even}}^{m-1} 4^s V_1 j V_2 k T^{l+s} q^{m-1-s} \right) \]

\[ \times \sigma(m - 1, s, \nu) \otimes 1 \]

\[ + i Y_i' \left( \frac{-jk}{2} V_1 j^{-1} V_2 k^{-1} T^l q^m \otimes 1 - i l V_1 j V_2 k T^{l-1} q^m \otimes 1 \right) \]

\[ - 2 \sum_{s=0, s \text{ odd}}^{m-1} 4^s V_1 j V_2 k T^{l+s} q^{m-1-s} \sigma(m - 1, s, \nu) \otimes 1 \]

Using the computations in the case \( n = 1, \) we have that for \( i = 1: \)

\[ X_1(V_1 j V_2 k T^l q^m \otimes 1) = j V_1 j^{-1} V_2 k T^l q^m (-j + 1 + k - l - m + \nu/2) \otimes 1 \]

\[ + k V_1 j V_2 k^{-1} T^l q^{m(k-1+m+\nu/2)} \otimes 1 \]

\[- l V_1 j V_2 k T^{l-1} q^m \otimes 1 - l V_1 j V_2 k T^{l-1} q^m \otimes 1 + V_1 j V_2 k T^l [X_1, q^m] \otimes 1 \]

Using that \( V_1 \) and \( V_2 \) commute with \( T \) and the identities

\[ [V_2 k, Y_1] = [V_2 k, V_1 - V_2] = k V_2 k^{-1} T + V_1 V_2 k - V_2 k + 1 \]

\[ [V_2 k, Y_1'] = [V_2 k, -i(V_1 + V_2)] = -i k V_2 k^{-1} T - i V_1 V_2 k - i V_2 k + 1 \]

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we conclude that

\begin{equation}
X_1.(V_1^j V_2^k T^l q^m \otimes 1) = jV_1^{i-1} V_2^k T^l q^m (-j + 1 + k - l - m + \nu/2) \otimes 1 \\
+ kV_1^{i-1} V_2^k - 1T^l q^m(k - 1 + m - \nu/2) \otimes 1 \\
- IV_1^{i+1} V_2^k T^l q^m \otimes 1 - IV_1^{i+1} V_2^k T^l q^m \otimes 1
\end{equation}

(3.24)

\begin{align*}
X_\nu.(V_1^j V_2^k T^l q^m \otimes 1) &= jV_1^{i-1} V_2^k T^l q^m (-j + 1 + k - l - m + \nu/2) \otimes 1 \\
+ kV_1^{i-1} V_2^k - 1T^l q^m(k - 1 + m - \nu/2) \otimes 1 \\
- IV_1^{i+1} V_2^k T^l q^m \otimes 1 - IV_1^{i+1} V_2^k T^l q^m \otimes 1
\end{align*}

(3.25)

where

\begin{align*}
\Sigma_e(T, q, \nu, m) &= 2 \sum_{s=0}^{m-1} 4^s T^s q^{m-1-s} \sigma(m - 1, s, \nu) \\
\Sigma_o(T, q, \nu, m) &= 2 \sum_{s=0}^{m-1} 4^s T^s q^{m-1-s} \sigma(m - 1, s, \nu)
\end{align*}

That is:

\begin{align*}
X_1.(V_1^j V_2^k T^l q^m \otimes 1) &= jV_1^{i-1} V_2^k T^l q^m (-j + 1 + k - l - m + \nu/2) \otimes 1 \\
+ kV_1^{i-1} V_2^k - 1T^l q^m(k - 1 + m - \nu/2) \otimes 1 \\
- IV_1^{i+1} V_2^k T^l q^m \otimes 1 - IV_1^{i+1} V_2^k T^l q^m \otimes 1
\end{align*}

(3.26)

