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On Complementary Extremum Principles

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Important complementary extremum principles are generated without recourse to general variational theory. The results are illustrated by an application to a class of boundary value problems in Magnetohydrodynamics.

The early work on complementary variational principles is due to Noble [1]. The method is concerned with the construction of upper and lower bounds for the solution of variational problems. The technique has been subsequently developed, in an abstract form, by Rall [2] and especially Arthurs ([3–7], for example). The latter author has given many interesting physical applications. In [3], general dual extremum principles are established for linear boundary value problems by use of the general canonical theory of variational calculus. Here, the results are established in a new direct manner. As an illustration, application is made to magnetohydrodynamic channel flow.

It is noted that a valuable account of dual extremum principles and their diversity of application is given by Noble and Sewell [8].

THE EXTREMUM PRINCIPLES

Consider the linear boundary value problem defined by

$$A\phi = f \qquad \text{in} \qquad V, \tag{1}$$

$$\sigma_T(\phi - \phi_B) = 0 \qquad \text{on} \qquad \partial V, \tag{2}$$

$$A = T^*T + Q, \tag{3}$$

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where $T: H_\phi \to H_u$ and its adjoint $T^*: H_u \to H_\phi$ are, in turn, linear operators on the real Hilbert spaces H_ϕ and H_u with inner products $\langle\ \rangle$ and $(\ )$, respectively, and are such that

$$(u, T\phi) = \langle T^*u, \phi\rangle + [u, \sigma_T\phi], \qquad \forall\phi \in D_T, \quad u \in D_{T^*}. \tag{4}$$

Here, $\sigma_T: H_\phi \to H_u$, while $[u, \sigma_T\phi]$ denotes boundary terms. Further, $Q: H_\phi \to H_\phi$ is a symmetric positive operator on D_Q; that is,

$$\langle\phi_1, Q\phi_2\rangle = \langle Q\phi_1, \phi_2\rangle, \qquad \phi_1, \phi_2 \in D_Q, \tag{5}$$

$$\langle\phi, Q\phi\rangle \geqslant 0, \qquad \phi \in D_Q. \tag{6}$$

Finally, $f \in H_\phi$ is specified while ϕ_B is a prescribed function on the boundary ∂V of the region V. D_A is dense in H_ϕ.

The complementary extremum principles state that

$$G(T\Psi) \leqslant I(\phi) \leqslant J(\Phi), \tag{7}$$

where ϕ is the exact solution of the boundary value problem defined by (1)–(3) and the functionals $G(T\Psi)$, $I(\phi)$, $J(\Phi)$ are given, in turn, by

$$\begin{aligned} G(T\Psi) &= -\tfrac{1}{2}(T\Psi, T\Psi) - \tfrac{1}{2}\langle Q\Psi_1, \Psi_1\rangle + [T\Psi, \sigma_T\phi_B], \\ &\qquad\qquad Q \neq 0 \quad (Q\Psi_1 = f - T^*T\Psi, \Psi \in D_T), \\ &= -\tfrac{1}{2}(T\Psi, T\Psi) + [T\Psi, \sigma_T\phi_B], \\ &\qquad\qquad Q = 0 \quad (\Psi \in \{\Psi: T^*T\Psi = f \text{ in } V\}), \end{aligned} \tag{8}$$

$$I(\phi) = -\tfrac{1}{2}\langle f, \phi\rangle + \tfrac{1}{2}[T\phi, \sigma_T\phi_B], \tag{9}$$

$$\begin{aligned} J(\Phi) &= \tfrac{1}{2}(T\Phi, T\Phi) + \tfrac{1}{2}\langle\Phi, Q\Phi\rangle - \langle f, \Phi\rangle - [T\Phi, \sigma_T(\Phi - \phi_B)] \\ &\qquad ([T(\Phi - \phi), \sigma_T(\Phi - \Phi_B)] \leqslant 0, \Phi \in D_A). \end{aligned} \tag{10}$$

