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Hyperbolic-like estimates for higher order equations

M. D'Abbicco [a,*], M.R. Ebert [b]

[a] Dipartimento di Matematica, Universitá di Bari, Via E. Orabona, 4 Bari, 70125, Italy

[b] Departamento de Computação e Matemática, Universidade de São Paulo (USP), FFCLRP, Av.Bandeirantes, 3900, Campus da USP, CEP 14040-901, Ribeirão Preto - SP, Brazil

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ABSTRACT

The main goal of this paper is to derive long time estimates of the energy for the higher order hyperbolic equations with time-dependent coefficients. In particular, we estimate the energy in the *hyperbolic* zone of the extended phase space by means of a function $f(t)$ which depends on the principal part and on the coefficients of the terms of order $m-1$. Then we look for sufficient conditions that guarantee the same energy estimate from above in all the extended phase space. We call this class of estimates *hyperbolic-like* since the energy behavior is deeply depending on the hyperbolic structure of the equation. In some cases, these estimates produce a dissipative effect on the energy.

1. Introduction

Let $m \geq 2$. We consider in $[0, \infty) \times \mathbb{R}$ the Cauchy problem for the following m-th order equation with time dependent, regular coefficients:

$$\begin{cases} \partial_t^m u - \displaystyle\sum_{j=0}^{m-1} a_j(t)\lambda(t)^{m-j}\partial_t^j\partial_x^{m-j}u + b_{m-1,0}(t)\partial_t^{m-1}u + \sum_{j=0}^{m-2}\sum_{k=1}^{m-1-j} b_{j,k}(t)\lambda(t)^k\partial_t^j\partial_x^k u = 0, \\ \partial_t^j u(0,x) = u_j(x), \quad j = 0, \ldots, m-1. \end{cases} \tag{1}$$

We assume that the coefficients $a_j(t)$ of the principal part are bounded whereas $\lambda(t) > 0$ may be unbounded. The function $\lambda(t)$ will describe the behavior of the speed of propagation. We remark that in (1) there are no terms with zero derivatives in x but $\partial_t^m u$ and $b_{m-1,0}(t)\partial_t^{m-1}u$. For the sake of simplicity, we consider (1) in one space dimension, but our arguments can be easily extended to $x \in \mathbb{R}^n$, $n \geq 2$.

It is well known that if the coefficients are sufficiently regular and the equation is strictly hyperbolic then the Cauchy problem (1) is $\mathcal{C}^\infty$ well-posed with no loss of regularity. Moreover, denoting by

$$E(t) = \sum_{j=0}^{m-1} \|\partial_t^j u(t,\cdot)\|_{H^{m-1-j}}^2, \tag{2}$$

the energy for the solution $u(t,x)$ to (1), then $E(t) \leq C_T E(0)$ for any $t \in [0,T]$, with $C_T > 0$ (see, for instance, [10]). Indeed, information on the long time behavior of the energy is interesting for many reasons. For instance, it can provide a basis to derive Strichartz decay estimates [15]. We refer the interested reader to [14] for dispersive and Strichartz estimates for solutions of higher order equations with constant coefficients.

* Corresponding author.
E-mail addresses: m.dabbicco@gmail.com, dabbicco@dm.uniba.it (M. D'Abbicco), ebert@ffclrp.usp.br (M.R. Ebert).
URL: http://www.dabbicco.com (M. D'Abbicco).

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On the other hand, it is interesting to study the behavior of the homogeneous λ-energy

$$E_\lambda(t) = \sum_{j=0}^{m-1} \lambda(t)^{2(m-1-j)} \|\partial_t^j \partial_x^{m-1-j} u(t,\cdot)\|_{L^2}^2, \tag{3}$$

deriving estimates in the form $E_\lambda(t) \le C d(t) E(0)$ uniformly in $[0,\infty)$, for some function $d(t)$ (see (25)).

The study of the λ-energy has been recently developed [11–13] for second-order wave-type equations of the form

$$u_{tt} - \lambda(t)^2 u_{xx} + b(t) u_t = 0, \tag{4}$$

by assuming regular coefficients with controlled oscillations. If $\lambda \equiv 1$ and $b(t) \ge 0$ then the presence of the damping term $b(t)u_t$ produces dissipative effects on the wave energy

$$\|u_t(t,\cdot)\|_{L^2}^2 + \|u_x(t,\cdot)\|_{L^2}^2,$$

which can be used as a basis to derive decay estimates [9,16–20]. These decay estimates are a useful tool in the study of nonlinear estimates [3,5].

On the other hand, if $b \equiv 0$ and $\lambda(t) \to \infty$ with $\lambda'(t) > 0$, then $E_\lambda(t) \le C\lambda(t)E(0)$ (see [6]). In particular, the *elastic* energy $\|u_x(t,\cdot)\|_{L^2}^2$ dissipates with a speed estimated by $\lambda(t)^{-1}$.

In the recent paper [2], we obtained estimates for the λ-energy of (4), which take into account effects coming from both $\lambda(t)$ and $b(t)$. We also derived results in the presence of a *drift term* $b_1(t)u_x$ and a small negative *mass term* $-m(t)^2u$. The approach in [2] gave hint on how to determinate and study possible dissipative effects for higher order equations of a special class what we called *hyperbolic-like*.

As far as we know, it is not yet clear how to determinate dissipative terms for higher order equations written in a general form. In this paper we delineate a strategy, stating some sufficient conditions which can be tested on different models.

We call the class of estimates that we study *hyperbolic-like* because the control that we derive for the energy is derived by estimating the pointwise energy in the *hyperbolic zone* of the extended phase space $[0,\infty) \times \mathbb{R}^n$. This zone contains frequencies ξ that are *large* with respect to some function depending on t related to the speed of propagation $\lambda(t)$. In some cases, the obtained estimates will represent a dissipative effect in a sense which will be clarified in Remark 10.

1.1. *The almost-positivity property*

In this paper, we are going to deal with long time integral inequalities. To deal with them we will make use of some assumption on the coefficients of (1). For the ease of readiness, in the next paragraph we introduce a property which comes into play in our hypotheses and which generalizes the positivity of a function in a way suitable for our purposes. This property has been recently introduced in [2].

Notation 1. Let $f, g : [0,\infty) \to (0,\infty)$ be two strictly positive functions. We use the notation $f \approx g$ if there exist constants C_1 and C_2 such that $C_1 g(t) \le f(t) \le C_2 g(t)$ for all $t \ge 0$. If the inequality is one-hand sided, namely, if $f(t) \le Cg(t)$ (resp. $f(t) \ge Cg(t)$) for all $t \ge 0$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

In particular $f \approx 1$ means that $C_1 \le f(t) \le C_2$ for some constants C_1, C_2.

Definition 1. Let $a : [0,\infty) \to \mathbb{R}$ be a continuous function. We say that $a(t)$ is *almost-zero*, and we denote it by $a(t) =_{(a)} 0$, if there exists a constant $C > 0$ such that

$$-C \le \int_0^t a(\tau)\, d\tau \le C, \tag{5}$$

that is, if each of its primitive integrals is bounded. We say that $a_1 : [0,\infty) \to \mathbb{R}$ is *almost-positive*, and we denote it by $a_1(t) \ge_{(a)} 0$, (or, respectively, *almost-negative*, $a_1(t) \le_{(a)} 0$) if there exists a *almost-zero* function $a(t)$ such that $a_1(t) - a(t) \ge 0$ (or, respectively, ≤ 0).

Clearly, we say that two functions $a_1, a_2 : [0,\infty) \to \mathbb{R}$ are *almost-equal* and we write $a_1(t) =_{(a)} a_2(t)$, if $a_1(t) - a_2(t)$ is *almost-zero*, whereas we say that $a_1(t)$ is *almost-greater* than $a_2(t)$ and we write $a_1(t) \ge_{(a)} a_2(t)$ if $a_1(t) - a_2(t)$ is *almost-positive*.

Remark 1. A continuous function $a : [0,\infty) \to \mathbb{R}$ is *almost-positive* if $a(t) \ge 0$ in $[T,\infty)$ for some $T \ge 0$.

Remark 2. Let $a : [0,\infty) \to \mathbb{R}$ be a continuous function, and let $A : [0,\infty) \to (0,\infty)$ be defined by

$$A(t) := \exp\left(\int_0^t a(\tau)\, d\tau\right). \tag{6}$$

Then $a(t)$ is *almost-positive* (respectively *almost-negative*) if, and only if, there exists an increasing (respectively decreasing) function $g : [0,\infty) \to (0,\infty)$ such that $A \approx g$. Trivially, $a(t) =_{(a)} 0$ if, and only if, $A \approx 1$.

In particular, it is clear that if $a(t) \geq_{(a)} 0$ (respectively $a(t) \leq_{(a)} 0$) then $A(t)$ in (6) is bounded by a positive constant from below, i.e. $A(t) \geq C_1 > 0$ (respectively from above, i.e. $A(t) \leq C_2$).

Remark 3. Let $a(t) \geq_{(a)} 0$ and let $A : [0, \infty) \to (0, \infty)$ be as in (6). Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. Then, for any $s \leq t$, we can estimate

$$\int_s^t A(\tau) f(\tau)\, d\tau \lesssim A(t) \int_s^t f(\tau)\, d\tau; \qquad A(s) \int_s^t f(\tau)\, d\tau \lesssim \int_s^t A(\tau) f(\tau)\, d\tau.$$

Analogously if $a(t) \leq_{(a)} 0$ or $a(t) =_{(a)} 0$.

2. Main result

First we introduce the instruments to construct the function $d(t)$ which will provide the estimate for the λ-energy $E_\lambda(t)$ in (3).

We assume that the equation in (1) is $\lambda(t)$-*scaled uniform strictly hyperbolic*.

Hypothesis 1. We assume that the m roots $\tau_i(t)$ of

$$P(t, \tau) := \frac{p(t, \lambda(t)\tau)}{\lambda(t)^m} \equiv \tau^m - \sum_{j=0}^{m-1} a_j(t) \tau^j = 0, \tag{7}$$

are real-valued and that they verify the following condition:

$$0 < C \leq \Delta(t) = \prod_{i \neq l} (\tau_i(t) - \tau_l(t))^2, \quad t \geq 0. \tag{8}$$

In particular, from (8) and from the boundedness of the coefficients $a_j(t)$, it follows that the speed of propagation of the equation in (1) is given by $\lambda(t)$.

Remark 4. We remark that the roots $\tau_i(t)$ in Hypothesis 1 are the eigenvalues of the matrix

$$A(t) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \\ a_0(t) & a_1(t) & \cdots & a_{m-1}(t) \end{pmatrix}. \tag{9}$$

We also assume a sufficient condition to exclude effects coming from the first derivatives of the roots of (7) on the long-time behavior of the energy. We notice that, by virtue of (8), the regularity of $\tau_i(t)$ will follow from the regularity of the coefficients $a_j(t)$ (see later in Hypothesis 4).