With similar computations we get the formula:

\begin{align*}
X_\nu.(V_1^j V_2^k T^l q^m \otimes 1) &= jV_1^{i-1} V_2^k T^l q^m (-j + 1 + k - l - m + \nu/2) \otimes 1 \\
+ kV_1^{i-1} V_2^k - 1T^l q^m(k - 1 + m - \nu/2) \otimes 1 \\
- IV_1^{i+1} V_2^k T^l q^m \otimes 1 - IV_1^{i+1} V_2^k T^l q^m \otimes 1
\end{align*}
Proposition 3.5. The coefficients $a_{j,k,l,m}(v, \lambda)$ for the $\chi$-Whittaker vector $u(\lambda, v + \rho)$ in $M(v + \rho)$ satisfy the following recurrence relations:

\[
\lambda a_{j,k,l,m} = (j + 1)(v - 2j + 2k - 2l - 2m) a_{j+1,k,l,m}
\]  
(3.27)

\[
-2(l+1) a_{j,k-1,l+1,m} - \sum_{s=0}^{l} 4^{s+1} \sigma(m+s,s,v) a_{j-1,k-l-s,m+1+s}
\]

\[
\lambda a_{j,k,k,m} = (k+1)(2k + 2m - v) a_{j,k+1,l,m}
\]  
(3.28)

\[
-2(l+1) a_{j-1,k,l+1,m} \times a_{j,k+1,l-1-s,m+1+s} \sigma(m+s,s,v)
\]

\[
+ \sum_{s=0}^{l} (-1)^{s} 4^{s+1} a_{j-1,k,l-s,m+1+s} \sigma(m+s,s,v)
\]

\[
(l+1) a_{j,k,l+1,m} = 2 \sum_{s=0}^{l} (-1)^{s} 4^{s} \sigma(m+s,s,v) a_{j,k,l-s,m+1+s}
\]  
(3.29)

\[
(j+1)(k+1) a_{j+1,k+1,l,m} = (l+1) a_{j,k,l+1,m}
\]  
(3.30)

Proof. The equations follow from the definition of the Whittaker vector (formula (1)), formulas (18), (20) and (20), and the fact that the monomials $Y_{i}V_{1}^{j}V_{2}^{k}T^{l}q^{m}$ and $Y_{i}V_{1}^{j}V_{2}^{k}T^{l}q^{m}$ are linearly independent. □

From equation (3.29) we see that $a_{j,k,l,m} = b_{j,k,l,m}(v) a_{j,k,0,l+m}$, where $b_{j,k,l,m}$ also depends on $n$. This function can be computed explicitly by the formulas:

\[
b_{j,k,0,m}(v) = 1
\]  
(3.31)

\[
(l+1) b_{j,k,l+1,m}(v) = 2 \sum_{s=0}^{l} (-1)^{s} 4^{s} \sigma(m+s,s,v) b_{j,k,l-s,m+1+s}(v)
\]  
(3.32)

Later on we shall prove that the factor $b_{j,k,l,m}$ actually does not depend on $j$ and $k$.

From formulas (3.30) and (3.27) we have:

\[
\lambda a_{j,k,l,m} = (j+1)(v - 2j - 2l - 2m) a_{j+1,k,l,m}
\]  
(3.33)

and hence it follows that

\[
a_{j,k,l,m} = \frac{\lambda^{j}}{j! \prod_{i=1}^{l+m-1} (v-2i)} a_{0,k,l,m}
\]  
(3.34)

From formulas (3.29) and (3.28) we get:

\[
\lambda a_{j,k,l,m} = (k+1)(2k + 2l + 2m - v) a_{j,k+1,l,m}
\]  
(3.35)

hence

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Finally, from (3.34) and (3.36) we get the formula:

\[
\begin{align*}
(3.37) \quad a_{j,k,l,m} &= \frac{(-1)^{k+j+k} \lambda^{k} \chi^{k}}{j! k! \prod_{i=1+m}^{k+l+m-1} (\nu - 2i)} a_{0,0,l,m} \\
&= \frac{(-1)^{k+j+k} \lambda^{k}}{j! k! \prod_{i=1+m}^{k+l+m-1} (\nu - 2i)} a_{0,0,l,m}
\end{align*}
\]

