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Note

Measure preserving homomorphisms and independent sets in tensor graph powers

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Abstract

In this note, we study the behavior of independent sets of maximum probability measure in tensor graph powers. To do this, we introduce an upper bound using measure preserving homomorphisms. This work extends some previous results concerning independence ratios of tensor graph powers.

Keywords: Tensor product; Homomorphism; Independent set

1. Introduction

The graphs in this note can have infinite number of vertices. A *homomorphism* from a graph H to a graph G is a map h from the vertices of H to the vertices of G such that $h(u)h(v)$ is an edge in G for every edge $uv \in E(H)$. For every graph G, we assume that there is a *probability measure* μ_G on the vertices of G. A homomorphism $h : V(H) \to V(G)$ is *measure preserving* if h is measurable and for every measurable $S \subseteq V(G)$, $\mu_H(h^{-1}(S)) = \mu_G(S)$. By $H \to G$, we mean that there exists a measure preserving homomorphism from H to G.

Definition 1. Let G be a graph with the probability measure μ_G on its vertices. We call G *vertex transitive* if

1. there exists a set S of measure preserving homomorphisms $\phi : V(G) \to V(G)$;
2. there exists a probability measure ν on S such that for almost every $v \in V(G)$, $\phi(v)$ has the same distribution as μ_G when ϕ is chosen according to ν.

Note that for a finite graph with the uniform measure, this definition coincides with the known definition of vertex transitivity of finite graphs (take S to be the group of automorphisms of G with the uniform measure).

The *tensor* product of two graphs, G and H, has vertex set $V(G) \times V(H)$, where (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. The measure on the new vertex set is the product measure. The characteristics of tensor products of graphs have been studied extensively (for example see [4,6]).

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Let G^n be the tensor product of n copies of G. For a graph G, define $\overline{\alpha}(G) := \sup_I \mu_G(I)$, where I is a measurable independent set. It is easy to see that if $H \to G$, then $\overline{\alpha}(H) \geq \overline{\alpha}(G)$ and $H^n \to G^n$. Since $G^{i+1} \to G^i$, this in particular implies that $\overline{\alpha}(G^n)$ is a nondecreasing sequence, and $\lim_{n\to\infty} \overline{\alpha}(G^n)$ exists. For finite graphs when the corresponding measure is the uniform probability measure on the vertices this limit has been previously studied (see [4,2]) under the name of *the ultimate categorical independence ratio*. For a finite vertex transitive graph H with the uniform measure, it is known that $\overline{\alpha}(H^n) = \overline{\alpha}(H)$ (see [1]). We prove an infinite version of this fact:

Lemma 1. *Let H be a (possibly infinite) vertex transitive graph. Then for any positive integer n,*

$$\overline{\alpha}(H^n) = \overline{\alpha}(H).$$

Proof. Since $\overline{\alpha}(H^n) \geq \overline{\alpha}(H)$, it is enough to prove that $\overline{\alpha}(H^n) \leq \overline{\alpha}(H)$. According to Definition 1, there exists a probability measure ν on a set S such that together they satisfy Definition 1 (properties 1, 2). Consider an arbitrary measurable independent set $I \subseteq H^n$ and for a vertex $w \in H^n$ denote by $[w \in I]$ the function that is 1 if $w \in I$ and 0 otherwise. Note that

$$\mu_{H^n}(I) = \Pr_{v_i \in V(H)}[(v_1, \dots, v_n) \in I] = \Pr_{\phi_i \in S, v \in V(H)}[(\phi_1(v), \dots, \phi_n(v)) \in I].$$

Thus, there exists a choice of $\overline{\phi}_1, \dots, \overline{\phi}_n$ such that

$$\begin{aligned}\mu_{H^n}(I) &\leq \Pr_{v \in V(H)}[(\overline{\phi}_1(v), \dots, \overline{\phi}_n(v)) \in I] \\ &= \mu(\{v : (\overline{\phi}_1(v), \dots, \overline{\phi}_n(v)) \in I, v \in V(H)\}).\end{aligned}$$