Hypothesis 2. We assume that the roots $\tau_i(t)$ of (7) satisfy the following condition:

$$\frac{\tau_i'(t)}{\tau_i(t) - \tau_k(t)} =_{(a)} 0, \quad \text{for any } k \neq i. \tag{10}$$

We may now construct the m scalar functions $f_i(t)$ which will determinate the estimate for $E_\lambda(t)$ in Theorem 1.

Definition 2. Let V_j be the m-th order Vandermonde vector related to τ_j, that is,

$$V_j := \left(1, \tau_j, \ldots, \tau_j^{m-1}\right)^T,$$

and let

$$\mathrm{Coeff}[p] := (\alpha_{m-1}, \ldots, \alpha_1, \alpha_0),$$

where $\mathrm{Coeff}[p]$ is the vector of the coefficients of a polynomial $p(\tau) = \sum_{k=0}^{m-1} \alpha_k \tau^{m-1-k}$. It follows that $p(\tau_j) = \mathrm{Coeff}[p] \cdot V_j$. We put

$$W_i(t) := \mathrm{Coeff}[P_i](t),$$

where $P_i(t,\tau)$ is the polynomial in τ given from

$$P_i(t,\tau) = \frac{P(t,\tau)}{\tau - \tau_i(t)} = \prod_{l\neq i} (\tau - \tau_l(t))\,.$$

Moreover, we define

$$W_i^\sharp(t) := \mathrm{Coeff}^\sharp[P_i](t), \quad \text{where } \mathrm{Coeff}^\sharp[p] := ((m-1)\alpha_{m-1}, (m-2)\alpha_{m-2}, \ldots, \alpha_1, 0)\,.$$

Remark 5. It is easy to check that $W_i(t)$ is a left eigenvector of $A(t)$ related to $\tau_i(t)$, whereas $V_j(t)$ is a right eigenvector of $A(t)$ related to $\tau_j(t)$. Since

$$W_i(t)\cdot V_j(t) = \delta_i^j P_i(t,\tau_i(t)),$$

if we put $\tilde{V}_j(t) = V_j(t)/P_j(t,\tau_j(t))$, it follows that

$$N(t) := (W_1(t), \ldots, W_m(t))^T\,, \qquad N^{-1}(t) = \left(\tilde{V}_1(t), \ldots, \tilde{V}_m(t)\right), \tag{11}$$

is a diagonalizer for $A(t)$, namely,

$$N(t)A(t)N^{-1}(t) = \mathcal{D}(t) := \mathrm{diag}(\tau_1(t), \ldots, \tau_m(t)).$$

The diagonalizer $N(t)$ is bounded as $A(t)$, and uniformly regular thanks to (8), being $\det N(t) = \sqrt{\Delta(t)}$.
Moreover, let A_0 be the $m\times m$ diagonal matrix defined by

$$A_0 := \mathrm{diag}(m-1, m-2, \ldots, 1, 0). \tag{12}$$

It follows that $W_j(t)\cdot A_0 = W_j^\sharp(t)$.

We refer the interested reader to [7,8] for more details concerning the construction of the diagonalizer in (11).

Definition 3. Let us define m scalar functions $f_i : [0,\infty) \to \mathbb{R}$ as the inner products

$$f_i(t) := \Big((\lambda'(t)/\lambda(t))W_i^\sharp(t) + (b_{m-1}(t))\Big)\cdot \tilde{V}_i(t), \tag{13}$$

for any $i = 1, \ldots, m$, where the vector $(b_{m-1}(t))$ depends on the coefficients of the terms of order $m-1$ in (1) and it is given by

$$(b_{m-1}(t)) = \left(-b_{0,m-1}(t), -b_{1,m-2}(t), \ldots, -b_{m-1,0}(t)\right).$$

We remark that the functions $f_i(t)$ depend only on the coefficients of the terms of order m and $m-1$ in (1).
To state our assumptions on the coefficients of the equation in (1) we introduce some auxiliary functions.

Hypothesis 3. Let $\lambda \in \mathcal{C}^2$ be a strictly positive function, with $\lambda \notin L^1$ and $\lambda(0) = 1$. We define

$$\Lambda(t) := 1 + \int_0^t \lambda(\tau)d\tau, \tag{14}$$

$$\eta(t) := \frac{\lambda(t)}{\Lambda(t)}, \tag{15}$$

and we assume that

$$\frac{|\lambda^{(k)}(t)|}{\lambda(t)} \lesssim \eta(t)^k, \quad \text{for } k = 1, 2. \tag{16}$$

We remark that the function $\eta(t)$ in (15) plays a fundamental role in our approach, and its properties are related to the speed of propagation $\lambda(t)$.

Remark 6. It is easy to prove that

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{\eta'(t)}{\eta(t)} + \eta(t). \tag{17}$$

We assume that the coefficients $b_{j,k}$ of the terms of order $j+k$ in the equation in (1) are bounded by $\eta(t)^{m-(j+k)}$. Since we have in mind to use a $\mathcal{C}^2$ diagonalization procedure, we also assume a similar condition on $a_j'(t)$, $a_j''(t)$, and on $b_{j,k}'(t)$ for $j+k = m-1$.

Hypothesis 4. We assume that the coefficients $a_j \in \mathcal{C}^2$ and $b_{j,m-1-j} \in \mathcal{C}^1$ are real-valued, whereas the coefficients $b_{j,k} \in \mathcal{C}$ for $j + k \le m - 2$ may be complex-valued. Moreover we assume the following:

$$|a_j^{(\ell)}(t)| \lesssim \eta(t)^\ell, \quad \text{for } \ell = 0, 1, 2, \tag{18}$$

$$|b_{j,m-1-j}^{(\ell-1)}(t)| \lesssim \eta(t)^\ell, \quad \text{for } \ell = 1, 2, \tag{19}$$

$$|b_{j,k}(t)| \lesssim \eta(t)^{m-(j+k)}, \quad \text{for any } j + k \le m - 2. \tag{20}$$

Hypothesis 5. We assume that

$$\frac{\eta'(t)}{\eta(t)} \le_{(a)} \delta\eta(t), \quad \text{for any } \delta > 0. \tag{21}$$

Remark 7. In fact, property (21) is very natural. In particular, it trivially holds if $\eta(t)$ is monotonic. Indeed, if $\eta(t)$ is decreasing then $\eta'(t) \le 0$ and (21) is trivially satisfied, whereas if $\eta(t)$ is increasing then $\eta'(t)/\eta(t)^2$ is integrable, since

$$\int_s^\infty \frac{\eta'(\tau)}{\eta(\tau)^2}\, d\tau \le \frac{1}{\eta(s)} \le \frac{1}{\eta(0)}.$$

In particular, $0 \le \eta'(t)/\eta(t)^2 \to 0$ as $t \to \infty$ therefore (21) follows (see Remark 1). Analogously if $\eta'(t)/\eta(t) \le_{(a)} 0$ or $\eta'(t)/\eta(t) \ge_{(a)} 0$.

We are now ready to state our first result.

Theorem 1. *Let us assume Hypotheses 1–5. Let $f : [0, \infty) \to \mathbb{R}$ be a function with constant sign, such that*

$$f_i(t) \le_{(a)} f(t), \quad \textit{for any } i = 1, \dots, m. \tag{22}$$

Moreover, let us assume that there exists $r \in [0, 1)$ such that

$$-b_{m-1,0}(t) \le_{(a)} \frac{\eta'(t)}{\eta(t)} + (1 - r)\,\eta(t) \le_{(a)} f(t). \tag{23}$$

Then the solution to (1) satisfies the following energy estimate:

$$E_\lambda(t) \le C\, d(t) E(0), \tag{24}$$

where $d(t)$ is given by

$$d(t) = \exp\left(2\int_0^t f(\sigma)\, d\sigma\right). \tag{25}$$

The proof of Theorem 1 will be divided in several steps (see Sections 4 and 6).

Remark 8. Condition (23) means that the estimate in (24), which is related to the function $f(t)$ satisfying (22), is *hyperbolic-like*. That is, the decay function $d(t)$ is related to the contribution coming from the functions $f_i(t)$ deriving from the diagonalization procedure. Then we assume condition (23) to exclude possible perturbations coming from the low-frequencies (see Section 6).

In fact, the range $r \in [0, 1)$ in condition (23) can be enlarged to $r \in [-q, 1)$ for some $q > 0$, if we have additional information on the structure of the equation in (1), more precisely if a part of the symbol of the equation vanishes at higher order as $\xi \to 0$ (see Remark 14 in Section 6).

Remark 9. Condition (22) means that the function $f(t)$ represents a control on the possible *strong* influence coming from $f_i(t)$ to the long-time behavior of $E_\lambda(t)$. It is clear that the choice of $f(t)$ is not unique. The estimate in Theorem 1 is as better as $f(t)$ will be smaller, provided that it satisfies (22) and (23).

Remark 10. We remark that, in general, $f(t)$ may be a positive function; in such a case, $d(t)$ is an increasing function. Nevertheless, from (24) we get

$$\|\partial_t^{m-1-k}\partial_x^k u(t,\cdot)\|_{L^2}^2 \le C\,\frac{d(t)}{\lambda(t)^{2k}}\, E(0), \quad \text{for any } k = 0, \dots, m-1.$$

In particular, if $\lambda(t)$ is increasing or, more in general, if $\lambda'(t)/\lambda(t) \geq_{(a)} 0$, and

$$f(t) \leq_{(a)} \ell \frac{\lambda'(t)}{\lambda(t)}, \quad \text{for some } \ell = 1, \dots, m-1,$$

then $\|\partial_t^{m-1-k}\partial_x^k u(t,\cdot)\|_{L^2}^2$ is bounded by a decreasing function for any $k = \ell, \dots, m-1$.
In such a case, the *dissipative* character of our model is described by $d(t)$ and (24).

Remark 11. In particular, thanks to (21), from the right-hand side of (23), it also follows that

$$f(t) \geq_{(a)} k \frac{\eta'(t)}{\eta(t)} + \epsilon\, \eta(t), \tag{26}$$

for any $k \geq 2$, for any $\epsilon \in (0, 1-r)$. Indeed, the right-hand side of (23) corresponds to (26) for $k = 1$ and $\epsilon = 1 - r$.

Remark 12. It is clear that (23) implies that

$$f(t) + b_{m-1,0}(t) \geq_{(a)} 0. \tag{27}$$

3. Examples

It is clear that condition (22) in Theorem 1 depends on the functions $f_i(t)$ in Definition 3, which are related to the structure of the equation in (1). These functions $f_i(t)$ are not easy to compute, in general. In the following we first present some examples in which we check condition (23) in Theorem 1 but we do not consider (22). Then we present two models for which we explicitly compute $f_i(t)$ and $f(t)$ which satisfy (22).