Now we are going to prove that \(b_{j,k,l,m}(\nu)\) does not depend on \(j\) and \(k\). We denote

\[
\begin{align*}
g_{j,k,l,m}(\nu) &= \frac{(-1)^{k+j+k} \lambda^{k}}{j! k! \prod_{i=1+m}^{k+l+m-1} (\nu - 2i)} a_{0,0,l,m} \\
&= \frac{(-1)^{k+j+k} \lambda^{k}}{j! k! \prod_{i=1+m}^{k+l+m-1} (\nu - 2i)} a_{0,0,l,m}
\end{align*}
\]

then by (3.29) and (3.36)

\[
\begin{align*}
a_{j,k,l,m} &= b_{j,k,l,m}(\nu) a_{j,k,0,l} \\
&= g_{j,k,l,m}(\nu) a_{0,0,l,m} \\
&= g_{j,k,l,m}(\nu) b_{0,0,l,m}(\nu) a_{0,0,l,m}
\end{align*}
\]

But \(a_{j,k,0,l} = g_{j,k,l,m}(\nu) a_{0,0,l,m}\), so \(b_{j,k,l,m}(\nu) = b_{0,0,l,m}(\nu), \forall j, k\). So we shall use the notation \(b_{l,m}\) instead of \(b_{j,k,l,m}\). Now, from formulas (3.29) and (3.30) it follows that:

\[
\begin{align*}
2 a_{j,k,0,m+1} + \sigma(m, 0) &= (j + 1)(k + 1) a_{j+1,k+1,0,m} - a_{j,k,1,m} \\
&= \frac{(-1)^{j-k} \lambda^{2}}{(\nu - 2j)(\nu - 2k - 2m)} a_{j,k,0,m} - 2\sigma(m, 0) a_{j,k,0,m+1}
\end{align*}
\]

Hence:

\[
\begin{align*}
a_{j,k,0,m} &= \frac{(-1)^{j-k} \lambda^{2}}{4\sigma(m-1,0)(\nu - 2j - 2m + 2)(\nu - 2k - 2m + 2)} a_{j,k,0,m+1} \\
&= \frac{(-1)^{m} \lambda^{2m}}{4m! \prod_{i=j}^{m+1} (\nu - 2i) \prod_{i=1}^{k+m-1} (\nu - 2i)} a_{j,k,0,m+1}
\end{align*}
\]

Finally, under the assumption that \(a_{0,0,0,0} = 1\) we obtain the formula:

\[
\begin{align*}
a_{j,k,l,m} &= b_{l,m}(\nu) a_{j,k,0,l} \\
(3.38) &= \frac{b_{l,m}(\nu)(-1)^{k+l+m} \chi^{j+k+l+2m}}{4^{l+m}(l+m)!j!k! \prod_{i=0}^{l+m} (\nu - 2i) \prod_{i=0}^{k+m-1} (\nu - 2i) \prod_{i=0}^{l+m-1} (n + \nu - i)}
\end{align*}
\]

Now let \(M(-\nu)\) be the Verma module \(M(-\nu) = U(g) \otimes U_{(\alpha)} \mathbb{C}_{-\nu - \rho}\), where
$C_{-\nu-\rho}$ is the $\mathfrak{p}$-module with $m \otimes n$ acting by 0 and $\alpha$ acting by $-\nu - \rho$. Let $u(-\nu) = \sum_{i} a_i(\lambda, \nu) Y(L) \otimes 1$ be the $\chi$-Whittaker vector on the Verma module $M(-\nu)$, with $\chi$ as in formula (2.3).

To obtain an explicit formula for the Whittaker vector $u(-\nu)$, we must change the parameter $\nu$ in the formulas (3.10) and (3.38) into $-\nu - \rho$, since $H$ acts by $(-\nu - \rho)(H)$. Remember that $\rho(H) = n + 1$ if $G \simeq SU(n + 1, 1)$. Then we get the formulas:

$$a_{j,k,l}(-\nu - \rho) = \frac{(-1)^{j+l}(\lambda/2)^{j+k+2l} \prod_{i=k+l+1}^{j+k+l} (\nu + i)}{j! k! l! \prod_{i=1}^{j+l} (\nu/2 + i) (\nu + i) \prod_{i=1}^{k} (\nu/2 + i)}$$

if $n = 1$

and

$$a_{j,k,l,m}(-\nu - \rho) = \frac{(-1)^{j}(\lambda/2)^{j+k+2l+2m}b_{j,l,m}(-\nu - \rho)}{4^l \cdot j! k! l! \prod_{i=1}^{j+l+m} (\nu/2 + i + 1) \prod_{i=1}^{k+l+m} (\nu + i) \prod_{i=1}^{l} (\nu + i)}$$

if $n > 1$.