But $\{v : (\overline{\phi}_1(v), \dots, \overline{\phi}_n(v)) \in I\}$ is an independent set in H because I is an independent set and $\{\overline{\phi}_i\}$ are homomorphisms. Thus we obtain that $\mu_{H^n}(I) \leq \overline{\alpha}(H)$ which completes the proof. □

We call a vertex transitive graph H a *descriptor* of G if $H \to G$. Thus, for a descriptor H, we have

$$\overline{\alpha}(H) = \lim_{n\to\infty} \overline{\alpha}(H^n) \geq \lim_{n\to\infty} \overline{\alpha}(G^n).$$

Now, define $\mathrm{u}(G)$ as below:

$$\mathrm{u}(G) = \inf_{\text{descriptor} H} \overline{\alpha}(H).$$

Trivially, we have

$$\lim_{n\to\infty} \overline{\alpha}(G^n) \leq \mathrm{u}(G). \quad (1)$$

This raises the following question:

Question 1. Does every finite graph G satisfy $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$?

This question is inspired by the work of Dinur and Friedgut [5], in which measure preserving homomorphisms are used to give a new proof for an Erdös–Ko–Rado-type theorem. We study the behavior of $\lim \overline{\alpha}(G^n)$ for graphs with probability measures. This is closely related to and can be considered as the generalization of some resultsin [4,2].

2. The results

For the following lemma an analogous form appears in [4] for graphs without measure, which is re-proved in [2] using ideas similar to what we are using here. This lemma is the generalization of the Brown et al. result to graphs with probability measures.

Lemma 2. *For every finite graph G, if* $\lim \overline{\alpha}(G^n) > \frac{1}{2}$, *then* $\lim \overline{\alpha}(G^n) = 1$.

Proof. If $\lim \overline{\alpha}(G^n) > \frac{1}{2}$, then there exists a positive integer i such that $\overline{\alpha}(G^i) > \frac{1}{2}$. Letting $H = G^i$, trivially $\lim \overline{\alpha}(H^n) = \lim \overline{\alpha}(G^n)$. Let I be an independent set of measure $\frac{1}{2} + \epsilon$ of H. Define $J \subseteq V(H^n)$ as the set of vertices with strictly more than half of the coordinates in I. Clearly, J is an independent set of H^n. To prove that $\overline{\alpha}(H^n) = 1$, it suffices to prove that as n goes to infinity a random vertex which is taken from H^n with respect to μ_{H^n} is in J almost surely. Let X_i be an indicator random variable such that $X_i = 1$ if the ith coordinate of the random vertex belongs to I and $X_i = 0$ otherwise. As a result, we have $E[X_i] = \overline{\alpha}(H)$ and the mean and variance of X_i are finite. Thus, by applying the weak law of large numbers for the random variable $X = \frac{1}{n}\sum_{i=1}^{n} X_i$, we obtain $\lim_{n\to\infty} P(|X - \overline{\alpha}(H)| < \epsilon') = 1$ for every positive real ϵ'. Therefore, X is greater than $\frac{1}{2}$ almost surely, as desired. $\square$

Now, we characterize the graphs for which $\lim \overline{\alpha}(G^n) = 1$ and, by using this, we present some classes of graphs satisfying $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$. It should be noted that for Lemmas 3 and 4, analogous forms appear in [2] for graphs without measures.

Lemma 3. *For every finite graph G, if $\mathrm{u}(G) = 1$ then there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$, where $N(I)$ is the set of the vertices in $V(G)$ that are adjacent to at least one vertex in I.*

Proof. Suppose that every independent set $I \subseteq V(G)$ satisfies $\mu_G(I) \le \mu_G(N(I))$. We claim that for all $Q \subseteq V(G)$, we have $\mu_G(Q) \le \mu_G(N(Q))$. Suppose that for a $Q \subseteq V(G)$, we have $\mu_G(Q) > \mu_G(N(Q))$. Let I be the set of all vertices of Q without any neighbor in Q. Clearly, I is an independent set and since $\mu_G(Q) > \mu_G(N(Q))$, I is nonempty. Let $Q' = Q \setminus I$. Hence, $Q' \subseteq N(Q)$ and $N(I) \subseteq N(Q) \setminus Q'$. Therefore,

$$\mu_G(N(I)) \le \mu_G(N(Q)) - \mu_G(Q') < \mu_G(Q) - \mu_G(Q') = \mu_G(I),$$

a contradiction.