Example 1. Let $\lambda(t) = (1+t)^p$, that is, $\Lambda \approx (1+t)^{p+1}$, for some $p > -1$. We remark that $\lambda(t)$ is strictly decreasing for $p \in (-1, 0)$ and constant $\lambda \equiv 1$ for $p = 0$. We can compute

$$\eta(t) =_{(a)} \frac{p+1}{1+t}, \qquad \frac{\eta'(t)}{\eta(t)} =_{(a)} -\frac{1}{1+t}.$$

Condition (21) trivially holds since $\eta'(t) \leq_{(a)} 0$. Let us assume that

$$b_{m-1,0}(t) \geq_{(a)} \mu(1+t)^{-1}, \qquad f(t) \geq_{(a)} \frac{\varphi}{1+t},$$

for some $\mu, \varphi \in \mathbb{R}$. Condition (23) holds if $-\mu \leq -1 + (1-r)(p+1) \leq \varphi$ for some $r \in [0, 1)$, that is, if

$$\mu \geq -p \quad \text{and} \quad \begin{cases} \varphi > -1, & \text{if } \mu \geq 1, \\ \varphi \geq -\mu & \text{if } \mu < 1. \end{cases} \tag{28}$$

It is easy to check that (23) is equivalent to (28). Indeed, if $\mu \geq 1$ and $\varphi > -1$, then the left-hand side of (23) holds for any $r \in [0, 1)$, whereas the right-hand side holds for some $r \in [0, 1)$, since $-1 + (1-r)(p+1) \to -1$ as $r \to 1$. On the other hand, if $\mu \in [-p, 1)$ and $\varphi \geq -\mu$, then (23) holds for

$$r = 1 - \frac{1-\mu}{p+1}, \quad \text{which satisfies } r \in [0, 1), \text{ due to } \mu \in [-p, 1).$$

An analogous reasoning holds for Examples 2 and 3.
The function $d(t)$ in (25) is given by $d(t) = (1+t)^{2\varphi}$. We remark that φ may be strictly negative if $\mu > 0$. On the other hand, if $p > 0$ and $\varphi \leq (m-1)p$ then Remark 10 is applicable.

Example 2. Let $\lambda(t) = e^{pt}$, that is, $\Lambda \approx \lambda$, for some $p > 0$. Then

$$\eta(t) =_{(a)} p, \qquad \frac{\eta'(t)}{\eta(t)} =_{(a)} 0.$$

Condition (21) trivially holds since $\eta'(t) =_{(a)} 0$. Let us assume that

$$b_{m-1,0}(t) \geq_{(a)} \mu, \qquad f(t) \geq_{(a)} \varphi,$$

for some $\mu, \varphi \in \mathbb{R}$. Then condition (23) holds if $-\mu \leq (1-r)p \leq \varphi$ for some $r \in [0, 1)$, that is, if

$$\mu \geq -p \quad \text{and} \quad \begin{cases} \varphi > 0, & \text{if } \mu \geq 0, \\ \varphi \geq -\mu & \text{if } \mu < 0. \end{cases}$$

The function $d(t)$ in (25) is given by $d(t) = e^{2\varphi t}$. We remark that φ may not be negative. Remark 10 is applicable if $\varphi \leq (m-1)p$.

Example 3. Let $\lambda(t) = e^t e^{e^t-1}$, that is, $\Lambda(t) = e^{e^t-1}$. Then

$$\eta(t) = e^t, \qquad \frac{\eta'(t)}{\eta(t)} = 1.$$

Condition (21) holds (see Remark 7). Let us assume that

$$b_{m-1,0}(t) \geq_{(a)} \mu e^t, \qquad f(t) \geq_{(a)} \varphi e^t,$$

for some $\mu, \varphi \in \mathbb{R}$. Then condition (23) holds if $-\mu \leq 1 - r < \varphi$ for some $r \in [0, 1)$, that is, if

$$\mu \geq -1 \quad \text{and} \quad \begin{cases} \varphi > 0, & \text{if } \mu \geq 0, \\ \varphi > -\mu & \text{if } \mu < 0. \end{cases}$$

The function $d(t)$ in (25) is given by $d(t) = e^{2\varphi e^t}$. We remark that φ may not be negative. Remark 10 is applicable if $\varphi \leq m-1$.

3.1. A third-order equation model

Let us consider the equation in (1) for $m = 3$, and let us assume that the three roots of (7) are α, 1, -1, where α is a real parameter, and $\alpha \neq \pm 1$ so that (8) is satisfied. This gives

$$u_{ttt} - \alpha\lambda(t)u_{ttx} - \lambda(t)^2 u_{txx} + \alpha\lambda(t)^3 u_{xxx} + b_{2,0}(t)u_{tt} + b_{1,1}(t)\lambda(t)u_{tx} + b_{0,2}(t)\lambda(t)^2 u_{xx} + b_{0,1}(t)\lambda(t)u_x = 0 \tag{29}$$

with initial data $(u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x)$. We study the λ-energy for the solution in (3), that is,

$$E_\lambda = \|u_{tt}(t,\cdot)\|_{L^2}^2 + \lambda(t)^2\|u_{tx}(t,\cdot)\|_{L^2}^2 + \lambda(t)^4\|u_{xx}(t,\cdot)\|_{L^2}^2. \tag{30}$$

For the coefficients we assume conditions (18)–(20), so we are now able to compute the functions $f_i(t)$ in (13). The diagonalizer for the matrix $A(t)$ in (11) is constant, therefore (10) trivially holds, and

$$\begin{aligned}
W_1^\sharp &= (-2, 0, 0), & \tilde{V}_1 &= -\frac{1}{1-\alpha^2}(1, \alpha, \alpha^2), \\
W_2^\sharp &= (-2\alpha, 1-\alpha, 0), & \tilde{V}_2 &= \frac{1}{2(1-\alpha)}(1, 1, 1), \\
W_3^\sharp &= (2\alpha, -(1+\alpha), 0), & \tilde{V}_3 &= \frac{1}{2(1+\alpha)}(1, -1, 1).
\end{aligned}$$

Therefore, since $(b_2(t)) = (-b_{02}(t), -b_{11}(t), -b_{20}(t))$, if we put $f_2 = f_-$ and $f_3 = f_+$, then the functions $f_i(t)$ in (13) can be written as

$$f_1(t) = \frac{1}{(1-\alpha^2)}\left[2\frac{\lambda'(t)}{\lambda(t)} + b_{0,2}(t) + \alpha b_{1,1}(t) + \alpha^2 b_{2,0}(t)\right], \tag{31}$$

$$f_\pm(t) = \frac{1}{2(1\pm\alpha)}\left[(1 \pm 3\alpha)\frac{\lambda'(t)}{\lambda(t)} - b_{2,0}(t) \pm b_{1,1}(t) - b_{0,2}(t)\right]. \tag{32}$$

The structure of the $f_i(t)$ greatly simplifies if we consider a special model. Let us apply on the left-hand side the first-order operator

$$L(\partial_t, \lambda(t)\partial_x, t) = \partial_t - \alpha\lambda(t)\partial_x + a(t), \tag{33}$$

where $\alpha \neq \pm 1$, to the second-order wave equation with time-dependent speed of propagation and damping

$$u_{tt} - \lambda(t)^2 u_{xx} + b(t)u_t = 0,$$

where $b(t)$ is real-valued, bounded, with $|b'| \lesssim \eta$ and $|b''| \lesssim \eta^2$. From

$$(\partial_t - \alpha\lambda(t)\partial_x + a(t))\left(\partial_t^2 - \lambda(t)^2\partial_x^2 + b(t)\partial_t\right)u = 0, \tag{34}$$

if we put $a(t) := -b'(t)/b(t)$ then we get (29) with

$$b_{2,0}(t) = b(t) - \frac{b'(t)}{b(t)}, \tag{35}$$

$$b_{1,1}(t) = -\alpha b(t), \tag{36}$$

$$b_{0,2}(t) = \frac{b'(t)}{b(t)} - 2\frac{\lambda'(t)}{\lambda(t)}, \tag{37}$$

and $b_{0,1} \equiv 0$. By computing the $f_i(t)$ as above, using (35)–(37), we immediately derive that

$$f_1(t) = \frac{b'(t)}{b(t)}, \qquad f_\pm(t) = \frac{1}{2}\left(3\frac{\lambda'(t)}{\lambda(t)} - b(t)\right). \tag{38}$$

We are now able to check condition (22) in Examples 1–3.

Example 4. Following Example 1, let $\lambda(t) = (1+t)^p$ for some $p > -1$. Let $b(t) = b_0(1+t)^{-1}$ for some $b_0 \geq -(p+1)$, that is, $a(t) = -(1+t)^{-1}$ in (33), so that

$$b_{2,0}(t) = \frac{\mu}{1+t}, \quad \text{where } \mu = b_0 + 1 \geq -p.$$

It follows that $f(t) = \varphi(1+t)^{-1}$, where

$$\varphi = \max\{-1, (3p - b_0)/2\}$$

satisfies (22). Since $\varphi > -1$ is a necessary condition to apply Theorem 1, we have to assume

$$p > -\frac{3}{4}, \quad b_0 \in [-(p+1), 3p+2), \text{ so that } \varphi = \frac{3p - b_0}{2}.$$

It remains to check $\varphi + \mu \geq 0$ in condition (28), that is, $b_0 \geq -(3p+2)$. Therefore (23) holds for $b_0 \in [-(3p+2), 3p+2)$ if $p \in (-2/3, -1/2]$ and for $b_0 \in [-(p+1), 3p+2)$ if $p \geq -1/2$.

We remark that $d(t) = (1+t)^{3p-b_0}$ in (25). In particular, let $p \geq 0$. The exponent is negative if $b_0 \in [3p, 3p+2)$. We remark that if $p = 0$, that is, the speed of propagation $\lambda \equiv 1$ is constant, this means $b_0 \in [0, 2)$. According to Remark 10, if $p > 0$ and $b_0 \geq -p$, then we obtain the dissipative estimates

$$\|u_{tx}(t,\cdot)\|_{L^2}^2 \leq C(1+t)^{-(b_0-p)}E(0), \quad \text{if } b_0 \in [p, 3p+2), \quad \text{and}$$
$$\|u_{xx}(t,\cdot)\|_{L^2}^2 \leq C(1+t)^{-(p+b_0)}E(0), \quad \text{for any } b_0 \in [-p, 3p+2).$$

We notice that Remark 14 is applicable in the special case $\alpha = 0$, since $b_{1,1} \equiv 0$ in (36). In such a case, we can also consider $b_0 \in [-2(p+1), -(p+1))$, but we cannot have a dissipative effect in the sense of Remark 10.