Furthermore, it will be convenient to multiply $u(-\nu - \rho)$ by the normalizing factor $I(\nu) = ([G((\nu + n + 1)/2)^2 I(\nu + 1)]^{-1}$ in order to obtain a holomorphic Whittaker vector $\tilde{u}(\lambda, -\nu - \rho)$. We may now state the main results in this section.

**Theorem 3.6.** Let $G$ be locally isomorphic with $SU(n + 1, 1)$. Then a holomorphic Whittaker vector is given by the following formula:

(i) If $n = 1$, then

$$\tilde{u}(\lambda, -\nu - \rho) = \sum_{j,k,l \geq 0} \frac{(-1)^{j+l}(\lambda/2)^{j+k+2l} \prod_{i=k+l+1}^{j+k+l} (\nu + i) V_{1}^{j} V_{2}^{k} T^{l}}{j! k! l! \prod_{i=1}^{j+l} (\nu/2 + i) (\nu + i) \prod_{i=1}^{k} (\nu/2 + i)} \times \frac{1}{\Gamma(\nu/2 + j + l + 1) \Gamma(\nu/2 + k + 1) \Gamma(\nu + j + l + 1)}$$

(ii) If $n > 1$, then

$$\tilde{u}(\lambda, -\nu - \rho) = \sum_{j,k,l,m \geq 0} \frac{(-1)^{j+l-m}(\lambda/2)^{j+k+2l+2m}b_{j,l,m}(-\nu - \rho) V_{1}^{j} V_{2}^{k} T^{l} q^{m}}{j! k! l! \prod_{i=1}^{j+l+m} (\nu + i + 1) \prod_{i=1}^{k+l+m} (\nu + i) \prod_{i=1}^{l+m} (\nu + i)} \times \frac{1}{\Gamma(\nu + n + 1/2 + j + l + m \prod_{i=1}^{j+l+m} (\nu + i + 1) \prod_{i=1}^{k+l+m} (\nu + i) \prod_{i=1}^{l+m} (\nu + i)}}$$

**Remark 1.** We observe from the formula (3.19) that
\[
\sigma(m,s,-\nu - \rho) = \sum_{j=s}^{m} \binom{m}{j} (-\nu - 1 - 2(m - j)) \\
= - \left( \frac{m + 1 + s}{s} \right) \frac{(\nu s + s + 2 \nu + 2m + 2)(m + 1)}{(s + 2)(s + 1)}
\]

and this together with (3.31) implies that \( b_{i,m}(-\nu - \rho) \) is a polynomial in \( \nu \) of degree \( l \).

4. AN EXPLICIT FORMULA FOR THE \( \tau \)-FUNCTION

Let \( G \) be a real rank one group locally isomorphic to \( \text{SU}(n + 1,1), n \geq 1 \). We recall from §2 the definition of the \( \tau \)-function:

\[
\tau(\chi_1, \chi, u\alpha, \nu) = \sum_{I \in \mathbb{N}^m} a_I(-\nu) d_{\chi_1} (\text{Ad}(u\alpha^\ast)^{-1} Y(I)^T)
\]

where \( u \in M, a \in A \) and \( u(\lambda, -\nu) = \sum_I a_I(\lambda, \nu) Y(I) \otimes 1 \) is the \( \chi \)-Whittaker vector on the Verma module \( M(-\nu) \).