Now let $G' = G \times K_2$, where $K_2 = uv$ has the uniform measure. It is clear that $X = \{(z, u) \in V(G') : z \in V(G)\}$ and $Y = V(G') - X$ is a bipartition of G'. Consider a flow network with vertices $V(G') \cup \{s, t\}$ and nonnegative capacities $c(s, x) = \mu_{G'}(x)$, and $c(y, t) = \mu_{G'}(y)$, for $x \in X$ and $y \in Y$, and $c(x, y) = \infty$ if $xy \in E(G')$. All the other capacities are 0. Let (S, T) be a minimum cut of this network with capacity $c(S, T)$. By the structure of the flow network, we have $c(S, T) \le \frac{1}{2}$. Now, let $X_1 = S \cap X$, $Y_1 = S \cap Y$, $X_2 = T \cap X$, and $Y_2 = T \cap Y$. Since $c(x, y) = \infty$ if $xy \in E(G')$, there is no edge between X_1 and Y_2. Therefore, $X_1 \cup Y_2$ is an independent set in G'. Since for all $Q \subseteq V(G)$, $\mu_G(Q) \le \mu_G(N(Q))$, we have $\mu_{G'}(X_1) \le \mu_{G'}(N(X_1))$ and $\mu_{G'}(Y_2) \le \mu_{G'}(N(Y_2))$, which yields $\mu_{G'}(X_1) + \mu_{G'}(Y_2) \le \mu_{G'}(N(X_1)) + \mu_{G'}(N(Y_2))$. Thus, we obtain $\mu_{G'}(X_1) + \mu_{G'}(Y_2) \le \frac{1}{2}$. Therefore, we have $\mu_{G'}(X_2) + \mu_{G'}(Y_1) \ge \frac{1}{2}$ and because $c(S, T) = \mu_{G'}(X_2) + \mu_{G'}(Y_1)$, we obtain $c(S, T) = \frac{1}{2}$. Thus by the max-flow min-cut theorem, the value of a maximum flow f must be equal to $\frac{1}{2}$.

Now by using the maximum flow f, we construct a descriptor graph H for G' together with the measure preserving homomorphism $h : H \to G'$ as follows. The vertices of H are the elements of the interval $[0, 1)$ endowed with the (uniform) Lebesgue measure, and $E(H) = \{\{a, a + \frac{1}{2}\} : a \in [0, \frac{1}{2})\}$. It is easy to see that H is vertex transitive. Now we have to specify h. For $xy \in E(G')$, let f_{xy} denote the amount of the flow that passes through this edge. Since the value of f is equal to $\frac{1}{2}$, we have $\sum_{xy\in E(G')} f_{xy} = \frac{1}{2}$. So it is possible to partition the interval $[0, \frac{1}{2})$ into *disjoint* intervals in the following way: $[0, \frac{1}{2}) = \bigcup_{xy\in E(G')}[a_{xy}, a_{xy} + f_{xy})$, where $a_{xy} \ge 0$. Now h is defined, as for every $z \in V(G') = X \cup Y$,

$$h^{-1}(z) = \begin{cases} \displaystyle\bigcup_{y:\, zy\in E(G')} [a_{zy}, a_{zy} + f_{zy}) & \text{if } z \in X \\ \displaystyle\bigcup_{x:\, xz\in E(G')} \left[\frac{1}{2} + a_{xz}, \frac{1}{2} + a_{xz} + f_{xz}\right) & \text{if } z \in Y. \end{cases}$$

It is not hard to see that h is a measure preserving homomorphism from H to G'. Since $G' \to G$, H is a descriptor of G. Hence, we have $\mathrm{u}(G) \le \frac{1}{2}$. $\square$

Lemma 4. *For every finite graph G, if there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$, then* $\lim \overline{\alpha}(G^n) = 1$.