Example 5. Following Example 2, let $\lambda(t) = e^{pt}$ for some $p > 0$. Let $b_{2,0}(t) = b(t) = \mu$ for some $\mu \in [-p, 3p)$, that is, $a \equiv 0$ in (33) and $f(t) = \varphi = (3p - \mu)/2$ satisfies (22). One can easily check that in this case condition (23) holds. We remark that $d(t) = e^{(3p-\mu)t}$ in (25). The exponent is always strictly positive, but according to Remark 10 we obtain the dissipative estimates

$$\|u_{tx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(\mu-p)t}E(0), \quad \text{if } \mu \in [p, 3p), \quad \text{and}$$
$$\|u_{xx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(p+\mu)t}E(0), \quad \text{for any } \mu \in [-p, 3p).$$

As in Example 4, Remark 14 is applicable in the special case $\alpha = 0$.

Example 6. Following Example 3, let $\lambda(t) = e^t e^{e^t - 1}$. Let $b(t) = b_0 e^t$ for some $b_0 \in (-1, 3)$, that is, $a \equiv -1$ in (33). Since $b_0 < 3$, the function $f(t) = \varphi e^t$ where $\varphi = (3 - b_0)/2$ satisfies (22). One can easily check that condition (23) holds.

We remark that $d(t) = e^{(3-b_0)e^t}$ in (25) is an increasing function. According to Remark 10 we obtain the dissipative estimates

$$\|u_{tx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(b_0-1)e^t}E(0), \quad \text{if } b_0 \in [1, 3), \quad \text{and}$$
$$\|u_{xx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(b_0+1)e^t}E(0), \quad \text{for any } b_0 \in (-1, 3).$$

As in Example 4, Remark 14 is applicable in the special case $\alpha = 0$.

3.2. A fourth-order equation model

Now we consider a fourth order equation. Let the four roots of (7) be $\alpha, -\alpha, 1, -1$, where α is a real parameter, and $\alpha \neq \pm 1$ so that (8) is satisfied. This gives

$$\partial_t^4 u - (1+\alpha^2)\lambda(t)^2\partial_t^2\partial_x^2 u + \alpha^2\lambda(t)^4\partial_x^4 u + \sum_{j=0}^{2}\sum_{k=1}^{3-j} b_{j,k}(t)\lambda(t)^k\partial_t^j\partial_x^k u = 0. \tag{39}$$

The diagonalizer for the matrix $A(t)$ in (11) is constant, therefore (10) trivially holds, and we get

$$W_1^\sharp = (-3\alpha, -2, \alpha, 0), \qquad \tilde{V}_1 = -\frac{1}{2\alpha(1-\alpha^2)}(1, \alpha, \alpha^2, \alpha^3),$$
$$W_2^\sharp = (3\alpha, -2, -\alpha, 0), \qquad \tilde{V}_2 = \frac{1}{2\alpha(1-\alpha^2)}(1, -\alpha, \alpha^2, -\alpha^3),$$
$$W_3^\sharp = (-3\alpha^2, -2\alpha^2, 1, 0), \qquad \tilde{V}_3 = \frac{1}{2(1-\alpha^2)}(1, 1, 1, 1),$$
$$W_4^\sharp = (3\alpha^2, -2\alpha^2, -1, 0), \qquad \tilde{V}_4 = -\frac{1}{\alpha(1-\alpha^2)}(1, -1, 1, -1).$$

As in Section 3.1, we present a model for (39), which comes from the composition of a wave-type equation and a wave-type damped wave equation. For the sake of simplicity, let $\lambda(t) = e^{pt}$ for some $p > 0$, as in Example 2; we consider

$$\left(\partial_t^2 - \alpha^2 e^{2pt}\partial_x^2\right)\left(\partial_t^2 - e^{2pt}\partial_x^2 + \mu\partial_t\right) u = 0, \tag{40}$$

where $\mu \in \mathbb{R}$. Eq. (40) can be written as in (39) if we put

$$b_{3,0} \equiv \mu, \qquad b_{1,2} \equiv -(\alpha^2\mu + 4p), \qquad b_{0,2} \equiv -4p^2$$

and $b_{2,1} = b_{0,3} = b_{2,0} = b_{1,1} = b_{1,0} = b_{0,1} = b_{0,0} = 0$. Straightforward calculations give a very easy expression for the functions $f_i(t)$ in (13):

$$f_1(t) = f_2(t) = \frac{p}{2}, \qquad f_3(t) = f_4(t) = \frac{5p-\mu}{2}.$$

Therefore we can take $f(t) = \varphi$, where $\varphi = \max\{5p - \mu, p\}/2$. Due to the simple structure of Eq. (40) we can apply Remark 14 with $q = 1$, that is, we can take $r \in [-1, 1)$ in (23). It is easy to check that (23) holds for any $\mu \geq -2p$; the function $d(t)$ in (25) is given by $d(t) = e^{(5p-\mu)t}$ if $\mu \in [-2p, 4p]$ and by $d(t) = e^{pt}$ if $\mu \geq 4p$. According to Remark 10 we obtain the dissipative estimates

$$\|u_{ttx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-pt}E(0), \qquad \|u_{txx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-3pt}E(0), \qquad \|u_{xxx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-5pt}E(0),$$

if $\mu \geq 4p$; otherwise, we have the following ones:

$$\|u_{ttx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(\mu-3p)t}E(0), \quad \text{if } \mu \in [3p, 4p],$$
$$\|u_{txx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(\mu-p)t}E(0), \quad \text{if } \mu \in [p, 4p), \quad \text{and}$$
$$\|u_{xxx}(t,\cdot)\|_{L^2}^2 \leq Ce^{-(p+\mu)t}E(0), \quad \text{if } \mu \in [-p, 4p).$$

4. The hyperbolic-like estimate

Here we describe the philosophy which leads to focus our attention on the function $f(t)$ in (22). We will use the following.

Notation 2. If $v = (v_1, \ldots, v_m)$ is a vector in $\mathbb{C}^m$, then we denote by $\operatorname{diag} v$ or $\operatorname{diag}(v_1, \ldots, v_m)$ the $m \times m$ diagonal matrix $M = (M_{ij})$ with entries $M_{ii} = v_i$ and $M_{ij} = 0$ for any $i \neq j$. On the other hand, if $M = (M_{ij})$ is a square matrix, then we denote the diagonal part of M by $\operatorname{Diag} M$, that is, $(\operatorname{Diag} M)_{ii} = M_{ii}$, and $(\operatorname{Diag} M)_{ij} = 0$ if $i \neq j$.

We perform the Fourier transform with respect to x of (1) and we introduce the Cauchy problem for the ordinary differential m by m system with parameter ξ

$$\partial_t U = i\lambda(t)\xi A(t)U + A_0\frac{\lambda'(t)}{\lambda(t)}U + \sum_{j=0}^{m-1}(i\lambda(t)\xi)^{-(m-1-j)}B_{m-1-j}(t)\,U, \tag{41}$$

with initial data $U(0,\xi) = U_0(\xi)$, where we put

$$U = \left((i\lambda(t)\xi)^{m-1}\widehat{u},\ (i\lambda(t)\xi)^{m-2}\widehat{u}_t,\ \ldots,\ i\lambda(t)\xi\,\partial_t^{m-2}\widehat{u},\ \partial_t^{m-1}\widehat{u}\right), \tag{42}$$

$$U_0(\xi) = \left((i\xi)^{m-1}\widehat{u_0}(\xi), (i\xi)^{m-2}\widehat{u_1}(\xi), \ldots, \widehat{u_{m-1}}(\xi)\right), \tag{43}$$

with $A(t)$ and A_0 as in (9) and (12), and the matrices $B_{m-1-j}(t)$ are given by

$$B_{m-1-j}(t) = \left((0), (0), \ldots, (0), (b_j(t))\right)^T,$$

where (0) is the null vector, and $(b_j(t))$ are the vectors

$$(b_j(t)) = (-b_{0,j}(t), -b_{1,j-1}(t), \ldots, -b_{j,0}(t), 0, \ldots, 0).$$

Hypothesis 1 corresponds to say that the system in (41), is *uniformly strictly hyperbolic*. In fact, the eigenvalues of the matrix $A(t)$ are given by $\tau_i(t)$ (see Remark 4). Moreover, it is clear that the λ-energy given in (3) for (1) is given by $\|U(t,\cdot)\|_{L^2}^2$ where U is as in (42). Analogously, it holds $\|U_0\|_{L^2}^2 \leq E(0)$, with U_0 as in (43).

Our aim is to prove that

$$\mathcal{E}(t,\xi) \leq C d(t) \mathcal{E}_0(\xi), \tag{44}$$

with $d(t)$ as in (25), uniformly with respect to $\xi \in \mathbb{R}$, where $\mathcal{E}(t,\xi)$ is the wave type pointwise energy given by

$$\mathcal{E}(t,\xi) := \sum_{j=0}^{m-1} (\lambda(t)|\xi|)^{2(m-1-j)} |\partial_t^j \widehat{u}(t,\xi)|^2, \tag{45}$$

whereas $\mathcal{E}_0(t,\xi)$ is the Klein–Gordon type pointwise energy given by

$$\mathcal{E}_0(\xi) := \sum_{j=0}^{m-1} (1+|\xi|^2)^{m-1-j} |\widehat{u_j}(\xi)|^2. \tag{46}$$

Indeed, by integrating this inequality with respect to ξ and by Plancherel's Theorem, the estimate (24) immediately follows from (44).

In order to prove (44) we divide, for some constant $N > 0$, the extended phase space $[0,\infty) \times \mathbb{R}^n$ into the *pseudo-differential* and the *hyperbolic* zone, defined by

$$Z_{\rm pd}(N) = \{(t,\xi) \in [0,\infty) \times \mathbb{R} : \Lambda(t)|\xi| \leq N\},$$
$$Z_{\rm hyp}(N) = \{(t,\xi) \in [0,\infty) \times \mathbb{R} : \Lambda(t)|\xi| \geq N\}.$$

Since $\Lambda : [0,\infty) \to [1,\infty)$ is strictly increasing and surjective, the *separating curve* is given by

$$\theta : (0,N] \to [0,\infty), \qquad \theta_{|\xi|} = \Lambda^{-1}(N/|\xi|).$$

We put also $\theta_0 = \infty$, and $\theta_{|\xi|} = 0$ for any $|\xi| > N$. The pair (t,ξ) in the extended phase space is in $Z_{\rm pd}(N)$ (resp. in $Z_{\rm hyp}(N)$) if, and only if, $t \leq \theta_{|\xi|}$ (resp. $t \geq \theta_{|\xi|}$).

In $Z_{\rm hyp}(N)$ the profile of the pointwise energy $\mathcal{E}(t,\xi)$ will be exactly described by the function $d(t)$ in (25). On the other hand, in $Z_{\rm pd}(N)$ we will only look for conditions which guarantee that the same function $d(t)$ appears in the estimate from above of the pointwise energy $\mathcal{E}(t,\xi)$. For this reason, we say that we are studying *hyperbolic-like* estimates.