This \( \tau \)-function appears in the \( \chi_1 \)-Fourier coefficient \( D_{\chi_1}^\chi \) of the Poincaré series studied in [MW] (see [MW], Proposition 1.2). We recall that, as proved in [MW1], \( D_{\chi_1}^\chi(P, P, \nu) \) has a meromorphic continuation to \( \mathbb{C} \) and the nonzero eigenvalues of the Casimir operator \( C \) on \( L^2_\chi(\Gamma \backslash G/K) \) have the form \( \nu_j(H)^2 - \rho(H)^2 \), where \( \nu_j \) ranges over the poles of \( \{ D_{\chi_1}^\chi(P, P, \nu) \mid \chi \in (\Gamma_N = \Gamma \cap N) \} \) in the closed right half plane. Our main goal in this section will be to give an explicit formula for \( \tau(\chi, \chi, u\alpha, \nu) \).

We have to compute \( \text{Ad}(u\alpha^\ast)^{-1} Y(I) \). If we take

\[
s^* = \begin{pmatrix}
0 & 0 & i \\
0 & I_n & 0 \\
i & 0 & 0
\end{pmatrix}
\]

where \( I_n \) denotes the \( n \times n \) identity matrix, we have that \( \text{Ad}(s^\ast)^{-1} Y_i = X_i^\prime \), \( \text{Ad}(s^\ast)^{-1} Y_i^\prime = -X_i \), and therefore \( \text{Ad}(s^\ast)^{-1} V_1 = 1/2(X_1^\prime - iX_1) \) and \( \text{Ad}(s^\ast)^{-1} V_2 = 1/2(-X_1^\prime - iX_1) \). Also, since

\[
\text{Ad}(u^{-1}) V_1 = c_1 V_1 + d_1 V_2 + \sum_{j=2}^{n} (c_j Y_j + d_j Y_j^\prime)
\]

for some coefficients \( c_j \) and \( d_j \) and using the fact that \( \langle V_1, V_1 \rangle = \langle V_2, V_2 \rangle = 1/2 \), we may write

\[
\text{Ad}(u^{-1}) V_1 = 2(\langle V_1, \text{Ad}(u) V_1 \rangle V_1 + \langle V_1, \text{Ad}(u) V_2 \rangle V_2) \\
+ \sum_{j=2}^{n} (c_j Y_j + d_j Y_j^\prime).
\]

In the same way we obtain

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\[ \text{Ad}(u^{-1}) V_2 = 2(\langle V_2, \text{Ad}(u)V_1 \rangle V_1 + \langle V_2, \text{Ad}(u)V_2 \rangle V_2) \]
\[ + \sum_{j=2}^{n} (\bar{c}_j Y_j + \bar{d}_j Y'_j). \]

As \( d\chi(\text{Ad}(s^*)^{-1} V_1) = d\chi(\text{Ad}(s^*)^{-1} V_2) = \frac{i\lambda}{2} \) we get the formula

\[ d\chi(\text{Ad}(us^*)^{-1} V_1, V_2^k) \]
\[ (4.2) = a(i+k)\alpha \left( \frac{-i\lambda}{2} \right)^{j+k} \langle V_1, \text{Ad}(u)(V_1 + V_2^k) \rangle^j \langle V_2, \text{Ad}(u)(V_1 + V_2^k) \rangle^k \]
\[ = (-i\lambda a^\alpha)^{j+k} \text{Ad}(u)(V_1 + V_2^k)^j \text{Ad}(u)(V_1 + V_2^k)^k \]

In the case of \( SU(2,1) \) the last formula can be simplified. If we take
\[ U = u_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \]

then \( \text{Ad}(u)V_1 = e^{3i\pi} V_1, \text{Ad}(u)V_2 = e^{-3i\pi} V_2 \) and so

\[ d\chi(\text{Ad}(us^*)^{-1} V_1, V_2^k) \]
\[ (4.3) = (-iXaa)^{j+k} e^{3i(\pi - k)_{1, V_1}} \langle V_2, V_2^k \rangle = (-iXaa)^{j+k} e^{3i(j-k)\theta} \]

\( T \) is \( M \)-invariant and it belongs to \( g_{-2\alpha} \). Then \( d\chi(\text{Ad}(s^*)^{-1} T^l) = 0 \) if \( l > 0 \).