Proof. Let $U = V(G) \setminus (I \cup N(I))$. Let $m_n = \overline{\alpha}(G^n)$. Trivially, $\mu_G(I) + \mu_G(N(I)) + \mu_G(U) = 1$, $m_1 \geq \mu_G(I)$ and $\mu_G(U) < 1$. Consider the union of the vertices with first coordinate in I and the vertices with first coordinate in U and last $n-1$ coordinates in the maximum measure independent set of G^{n-1}. It can be seen that this is an independent set and we have $m_n \geq \mu_G(I) + \mu_G(U)m_{n-1}$. By applying this inequality repeatedly, we obtain

$$\begin{aligned} m_n &\geq \mu_G(I) + \mu_G(I)\mu_G(U) + \cdots + \mu_G(U)^{n-1}.m_1 \\ &\geq \mu_G(I) + \mu_G(I)\mu_G(U) + \cdots + \mu_G(I)\mu_G(U)^{n-1} = \frac{\mu_G(I) - \mu_G(I)\mu_G(U)^n}{1 - \mu_G(U)}. \end{aligned}$$

Thus, we have $\lim_{n\to\infty} m_n \geq \frac{\mu_G(I)}{1-\mu_G(U)} = \frac{\mu_G(I)}{\mu_G(I)+\mu_G(N(I))} > \frac{1}{2}$, and by Lemma 2, we have $\lim \overline{\alpha}(G^n) = 1$. □

Theorem 1. *For every finite graph G, the following are equivalent:*

(i) $\lim \overline{\alpha}(G^n) = 1$*;*
(ii) $\mathrm{u}(G) = 1$*;*
(iii) *there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$.*

Proof. (i) implies (ii) by the inequality (1), (ii) implies (iii) by Lemma 3, and (iii) implies (i) by Lemma 4. □

Corollary 1. *For every finite graph G, if* $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$ *then* $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$.

Remark 1. It is not hard to see that for graphs with rational measures, Theorem 1(i) directly yields Theorem 1(iii). To prove this, it can be shown that if the condition Theorem 1(iii) does not hold, a graph H could be found which satisfies both $H \to G$ and Tutte's 1-factor theorem. Thus, this graph has a perfect matching M and $\lim \overline{\alpha}(G^n) \leq \overline{\alpha}(M) \leq \frac{1}{2}$. To generalize this to graphs with real measures, a density argument by Noga Alon can be used [3]. The sketch of this proof is as follows: Suppose that the statement of Theorem 1(iii) does not hold, and let $V = \{v_1, v_2, \ldots, v_m\}$. The set of all points $(\nu(v_1), \nu(v_2), \ldots, \nu(v_m))$ in $\mathbb{R}^m$ satisfy $\nu(v_i) \geq 0$ for $i = 1, \ldots, m$, $\sum_{i=1}^m \nu(v_i) = 1$, and for every independent set $I \subseteq V(G)$, $\nu(I) \leq \nu(N(I))$ is a nonempty convex polytope with rational vertices. Thus, μ_G is a convex combination of rational measures and for fixed integer $n > 0$ and $\epsilon > 0$, there exists a rational measure ν in this polytope satisfying $|\mu_{G^n}(T) - \nu(T)| \leq \epsilon$ for every independent set $T \subseteq V(G^n)$. Now by the result for the rational case (mentioned above), it is not hard to see that for every n, $\lim \overline{\alpha}(G^n) \leq \frac{1}{2}$.

Corollary 1 presents a family of graphs for which equality holds in Question 1. Trivially, finite vertex transitive graphs are another family of graphs for which equality holds in Question 1. In the next proposition, we show that this family also contains bipartite graphs. For Proposition 1 an analogous form appears in [4] for graphs without measure.

Proposition 1. *For a finite bipartite graph G, we have* $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$.

Proof. Let X and Y be a bipartition of G. The set of the vertices of G^n whose first coordinates are in X and the set of the vertices of G^n whose first coordinates are in Y is a bipartition of G^n. Thus, for the bipartite graph G^n, $\overline{\alpha}(G^n) \geq \frac{1}{2}$. Therefore, by Lemma 2, we obtain $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$. □

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