5. The hyperbolic zone

In $Z_{\rm hyp}(N)$ we use a $\mathcal{C}^2$ diagonalization procedure. We want to prove the *a priori* estimate

$$|U(t,\xi)|^2 \leq C \frac{d(t)}{d(\theta_{|\xi|})} |U(\theta_{|\xi|},\xi)|^2, \quad t \geq \theta_{|\xi|}, \tag{47}$$

with $d(t)$ as in (25). Let $N(t)$ be the diagonalizer for $A(t)$ introduced in Remark 5 and let $V(t,\xi) = N(t)U(t,\xi)$. Then (41) becomes

$$\partial_t V = i\lambda(t)\xi \mathcal{D}(t) V + \mathcal{B}_0(t) V + \sum_{j=0}^{m-2} (i\lambda(t)\xi)^{-(m-1-j)} \mathcal{B}_{m-1-j}(t)\, V, \tag{48}$$

where

$$\mathcal{B}_0(t) = \left(N'(t) + \frac{\lambda'(t)}{\lambda(t)} N(t) A_0 + N(t) B_0(t) \right) N^{-1}(t), \tag{49}$$

$$\mathcal{B}_{m-1-j}(t) = N(t) B_{m-1-j}(t) N^{-1}(t), \quad j = 0, \ldots, m-2. \tag{50}$$

Thanks to Hypotheses 3 and 4, it is clear that $\|\mathcal{B}_0\| \lesssim \eta$ and $\|\mathcal{B}_0'\| \lesssim \eta^2$, whereas $\|\mathcal{B}_{m-1-j}\| \lesssim \eta^{m-j}$ for $j \leq m-2$. Indeed:

- the function $\lambda'(t)/\lambda(t)$ is bounded by $\eta(t)$ and its derivative is bounded by $\eta(t)^2$ by virtue of (16),
- the matrices $N(t)$ and $N^{-1}(t)$ are bounded (see Remark 5), and their derivatives are bounded by $\eta(t)$ thanks to (18),
- the matrix $B_0(t)$ is bounded by $\eta(t)$ and its derivative is bounded by $\eta(t)^2$ by virtue of (19),
- the matrices $B_{m-1-j}(t)$ are bounded by $\eta(t)^{m-j}$ by virtue of (20).

To derive our estimate in $Z_{\text{hyp}}(N)$, we need to control the diagonal part of $\mathcal{B}_0(t)$ in (49). Thanks to Remark 5, it is clear that the rows of the matrix $N(t)A_0$ are given by the vectors $W_i^\sharp(t)$, for $i = 1, \ldots, m$, whereas each row of the matrix $N(t)B_0(t)$ is given by the vector $(b_{m-1}(t))$, since all the entries in the last column of $N(t)$ are 1. Therefore we get

$$\left(\frac{\lambda'(t)}{\lambda(t)}N(t)A_0 + N(t)B_0(t)\right)N^{-1}(t) = \begin{pmatrix} (\lambda'(t)/\lambda(t))W_1^\sharp(t) + (b_{m-1}(t)) \\ \vdots \\ (\lambda'(t)/\lambda(t))W_m^\sharp(t) + (b_{m-1}(t)) \end{pmatrix} \left(\tilde V_1(t)^T, \ldots, \tilde V_m(t)^T\right)$$

whose diagonal entries are given by $f_i(t)$ as in (13). Let $f(t)$ be as in (22); since $f(t) - f_i(t) \geq_{(a)} 0$, it follows from Definition 1 that there exist m *almost-zero* functions $f_{i,w}(t)$ such that

$$f_{i,s}(t) := f(t) - f_i(t) - f_{i,w}(t) \geq 0.$$

Here we use the notation $f_{i,w}(t)$ and $f_{i,s}(t)$ to distinguish between *weak* and *strong* components of each difference $f(t) - f_i(t)$. We can now construct the *second diagonalizer* $K(t, \xi)$, that depends on the not diagonal entries of $\mathcal{B}_0(t)$:

$$K_{ii}(t,\xi) = 1; \qquad K_{ij}(t,\xi) = \left(\xi\lambda(t)(\tau_i(t) - \tau_j(t))\right)^{-1}(\mathcal{B}_0(t))_{ij} \quad \text{if } i \neq j. \tag{51}$$

Thanks to (18)–(20), we derive

$$|K_{ij}(t,\xi)| \leq \frac{C\eta(t)}{|\xi|\,\lambda(t)} = \frac{C}{|\xi|\,\Lambda(t)} \leq \frac{C}{N}, \quad \text{for any } i \neq j,$$

that is, $K(t, \xi)$ is uniformly regular and bounded for a sufficiently large N. We replace $V(t, \xi) = K(t, \xi)W(t, \xi)$ in (48) and we get

$$\partial_t W = \mathcal{D}(t)i\lambda(t)\xi W + \mathcal{F}_0(t)W + J(t,\xi)W, \tag{52}$$

where

$$\mathcal{F}_0(t) = \operatorname{Diag}\mathcal{B}_0(t) = \operatorname{diag}(f_i(t)), \tag{53}$$

and $J(t, \xi)$ satisfies (see [2])

$$\|J(t,\xi)\| \leq C\frac{\eta(t)^2}{|\xi|\,\lambda(t)}. \tag{54}$$

Now let

$$D(t,\xi) = \exp\left(\int_{\theta_{|\xi|}}^t \operatorname{Diag}(N'(\sigma)N^{-1}(\sigma))d\sigma\right) \operatorname{diag}\left(\exp\left(\int_{\theta_{|\xi|}}^t \left(i\tau_j(\sigma)\lambda(\sigma)\xi + f_{j,w}(\sigma)\right)\, d\sigma\right)\right).$$

Thanks to (10), we obtain (see [4]) that each entry of the diagonal matrix $\operatorname{Diag}\left(N'(t)N(t)^{-1}\right)$ is a *almost-zero* function. Since $\tau_j(t)$ are real and $f_{j,w}(t) =_{(a)} 0$, the matrix $D(t, \xi)$ is uniformly regular and bounded, i.e. $\|D(t, \xi)\|, \|D^{-1}(t, \xi)\| \leq C$. By making the substitution $W(t,\xi) = \sqrt{d(t)d(\theta_{|\xi|})^{-1}}D(t,\xi)Z(t,\xi)$, we obtain in $Z_{\text{hyp}}(N)$,

$$\begin{cases} \partial_t Z = G(t)Z + \widetilde{J}(t,\xi)Z, & t \geq \theta_{|\xi|}, \\ Z(\theta_{|\xi|},\xi) = K^{-1}(\theta_{|\xi|},\xi)N^{-1}(\theta_{|\xi|})\,U(\theta_{|\xi|},\xi), \end{cases} \tag{55}$$

where $G(t)$ is a diagonal matrix with negative entries $-f_{i,s}(t)$ and $\widetilde{J}(t,\xi) = D^{-1}(t,\xi)J(t,\xi)D(t,\xi)$ satisfies again (54). It is easy to prove that $|Z(t, \xi)| \leq C|Z(\theta_{|\xi|}, \xi)|$, and this concludes the proof of (47).

Remark 13. It is now clear that the functions $f_i(t)$ introduced in Definition 3 came into play in $Z_{\text{hyp}}(N)$. Moreover, we remark that the estimate given in (47) holds also from below, that is,

$$|U(t,\xi)|^2 \approx \frac{d(t)}{d(\theta_{|\xi|})}\,|U(\theta_{|\xi|},\xi)|^2, \quad t \geq \theta_{|\xi|},$$

if $f_i(t) =_{(a)} f(t)$ for any $i = 1, \ldots, m$. Moreover, if $g(t)$ is a function with constant sign such that $g(t) \leq_{(a)} f_i(t)$ for any $i = 1, \ldots, m$, then one can easily prove that

$$|U(t,\xi)|^2 \gtrsim \exp\left(\int_{\theta_{|\xi|}}^t 2g(\sigma)\,d\sigma\right)|U(\theta_{|\xi|},\xi)|^2, \quad t \geq \theta_{|\xi|}.$$

6. The pseudo-differential zone

Having in mind (47), we can conclude the proof of our claim (44) for any $(t, \xi) \in [0, \infty) \times \mathbb{R}^n$ if we prove it in $Z_{\text{pd}}(N)$, that is, for any $|\xi| \leq N$ and $t \leq \theta_{|\xi|}$.

Here we present a strategy to derive (44) in $Z_{\mathrm{pd}}(N)$, in which we reduce our problem to a system of one Volterra-type integral equation and $m-1$ integral inequalities. This strategy is particularly successful due to the special structure in (1), that is, there are no terms with zero derivatives in x, but $\partial_t^m u$ and $b_{m-1,0}(t)\partial_t^{m-1}u$.

Since in $Z_{\mathrm{pd}}(N)$ we can estimate $\lambda(t)|\xi| \le N\eta(t)$, we put

$$V = \left((i\eta(t))^{m-1}\widehat{u}, (i\eta(t))^{m-2}\widehat{u}_t, \ldots, \partial_t^{m-1}\widehat{u}\right)^T,$$

and $V = \sqrt{d(t)}\widetilde{V}$ with $d(t)$ as in (25), and we study the Cauchy problem

$$\partial_t\widetilde{V} = \mathcal{A}(t,\xi)\widetilde{V}, \qquad \widetilde{V}(0,\xi) = V_0(\xi) := \left(i^{m-1}\widehat{u_0}, i^{m-2}\widehat{u_1}, \ldots, \widehat{u_{m-1}}\right)^T, \tag{56}$$

with

$$\mathcal{A} := \begin{pmatrix} (m-1)\eta'/\eta & i\eta & 0 & 0 & \ldots & 0 \\ 0 & (m-2)\eta'/\eta & i\eta & 0 & \ldots & 0 \\ 0 & 0 & (m-3)\eta'/\eta & i\eta & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \eta'/\eta & i\eta \\ \beta_0/\eta^{m-1} & \beta_1/\eta^{m-2} & \beta_2/\eta^{m-3} & \ldots & \beta_{m-2}/\eta & \beta_{m-1} \end{pmatrix} - \frac{d'}{2d} I_m,$$

where I_m denotes the identity m by m matrix and

$$\beta_{m-1}(t,\xi) := i\xi\lambda(t)a_{m-1}(t) - b_{m-1,0}(t), \tag{57}$$

$$\beta_j(t,\xi) := i\, a_j(t)(\xi\lambda(t))^{m-j} - \sum_{k=1}^{m-1-j} i^{-(m-1-j-k)} b_{j,k}(t)(\xi\lambda(t))^k, \quad j = 0,\ldots,m-2. \tag{58}$$