Also, from the computations above we get:

\[ d\chi(\text{Ad}(us^*)^{-1} q^m) = \alpha^{2\alpha} d\chi(X_1^2)^m = (\lambda a^\alpha)^{2m}. \]

Now we can give a first formula for the \( \tau \)-function in the case that \( G \) is locally isomorphic to \( SU(2,1) \). By (3.39) and (4.1) we get:

\[ \tau(\chi, \chi, \alpha, \nu) = \sum_{j,k \geq 0} (-1)^j \prod_{s=1}^{j+k} (\nu + s) (-i\lambda a^\alpha / 2)^{j+k} e^{3i(j-k)\theta} \]
\[ \left( \begin{array}{c} j+k \\ j+k \end{array} \right) \frac{\left( \begin{array}{c} j+k \\ j+k \end{array} \right) \prod_{s=1}^{j+k} (\nu + s)(\nu/2 + s) \prod_{s=1}^{j+k} (\nu + s)(\nu/2 + s) \right. \]

Remark 2. We note that this formula coincides with that given in [MW], Proposition A4 except by the factor \((-1)^j\).

In the case that \( G \) is locally isomorphic to \( SU(n+1,1) \), \( n > 1 \), and with the notations above we get:

\[ \tau(\chi, \chi, uu, \nu) \]
\[ (4.4) = \sum_{j,k,m \geq 0} \frac{(i\lambda a^\alpha / 2)^{j+k+2} \langle V_1, \text{Ad}(u)(V_1 + V_2)^j \langle V_2, \text{Ad}(u)(V_1 + V_2)^k \rangle}{4^m j! k! m! \prod_{s=1}^{j+m} (\nu + q - 1 + s) \prod_{s=1}^{k+m} (\nu + q - 1 + s) \prod_{s=1}^{m} (\nu + s)} \]

We will now try to get a simpler formula for the \( \tau \)-function. More precisely, we
want to have an expression of the $\tau$-function in terms of generalized hypergeometric functions. We shall follow the notation of [Sl]. Let $(a) = (a_1, a_2, \ldots, a_A)$ and $(b) = (b_1, \ldots, b_B)$. The generalized hypergeometric function $A^F_B((a); (b); y)$ is defined as follows:

$$A^F_B((a); (b); y) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_A)_n}{n!(b_1)_n \cdots (b_B)_n} y^n$$

where $(c)_n = c(c+1) \ldots (c+n-1)$ if $n \geq 1$ and $(c)_0 = 1$. We recall that if $A \leq B$ then $A^F_B((a); (b); y)$ converges for every finite value of $y$. With this new notation we can rewrite the $\tau$-function as follows:

(4.6) \hspace{1cm} \tau(x, x, u\theta, \nu) = \sum_{j,k \geq 0} \frac{(-1)^k(\nu + 1)_{j+k}(i\lambda^2 a^2/2)^j e^{3i(j-k)\theta}}{j!k!(\nu/2 + 1)_{j+k}(\nu/2 + 1)_k} \text{ if } n = 1

and

$$\tau(x, x, u\theta, \nu) \quad (4.7) \hspace{1cm} = \sum_{j,k,m \geq 0} \frac{(i\lambda^2/2a^2)^{j+k+2m}(V_1, \text{Ad}(u)(V_1 + V_2))^j(V_2, \text{Ad}(u)(V_1 + V_2))^k}{4^m j!k!(\nu+m+1)_j m(\nu+m+1)_k}$$

if $n > 1$.

Let us first rewrite formula (4.6). We note that $(\nu + 1)_{j+k} = (\nu + 1)_{j}(\nu + j + 1)_k$. Thus (4.6) is equal to:

$$\sum_{j=0}^{\infty} \frac{(\nu + 1)_j(i\lambda^2 a^2/2)^j e^{3ij\theta}}{j!(\nu/2 + 1)_j} \sum_{k=0}^{\infty} \frac{(-i\lambda^2 a^2/2)^k e^{-3ik\theta}}{k!(\nu/2 + 1)_k}$$

If $z = (i\lambda^2 a^2/2)e^{3i\theta}$ then, with the above notation, we get the formula:

$$\tau(x, x, u\theta, \nu) = \sum_{j=0}^{\infty} \frac{(z)^j}{j!(\nu/2 + 1)_j} F_2(\nu + j + 1; \nu + 1, \nu/2 + 1; z)$$