Our purpose is to prove that the fundamental solution $E(t,\xi)$ to (56), that is, the solution to

$$\partial_t E = \mathcal{A}(t,\xi)E, \qquad E(0,\xi) = I_m, \quad \text{where } E(t,\xi) = (E_{j,k})_{j,k=1,\ldots,m} \text{ is an } m \text{ by } m \text{ matrix,}$$

is bounded. Indeed, since $|U(t,\xi)| \lesssim |V(t,\xi)|$ for any $t \le \theta_{|\xi|}$ and $|V_0(\xi)| \le \mathcal{E}_0(\xi)$, our claim (44) immediately follows if $\|E(t,\xi)\| \le C$, uniformly in $Z_{\mathrm{pd}}(N)$. We can write the integral equations

$$E_{\ell,k} = \frac{\eta(t)^{m-\ell}}{\sqrt{d(t)}}\left(\delta_{\ell k} + i\int_0^t \frac{\sqrt{d(\tau)}}{\eta(\tau)^{m-1-\ell}} E_{\ell+1,k}(\tau)\,d\tau\right), \quad \text{for any } \ell = 1,\ldots,m-1, \tag{59}$$

$$E_{m,k} = \frac{\Theta(t,\xi)}{\sqrt{d(t)}}\left(\delta_{mk} + \int_0^t \frac{\sqrt{d(\sigma)}}{\Theta(\sigma,\xi)}\left(\sum_{j=0}^{m-2}\frac{\beta_j(\sigma,\xi)}{\eta(\sigma)^{m-1-j}}E_{j+1,k}(\sigma,\xi)\right)d\sigma\right), \tag{60}$$

for $k = 1,\ldots,m$, where we defined

$$\Theta(t,\xi) := \exp\left(\int_0^t \left(i\xi\lambda(\tau)a_{m-1}(\tau) - b_{m-1,0}(\tau)\right)d\tau\right).$$

We want to write a Volterra-type integral equation for $E_{m-1,k}$, independent of the other $E_{j,k}$:

$$E_{m-1,k}(t,\xi) = f_k(t,\xi) + \int_0^t g(t,\tau,\xi)\,E_{m-1,k}(\tau,\xi)\,d\tau. \tag{61}$$

By using (59) and (60) and integrating by parts we get

$$\begin{aligned} E_{m-1,k}(t,\xi) &= \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\delta_{mk}\int_0^t \Theta(\tau,\xi)\,d\tau\right) \\ &\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta(\tau,\xi)\int_0^\tau \frac{\sqrt{d(\sigma)}}{\Theta(\sigma,\xi)}\left(\sum_{j=0}^{m-2}\frac{\beta_j(\sigma,\xi)}{\eta(\sigma)^{m-1-j}}E_{j+1,k}(\sigma,\xi)\right)d\sigma\,d\tau \\ &= \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\delta_{mk}\Theta_1(t,0,\xi)\right) \\ &\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_1(t,\sigma,\xi)\sqrt{d(\sigma)}\left(\sum_{j=0}^{m-2}\frac{\beta_j(\sigma,\xi)}{\eta(\sigma)^{m-1-j}}E_{j+1,k}(\sigma,\xi)\right)d\sigma, \end{aligned}$$

where we put

$$\Theta_1(t,\sigma,\xi) := \Theta(\sigma,\xi)^{-1}\int_\sigma^t \Theta(\tau,\xi)\,d\tau \equiv \int_\sigma^t \exp\left(\int_\sigma^\tau \left(i\xi\lambda(t)a_{m-1}(s) - b_{m-1,0}(s)\right)ds\right)d\tau.$$

By using (59) for $E_{j+1,k}$, for any $j \le m-3$ and integrating by parts again, we obtain

$$\begin{aligned}
E_{m-1,k}(t,\xi) &= \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\delta_{mk}\Theta_1(t,0,\xi)\right) \\
&\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_1(t,\sigma,\xi)\,\frac{\sqrt{d(\sigma)}}{\eta(\sigma)}\,\beta_{m-2}(\sigma,\xi)E_{m-1,k}(\sigma,\xi)\,d\sigma \\
&\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_1(t,\sigma,\xi)\sum_{j=0}^{m-3}\beta_j(\sigma,\xi)\left(\delta_{j+1,k} + i\int_0^\sigma \frac{\sqrt{d(\tau)}}{\eta(\tau)^{m-2-j}}\,E_{j+2,k}(\tau)\,d\tau\right)d\sigma \\
&= \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\Theta_1(t,0,\xi)\delta_{mk} + i\sum_{j=0}^{m-3}\Theta_{2,j}(t,0,\xi)\delta_{j+1,k}\right) \\
&\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_{1,m-2}(t,\sigma,\xi)\,\frac{\sqrt{d(\sigma)}}{\eta(\sigma)}E_{m-1,k}(\sigma,\xi)\,d\sigma \\
&\quad - \frac{\eta(t)}{\sqrt{d(t)}}\int_0^t\left(\sum_{j=0}^{m-3}\Theta_{2,j}(t,\tau,\xi)\frac{\sqrt{d(\tau)}}{\eta(\tau)^{m-2-j}}\,E_{j+2,k}(\tau)\,d\tau\right),
\end{aligned}$$

where we put

$$\begin{aligned}
\Theta_{1,j}(t,\sigma,\xi) &:= \Theta_1(t,\sigma,\xi)\,\beta_j(\sigma,\xi), \\
\Theta_{2,j}(t,\tau,\xi) &:= \int_\tau^t \Theta_{1,j}(t,\sigma,\xi)\,d\sigma.
\end{aligned}$$

Using again (59) for $E_{j+2,k}$, for any $j \le m-4$ and integrating by parts, we get

$$\begin{aligned}
E_{m-1,k}(t,\xi) &= \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\Theta_1(t,0,\xi)\delta_{mk} + i\sum_{j=0}^{m-3}\Theta_{2,j}(t,0,\xi)\delta_{j+1,k} - \sum_{j=0}^{m-4}\Theta_{3,j}(t,0,\xi)\delta_{j+2,k}\right) \\
&\quad + i\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_{1,m-2}(t,\sigma,\xi)\,\frac{\sqrt{d(\sigma)}}{\eta(\sigma)}E_{m-1,k}(\sigma,\xi)\,d\sigma \\
&\quad - \frac{\eta(t)}{\sqrt{d(t)}}\int_0^t \Theta_{2,m-3}(t,\tau,\xi)\frac{\sqrt{d(\tau)}}{\eta(\tau)}\,E_{m-1,k}(\tau)\,d\tau \\
&\quad - i\,\frac{\eta(t)}{\sqrt{d(t)}}\int_0^t\left(\sum_{j=0}^{m-4}\Theta_{3,j}(t,\sigma,\xi)\frac{\sqrt{d(\sigma)}}{\eta(\sigma)^{m-3-j}}\,E_{j+3,k}(\sigma)\,d\sigma\right),
\end{aligned}$$

where we put

$$\Theta_{3,j}(t,\sigma,\xi) := \int_\sigma^t \Theta_{2,j}(t,\tau,\xi)\,d\tau.$$

It is clear that we can iterate the procedure, so that we obtain (61), where

$$f_k(t,\xi) = \frac{\eta(t)}{\sqrt{d(t)}}\left(\delta_{m-1,k} + i\Theta_1(t,0,\xi)\delta_{mk} + \sum_{\ell=1}^{m-2} i^\ell\left(\sum_{j=0}^{m-2-\ell}\Theta_{\ell+1,j}(t,0,\xi)\delta_{j+\ell,k}\right)\right), \tag{62}$$

and

$$g(t,\tau,\xi) = \frac{\eta(t)}{\sqrt{d(t)}}\,\frac{\sqrt{d(\tau)}}{\eta(\tau)}\sum_{\ell=1}^{m-1} i^\ell\Theta_{\ell,m-1-\ell}(t,\tau,\xi). \tag{63}$$

We have in mind to estimate $E^\sharp_{m-1,k}(t,\xi) := |E_{m-1,k}(t,\xi)|$. Since $\lambda(t)a_{m-1}(t)$ is real, it holds $|\Theta(t,\xi)| = \gamma(t)$, where

$$\gamma(t) := \exp\left(-\int_0^t b_{m-1,0}(\tau)d\tau\right), \tag{64}$$

is a positive function which does not depend on ξ. We remark that $\Theta_1(t,0,\xi) = \int_0^t \Theta(\tau,\xi)\gamma(\tau)\,d\tau$ since $\Theta(0,\xi) = \gamma(0) = 1$, and that

$$|\Theta_1(t,\sigma,\xi)| \le \gamma(\sigma)^{-1}\int_\sigma^t \gamma(\tau)\,d\tau.$$

In particular, $|\Theta_1(t,0,\xi)| \le \Gamma(t)$, where we put

$$\Gamma(t) := \int_0^t \gamma(\tau)\,d\tau. \tag{65}$$

Then we can estimate

$$|\Theta_{1,j}(t,\tau,\xi)| \le |\beta_j(\tau,\xi)|\,\gamma(\tau)^{-1}\int_\tau^t \gamma(\sigma)\,d\sigma =: \Theta_{1,j}^\sharp(t,\tau,\xi),$$

and, analogously,

$$|\Theta_{\ell,j}(t,\tau,\xi)| \le \int_\tau^t \Theta_{\ell-1,j}^\sharp(t,\sigma,\xi)\,d\sigma =: \Theta_{\ell,j}^\sharp(t,\tau,\xi),$$

for any $\ell = 2,\dots,m-1$ and $j \le m-1-\ell$. This gives us

$$|f_k(t,\xi)| \le \frac{\eta(t)}{\sqrt{d(t)}}\left(1+\Gamma(t)+C\max\Theta_{\ell,j}^\sharp(t,0,\xi)\right) =: f_k^\sharp(t,\xi), \tag{66}$$

where the maximum is taken over $\ell = 2,\dots,m-1$ and $j \le m-1-\ell$, and

$$|g(t,\tau,\xi)| \le \frac{\eta(t)}{\sqrt{d(t)}}\,\frac{\sqrt{d(\tau)}}{\eta(\tau)}\sum_{\ell=1}^{m-1}\Theta_{\ell,m-1-\ell}^\sharp(t,\tau,\xi) =: g^\sharp(t,\tau,\xi). \tag{67}$$

Therefore, to prove that $E_{m-1,k}(t,\xi)$ is bounded (with respect to (t,ξ)) in $Z_{\mathrm{pd}}(N)$, we can apply a Gronwall-like lemma to the following inequality:

$$E_{m-1,k}^\sharp(t,\xi) \le f_k^\sharp(t,\xi) + \int_0^t g^\sharp(t,\tau,\xi)\,E_{m-1,k}^\sharp(\tau,\xi)\,d\tau. \tag{68}$$

In order to do this, since the kernel $g^\sharp(t,\tau,\xi)$ depends on both t and τ and on the parameter ξ, we look for two positive, continuous functions $q(t)$ and $\phi_N(\tau,\xi)$, with $q(t)$ bounded, such that

$$f_k^\sharp(t,\xi) \le C_{k,N}\,q(t), \quad \text{for any } t \le \theta_{|\xi|} \text{ and } k=1,\dots,m, \tag{69}$$

$$g^\sharp(t,\tau,\xi) \le q(t)\,\phi_N(\tau,\xi), \quad \text{for any } t \le \theta_{|\xi|}, \tag{70}$$

$$\int_0^t q(\tau)\phi_N(\tau,\xi)\,d\tau \le C_N, \quad \text{for any } t \le \theta_{|\xi|}. \tag{71}$$

Indeed, thanks to (69) and (70), from (68) we obtain

$$E_{m-1,k}^\sharp(t,\xi) \le q(t)\left(C_{k,N} + \int_0^t \phi_N(\tau,\xi)\,E_{m-1,k}^\sharp(\tau,\xi)\,d\tau\right)$$

and therefore, using (71), it follows that $E_{m-1,k}^\sharp(t,\xi)$ is bounded by virtue of the following Gronwall-type estimate (which follows as corollary of Theorem 1.5 in [1]).

Lemma 1. *Let $u(t), q(t), \phi(t)$ be continuous, non negative functions in $[0,\infty)$. If*

$$u(t) \le Cq(t) + q(t)\int_0^t b(\sigma)u(\sigma)\,d\sigma, \tag{72}$$

$$q(t)\exp\left(\int_0^t q(\tau)b(\tau)\,d\tau\right) \le C', \tag{73}$$

for some $C, C' > 0$, then $u(t)$ is bounded.

Let us assume that we found $q(t)$ and $\phi_N(\tau,\xi)$ such that (69)–(71) hold, so that $E_{m-1,k}$ is bounded. Then the boundedness of $E_{\ell k}(t,\xi)$ for any $\ell = 1,\dots,m-2$, follows if

$$\frac{\eta(t)^{m-\ell}}{\sqrt{d(t)}}\left(1+\int_0^t \frac{\sqrt{d(\tau)}}{\eta(\tau)^{m-1-\ell}}\,d\tau\right) \le C, \quad \text{for any } \ell = 1,\dots,m-2. \tag{74}$$

Condition (74) immediately follows from

$$f(t) - (m-\ell)\,\frac{\eta'(t)}{\eta(t)} - \epsilon\,\eta(t) \ge_{(a)} 0, \quad \ell = 1,\dots,m-2, \tag{75}$$

which is a consequence of (26) (see Remark 11). On the one hand, by virtue of Remark 2, from (75) it follows that

$$\frac{\eta(t)^{m-\ell}}{\sqrt{d(t)}} \le C\Lambda(t)^{-\epsilon} \to 0 \quad \text{as } t \to \infty.$$

On the other hand, thanks to (75), using the notation in Remark 3 we put

$$a(t) = f(t) - (m-\ell)\,\frac{\eta'(t)}{\eta(t)} - \epsilon\,\eta(t) \ge_{(a)} 0, \qquad A(t) \equiv \exp\left(\int_0^t a(\tau)\,d\tau\right) = \frac{\sqrt{d(t)}}{\eta(t)^{m-\ell}}\,\Lambda(t)^{-\epsilon},$$

so that we get

$$\frac{\eta(t)^{m-\ell}}{\sqrt{d(t)}}\int_0^t \frac{\sqrt{d(\tau)}}{\eta(\tau)^{m-1-\ell}}\,d\tau \le C\Lambda(t)^{-\epsilon}\int_0^t \eta(\tau)\Lambda(\tau)^{\epsilon}\,d\tau \equiv C\Lambda(t)^{-\epsilon}\int_0^t \lambda(\tau)\Lambda(\tau)^{\epsilon-1}\,d\tau \le C\epsilon^{-1}.$$

Moreover, if $E_{m-1,k}$ is bounded as well as $E_{j,k}$ for any $j = 1,\dots,m-2$, then $E_{mk}(t,\xi)$ is bounded too if

$$\frac{\gamma(t)}{\sqrt{d(t)}}\left(1 + \int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\left(\sum_{j=0}^{m-2}\frac{|\beta_j(\sigma,\xi)|}{\eta(\sigma)^{m-1-j}}\right)d\sigma\right) \le C_N, \quad \text{for any } t \le \theta_{|\xi|}. \tag{76}$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. It is clear that the proof of Theorem 1 follows if we construct two functions $q(t)$ and $\phi_N(\tau,\xi)$ which satisfy (69)–(71) and if we prove (76). Let $r \in [0,1)$ be as in (23). We claim that

$$q(t) = C_{m,\delta,N}\,\frac{\lambda(t)}{\sqrt{d(t)}}\,\Lambda(t)^{-r}, \qquad \phi_N(\tau,\xi) := |\xi|^{1-(r+\delta)}\Lambda(\tau)^{-\delta}\sqrt{d(\tau)},$$

verify (69)–(70) for any $\delta \in (0, 1-r)$. Indeed, in such a case condition (71) holds since

$$|\xi|^{1-(r+\delta)}\int_0^t \lambda(\tau)\,\Lambda(\tau)^{-r-\delta}\,d\tau \le \frac{(|\xi|\,\Lambda(t))^{1-(r+\delta)}}{1-(r+\delta)} \le \frac{N^{1-(r+\delta)}}{1-(r+\delta)}.$$

Moreover, the boundedness of $q(t)$ follows from the right-hand side of (23), which implies $\lambda\Lambda^{-r} \lesssim \sqrt{d}$.

Let us prove (69)–(70). Thanks to (19) and (20), we can estimate $\beta_j(t,\xi)$ in (58) by

$$|\beta_j(t,\xi)| \le C\sum_{k=1}^{m-j}(|\xi|\lambda(t))^k\eta(t)^{m-j-k} \le C_1|\xi|\lambda(t)\eta(t)^{m-1-j}, \tag{77}$$

for any $j = 0,\dots,m-2$. By using the left-hand side of (23) we get

$$\lambda(s)\Lambda(s)^{-r}\gamma(s)^{-1}\int_s^t \gamma(\tau)\,d\tau \le C\int_s^t \lambda(\tau)\Lambda(\tau)^{-r}\,d\tau \le \frac{C}{1-r}\Lambda(t)^{1-r}.$$

Since $\Lambda(s)\,|\xi| \le N$ for any $s \le \theta_{|\xi|}$, we can estimate

$$|\xi| \le N^{r+\delta}\,|\xi|^{1-(r+\delta)}\,\Lambda(s)^{-(r+\delta)},$$

so that we obtain

$$\Theta^{\sharp}_{1,j}(t,s,\xi) \le C_N\,|\xi|^{1-(r+\delta)}\,\eta(s)^{m-1-j}\Lambda(s)^{-\delta}\Lambda(t)^{1-r},$$

for any $j = 0,\dots,m-2$. Moreover, we remark that $1 + \Gamma(t) \le C\Lambda(t)^{1-r}$ using again the left-hand side of (23), therefore we can estimate

$$\frac{\eta(t)}{\sqrt{d(t)}}(1+\Gamma(t)) \le C\frac{\lambda(t)}{\sqrt{d(t)}}\,\Lambda(t)^{-r} \le q(t),$$

in (66). Since $\delta > 0$, thanks to (21) we can estimate

$$\int_s^t \eta(\tau)^k\Lambda(\tau)^{-\delta}\,d\tau \le C_{\delta-\delta_1}\eta(s)^{k-1}\Lambda(s)^{-(\delta-\delta_1)}\int_s^t \eta(\tau)\Lambda(\tau)^{-\delta_1}\,d\tau \le \frac{C_{\delta-\delta_1}}{\delta_1}\eta(s)^{k-1}\Lambda(s)^{-\delta}, \tag{78}$$

for any $k \geq 1$ and for any $\delta_1 \in (0, \delta)$. Therefore we can estimate

$$\Theta^\sharp_{\ell,m-1-\ell}(t,\tau,\xi) \leq C_{m,\delta,N}\,|\xi|^{1-(r+\delta)}\,\Lambda(t)^{1-r}\eta(\tau)\Lambda(\tau)^{-\delta},$$

for any $\ell = 1,\ldots,m-1$ and

$$\Theta^\sharp_{\ell,j}(t,0,\xi) \leq C_{m,\delta,N}\,|\xi|^{1-(r+\delta)}\,\Lambda(t)^{1-r} \leq NC_{m,\delta}\Lambda(t)^{\delta},$$

for any $\ell = 2,\ldots,m-1$ and $j = 0,\ldots,m-1-\ell$. We remark that in the estimate above we could use $|\xi|^{1-(r+\delta)} \leq N^{1-(r+\delta)}\Lambda(t)^{r+\delta-1}$ since $1-(r+\delta) \geq 0$.

Conditions (69) and (70) immediately follow.

Now we prove (76). On the one hand, using (27), by virtue of Remark 2, we can estimate the function γ in (64) by $\gamma \lesssim \sqrt{d}$. On the other hand, thanks to (77) and (27), using the notation in Remark 3 we put

$$a(t) = f(t) + b_{m-1}(t) \geq_{(a)} 0, \qquad A(t) \equiv \exp\left(\int_0^t a(\tau)\,d\tau\right) = \frac{\sqrt{d(t)}}{\gamma(t)},$$

so that we get

$$\begin{aligned}\frac{\gamma(t)}{\sqrt{d(t)}}\int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\left(\sum_{j=0}^{m-2}\frac{|\beta_j(\sigma,\xi)|}{\eta(\sigma)^{m-1-j}}\right)d\sigma &\leq C\frac{\gamma(t)}{\sqrt{d(t)}}\int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\,|\xi|\,\lambda(\sigma)\,d\sigma\\ &\leq C_1\int_0^t |\xi|\,\lambda(\sigma)\,d\sigma \leq C_1 N.\end{aligned}$$

This completes the proof of Theorem 1. $\square$

Remark 14. In the same assumptions of Theorem 1, let $q \in \{1,\ldots,m-2\}$ be such that

$$\begin{cases} a_{m-k} \equiv 0 & \text{for any } 2 \leq k \leq q,\\ b_{j,k} \equiv 0 & \text{for any } 1 \leq k \leq q \end{cases} \tag{79}$$

that is, we have no term with a number of x derivatives lesser than or equal to q, but $\partial_t^m u$, $b_{m-1,0}(t)\partial_t^{m-1}u$ and $a_{m-1}(t)\lambda(t)\partial_t^{m-1}\partial_x u$.

Then we can enlarge the range $r \in [0,1)$ in (23) to $r \in [-q,1)$. Indeed, thanks to (79) it follows that $\beta_j \equiv 0$ for any $m-q \leq j \leq m-2$ and we can refine estimate (77) to the following:

$$\begin{aligned}|\beta_j(t,\xi)| &\leq C\sum_{\ell=q+1}^{m-j}(|\xi|\lambda(t))^{\ell}\eta(t)^{m-j-\ell} \leq (|\xi|\lambda(t))^{q+1}\eta(t)^{m-1-(j+q)}\\ &= |\xi|^{q+1}\,\lambda(t)\Lambda(t)^q\eta(t)^{m-1-j} \leq N^{q-r}\,|\xi|^{1-r}\,\lambda(t)\Lambda(t)^{-r}\eta(t)^{m-1-j},\end{aligned} \tag{80}$$

for any $j = 0,\ldots,m-(q+1)$. We can now follow the proof of Theorem 1.