With respect to the formula (4.7), we first sum over $m$, so to obtain the formula:

$$\sum_{m=0}^{\infty} \frac{(-1)^m(-i\lambda^2 a^2/2)^{2m}}{4^m m!(\nu+1)m(\nu+m+1)_m} \times$$

$$\sum_{j,k \geq 0} \frac{(-1)^j(-i\lambda^2 a^2/2)^{j+k}(V_1, \text{Ad}(u)(V_1 + V_2))^j(V_2, \text{Ad}(u)(V_1 + V_2))^k}{j!k!(\nu+m+1)_j m(\nu+m+1)_k}$$

We note that the sum over $j$ and $k$ is a product of two generalized hypergeometric functions of the type $\phi F_1$, that also can be interpreted as a product of classical generalized Bessel functions. That is, if $z = i\lambda^2 a^2/2$, $\omega_1 = \langle V_1, \text{Ad}(u)(V_1 + V_2) \rangle$ and $\omega_2 = \langle V_2, \text{Ad}(u)(V_1 + V_2) \rangle$ then this double sum is equal to
Theorem 4.1. Let \( G \) be isomorphic to \( SU(n + 1, 1) \). Let \( \chi \) be a a nontrivial character of \( N \).

If \( n = 1 \), let \( u = u_\theta \) as above, let \( a \in A \) and let \( z = (i\lambda^2 a^\alpha/2) \). Then with the notations above:

\[
\tau(\chi, \chi, u_\theta u, \nu) = \sum_{j=0}^{\infty} \frac{(ze^{3i\theta})^j}{j!(\nu/2 + 1)^j} _0F_1(\nu + j + 1; \nu + 1, \nu/2 + 1; ze^{3i\theta})
\]

If \( n > 1 \), let \( z = i\lambda^2/2a^\alpha \). Then with the notations above:

\[
\tau(\chi, \chi, ua, \nu) = \sum_{m=0}^{\infty} \frac{(-1)^m(z/2)^m}{m!(\nu + 1)m[\nu + (n-1)/2 + 1]^m} \times
\]

\[
_0F_1(\frac{\nu + n - 1}{2} + m + 1; 2\omega_1)_0 \times F_1(\frac{\nu + n - 1}{2} + m + 1; 2\omega_2)
\]

To end this section we will try to simplify the formula obtained by [MW], Appendix, in the case that \( G \) is locally isomorphic to \( SO(n + 1, 1) \), \( n > 1 \). We follow the notation of [MW]. In this case the Whittaker vector \( u(\lambda, -\nu - \rho) \) is of the form: \( u(\lambda, -\nu - \rho) = \sum_{j,k \geq 0} a_{j,k}(\nu) Y_1 q^k \) where

\[
a_{j,k} = \frac{(-1)^{j+2k} \lambda^{j+2k}}{4^k j! k!(\nu + \frac{n}{2})^j+2k(\nu)_k}
\]

This formula differs from that in [MW], as we have corrected the following mistakes in the computations: first, formula A.4 must be replaced by:

\[
[X_i, q^k] \equiv k Y_i q^{k-1} (2H + n - 2k) \mod U(g) \cdot m.
\]

In formula (A.10) there is missing power of 2, and finally, in the formulas given in the proposition A.1 it is written \( q^{2k} \) instead of \( q^k \).

Now it follows that the \( \tau \)-function has the following formula:

\[
\tau(\chi, \chi, ua, \nu) = \sum_{j,k \geq 0} (-1)^j (\lambda^2 a^\alpha)^j+2k(Y_1, \text{Ad}(u) Y_1)\frac{1}{4^k j! k!(\nu + \frac{n}{2})^j+2k(\nu)_k}.
\]

This means that if \( z = \lambda^2 a^\alpha \), then

\[
\tau(\chi, \chi, ua, \nu) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!(\nu)_k(\nu + n/2)^{2k}} \times 0 \times F_1(\nu + n/2 + 2k; -z(Y_1, \text{Ad}(u) Y_1))
\]

REFERENCES


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