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Appendix. The complete *m*-th order equation

In (1) we considered an equation with no terms with zero derivatives in x but $\partial_t^m u$ and $b_{m-1,0}(t)\partial_t^{m-1}u$. In order to consider the Cauchy problem for the complete m-th order equation

$$\begin{cases}\partial_t^m u - \displaystyle\sum_{j=0}^{m-1} a_j(t)\lambda(t)^{m-j}\partial_t^j\partial_x^{m-j}u + \sum_{j+k\leq m-1} b_{j,k}(t)\lambda(t)^k\partial_t^j\partial_x^k u = 0,\\ \partial_t^j u(0,x) = u_j(x), \qquad j = 0,\ldots,m-1,\end{cases} \tag{A.1}$$

we should manage very carefully the new terms, since their influence is not easily controlled in $Z_{\mathrm{pd}}(N)$ (see (77) in Section 6). In particular, here we state a rough condition which allows to exclude any influence coming from them to the energy estimate. Nevertheless, it is reasonable to expect that in some cases the λ-energy in (3) should be modified to obtain good results (for instance, if we consider the Klein–Gordon equation).

If we consider the Cauchy problem (A.1), then we can follow the proof of Theorem 1, but we should replace (77) with the following:

$$|\beta_j(t,\xi)| \leq C|\xi|\lambda(t)\eta(t)^{m-1-j} + |b_{j,0}(t)|. \tag{A.2}$$

We now look for three functions $q(t)$, $\phi_N(\tau,\xi)$, $\psi(\tau)$, with $q(t)$ bounded, such that (69) and (71) hold together with

$$g^\sharp(t,\tau,\xi) \leq q(t)\ (\phi_N(\tau,\xi) + \psi(\tau)), \quad \text{for any } t \leq \theta_{|\xi|}, \tag{A.3}$$

$$q(t)\exp\left(\int_0^t q(\tau)\psi(\tau)\,d\tau\right) \leq C, \quad \text{for any } t \leq \theta_{|\xi|}. \tag{A.4}$$

In fact, thanks to (69)–(A.3) and (71)–(A.4), using again Lemma 1 in Section 6, the boundedness of $E^\sharp_{m-1,k}(t,\xi)$ would follow from (68). We remark that $q(t)$ and $\psi(\tau)$ are independent of the parameter N, as well as the constant $C > 0$ in (A.4).

If we prove that $E^\sharp_{m-1,k}(t,\xi)$ is bounded, then the boundedness of $E^\sharp_{\ell,k}(t,\xi)$ still follows from (26) for any $\ell = 1,\dots,m-2$, whereas (76) holds if we replace (27) with the stronger condition:

$$f(t) + b_{m-1,0}(t) \geq_{(a)} \epsilon\,\eta(t). \tag{A.5}$$

Indeed, thanks to (20) and (A.2), it is sufficient to estimate

$$\frac{\gamma(t)}{\sqrt{d(t)}}\left(1 + \int_0^t \frac{\sqrt{d(\sigma)}}{\gamma(\sigma)}\,\eta(\sigma)\,d\sigma\right) \leq C. \tag{A.6}$$

In order to construct the new functions $q(t)$, $\phi_N(\tau,\xi)$, $\psi(\tau)$ we should take into account the influence of $|b_{j,0}(t)|$ in (A.2) on the functions $\Theta^\sharp_{\ell,j}(t,s,\xi)$. Moreover, we should consider all constants in our estimates since, in general, the product $q(\tau)\psi(\tau)$ in (A.4) would be not integrable. We will compensate the possible increasing behavior of the exponential in (A.4) with the decreasing behavior of the function $q(t)$.

Let us assume that $\eta(t)$ is monotonic or, more in general (see Remark 7) that

$$\limsup_{t\to\infty} \frac{\eta'(t)}{\eta(t)^2} \leq 0. \tag{A.7}$$

Let $p > 0$ be such that

$$p \geq -\liminf_{t\to\infty}\left(\frac{\eta'(t) + b_{m-1,0}(t)\eta(t)}{\eta(t)^2}\right). \tag{A.8}$$

Let $M_j \geq 0$ be defined by

$$M_j := \limsup_{t\to\infty} \frac{|b_{j,0}(t)|}{\eta(t)^{m-j}}. \tag{A.9}$$

Thanks to Hypothesis 4, p and M_j are finite numbers. Conditions (A.7)–(A.9) imply that for any $\epsilon > 0$ there exists $T = T(\epsilon) \geq 0$ such that

$$\frac{\eta'(t)}{\eta(t)} \leq \epsilon\eta(t), \qquad \frac{\eta'(t)}{\eta(t)} + (p+\epsilon)\eta(t) + b_{m-1,0}(t) \geq 0, \qquad |b_{j,0}(t)| \leq (M_j+\epsilon)\,\eta(t)^{m-j},$$

for any $t \in [T,\infty)$. We remark that if (A.8) holds for some $p \in (0,1)$ then the left-hand side of (23) holds for any $r \in [0, 1-p)$. Otherwise, we assume that the left-hand side of (23) holds for $r = 0$, that is,

$$\frac{\eta'(t)}{\eta(t)} + \eta(t) + b_{m-1,0}(t) \geq_{(a)} 0. \tag{A.10}$$

In both cases, in the following it will be $p \geq 1 - r$.

The subzone $Z_{\mathrm{pd}}(N) \cap \{0 \leq t \leq T\}$ is compact, therefore we can assume with no loss of generality $t \geq T$ in $Z_{\mathrm{pd}}(N)$. We can now estimate

$$\eta(s)\Lambda(s)^{p+\epsilon}\gamma(s)^{-1}\int_s^t \gamma(\tau)\,d\tau \leq \int_s^t \eta(\tau)\Lambda(\tau)^{p+\epsilon}\,d\tau \leq \frac{1}{p+\epsilon}\Lambda(t)^{p+\epsilon},$$

for any $t \geq s \geq T$, that is,

$$\Theta^\sharp_{1,j}(t,s,\xi) \leq C_N\,|\xi|^{1-(r+\delta)}\,\eta(s)^{m-1-j}\Lambda(s)^{-\delta}\Lambda(t)^{1-r} + \frac{M_j+\epsilon}{p+\epsilon}\,\eta(s)^{m-1-j}\Lambda(s)^{-(p+\epsilon)}\Lambda(t)^{p+\epsilon}.$$

Thanks to (A.7) we can estimate

$$\int_s^t \eta(\tau)^k \Lambda(\tau)^{-(p+\epsilon)}\,d\tau \le \eta(s)^{k-1}\Lambda(s)^{-\epsilon}\int_s^t \eta(\tau)\Lambda(\tau)^{-p}\,d\tau \le \frac{1}{p}\eta(s)^{k-1}\Lambda(s)^{-(p+\epsilon)},$$

for any $k \ge 1$, therefore

$$\Theta^{\sharp}_{\ell,m-1-\ell}(t,\tau,\xi) \le C_{m,\delta,N}\,|\xi|^{1-(r+\delta)}\,\Lambda(t)^{1-r}\eta(\tau)\Lambda(\tau)^{-\delta} + \frac{M_{m-1-\ell}+\epsilon}{p^{\ell}}\,\eta(\tau)\Lambda(\tau)^{-(p+\epsilon)}\Lambda(t)^{p+\epsilon},$$

for any $\ell = 1,\dots,m-2$, and

$$\Theta^{\sharp}_{\ell,j}(t,0,\xi) \le NC_{m,\delta}\Lambda(t)^{\delta} + C_{p,m}\Lambda(t)^{p+\epsilon},$$

for any $\ell = 2,\dots,m-1$ and $j = 0,\dots,m-1-\ell$. Since $p \ge 1-r$, if we define

$$M_p := \sum_{\ell=1}^{m-2}\frac{M_{m-1-\ell}}{p^{\ell}},$$

then we have proved that we can estimate

$$g^{\sharp}(t,\tau,\xi) \le \frac{\eta(t)}{\sqrt{d(t)}}\sqrt{d(\tau)}\,\Lambda(t)^{p+\epsilon}\,\left(C_{m,\delta,N}\,|\xi|^{1-(r+\delta)}\,\Lambda(\tau)^{-(p+\epsilon+r+\delta-1)} + (M_p+\epsilon_1)\Lambda(\tau)^{-(p+\epsilon)}\right),$$

for any $\epsilon_1 > 0$. Therefore, if we take

$$q(t) = \frac{\eta(t)}{\sqrt{d(t)}}\,\Lambda(t)^{p+\epsilon},$$
$$\psi(\tau) = (M_p+\epsilon_1)\sqrt{d(\tau)}\,\Lambda(\tau)^{-(p+\epsilon)},$$
$$\phi_N(\tau,\xi) = C_{m,\delta,N}\,\sqrt{d(\tau)}\,|\xi|^{1-(r+\delta)}\,\Lambda(\tau)^{-(p+\epsilon+r+\delta-1)},$$

then conditions (A.3), (69) and (71) are satisfied. Moreover, $q(t)$ is bounded and condition (A.4) holds if we replace the right-hand side of (23) with the stronger condition

$$f(t) \ge_{(a)} \frac{\eta'(t)}{\eta(t)} + (p+M_p+\varepsilon)\eta(t) \tag{A.11}$$

for some $\varepsilon > 0$. Indeed, it holds:

$$q(t)\exp\left(\int_T^t q(\tau)\,\psi(\tau)\,d\tau\right) \le \frac{\eta(t)}{\sqrt{d(t)}}\,\Lambda(t)^{p+\epsilon}\exp\left((M_p+\epsilon_1)\int_T^t \eta(\tau)\,d\tau\right) \le \frac{\eta(t)}{\sqrt{d(t)}}\,\Lambda(t)^{p+M_p+\epsilon+\epsilon_1}.$$

We remark that (A.5) follows from (A.8) and (A.11).

Summarizing we have the following.

Theorem 2. *Let us assume Hypotheses 1–4 together with (A.7). Let us assume (A.10) and let $\overline{M}$ be the minimum of the sum $p+M_p$ over the parameters $p > 0$ which satisfy (A.8). Let $f : [0,\infty) \to \mathbb{R}$ be a function with constant sign which satisfies (22). If $f(t)$ satisfies*

$$f(t) \ge_{(a)} \frac{\eta'(t)}{\eta(t)} + (\overline{M}+\epsilon)\eta(t), \tag{A.12}$$

for some $\epsilon > 0$, then the solution to (A.1) satisfies the energy estimate (24).

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