Proof. (a) $I(\phi) \leqslant J(\Phi)$. It is given that $\Phi \in D_A$ and

$$[T(\Phi - \phi), \sigma_T(\Phi - \phi_B)] \leqslant 0. \tag{11}$$

Now,

$$\begin{aligned} 0 &\leqslant [T(\Phi - \phi), T(\Phi - \phi)] \\ &= (T\Phi, T\Phi) - 2(T\phi, T\Phi) + (T\phi, T\phi) \\ &= (T\Phi, T\Phi) - 2\langle T^*T\phi, \Phi\rangle - 2[T\phi, \sigma_T\Phi] + \langle T^*T\phi, \phi\rangle + [T\phi, \sigma_T\phi_B] \\ &\qquad\qquad \text{(using (2) and (4))} \\ &= (T\Phi, T\Phi) - 2\langle f, \Phi\rangle + 2\langle Q\phi, \Phi\rangle + \langle f, \phi\rangle - \langle Q\phi, \phi\rangle - [T\phi, \sigma_T\phi_B] \\ &\quad + 2\{[T\phi, \sigma_T\phi_B] - [T\phi, \sigma_T\Phi]\} \qquad \text{(using (1) and (3))} \\ &= \{(T\Phi, T\Phi) + \langle\Phi, Q\Phi\rangle - 2\langle f, \Phi\rangle - 2[T\Phi, \sigma_T(\Phi - \phi_B)] \\ &\quad + \langle f, \phi\rangle - [T\phi, \sigma_T\phi_B]\} + 2\langle Q\phi, \Phi\rangle - \langle\Phi, Q\Phi\rangle - \langle Q\phi, \phi\rangle \\ &\quad + 2\{[T\Phi, \sigma_T(\Phi - \phi_B)] + [T\phi, \sigma_T\phi_B] - [T\phi, \sigma_T\Phi]\}. \end{aligned} \tag{12}$$

But, from (5) and (6) it is seen that

$$2\langle Q\phi, \Phi\rangle - \langle \Phi, Q\Phi\rangle - \langle Q\phi, \phi\rangle = -\langle Q(\Phi - \phi), \Phi - \phi\rangle \leqslant 0. \quad (13)$$

Further,

$$[T\Phi, \sigma_T(\Phi - \phi_B)] + [T\phi, \sigma_T\phi_B] - [T\phi, \sigma_T\phi] = [T(\Phi - \phi), \sigma_T(\Phi - \phi_B)]. \quad (14)$$

Use of (13) and (14) in (12) shows that

$$\begin{aligned} &-\tfrac{1}{2}\langle f, \phi\rangle + \tfrac{1}{2}[T\phi, \sigma_T\phi_B] \\ &\quad\leqslant \tfrac{1}{2}(T\Phi, T\Phi) + \tfrac{1}{2}\langle \Phi, Q\Phi\rangle - \langle f, \Phi\rangle - [T\Phi, \sigma_T(\Phi - \phi_B)] \\ &\qquad - \langle Q(\Phi - \phi), \Phi - \phi\rangle + [T(\Phi - \phi), \sigma_T(\Phi - \phi_B)]. \end{aligned} \quad (15)$$

In view of (11) and (13), relation (15) implies the complementary variational principle $I(\phi) \leqslant J(\Phi)$.

(b) $G(T\Psi) \leqslant I(\phi)$.

(i) $Q \neq 0$. Now,

$$\begin{aligned} 0 &\leqslant (T(\Psi - \phi), T(\Psi - \phi)) \\ &= (T\Psi, T\Psi) - 2(T\Psi, T\phi) + (T\phi, T\phi) \\ &= (T\Psi, T\Psi) - 2\{\langle T^*T\Psi, \phi\rangle + [T\Psi, \sigma_T\phi_B]\} + \langle T^*T\phi, \phi\rangle + [T\phi, \sigma_T\phi_B] \\ &\qquad\qquad \text{(using (2) and (4)).} \end{aligned} \quad (16)$$

But,

$$Q\Psi_1 = f - T^*T\Psi, \qquad \Psi \in D_T,$$

so that, from (16), (1), and (3),

$$\begin{aligned} 0 &\leqslant (T\Psi, T\Psi) - \langle f, \phi\rangle + [T\phi, \sigma_T\phi_B] - \langle Q\phi, \phi\rangle + 2\langle Q\Psi_1, \phi\rangle - 2[T\Psi, \sigma_T\phi_B] \\ &= \{(T\Psi, T\Psi) + \langle Q\Psi_1, \Psi_1\rangle - 2[T\Psi, \sigma_T\phi_B] - \langle f, \phi\rangle + [T\phi, \sigma_T\phi_B]\} \\ &\quad - \langle Q(\Psi_1 - \phi), \Psi_1 - \phi\rangle \qquad \text{(using (5))} \\ &\leqslant \{(T\Psi, T\Psi) + \langle Q\Psi_1, \Psi_1\rangle - 2[T\Psi, \sigma_T\phi_B] - \langle f, \phi\rangle + [T\phi, \sigma_T\phi_B]\}. \end{aligned}$$

Hence, $G(T\Psi) \leqslant I(\phi)$, $Q \neq 0$.

(ii) $Q = 0$. Relation (16) is derived as above. But now,

$$T^*T\Psi = f \qquad \text{in} \qquad V$$

so that

$$0 \leqslant (T\Psi, T\Psi) - 2[T\Psi, \sigma_T\phi_B] - \langle f, \phi\rangle + [T\phi, \sigma_T\phi_B]$$

and the result $G(T\Psi) \leqslant I(\phi)$, $Q = 0$, follows.

Magnetohydrodynamic Channel Flow

Extremum principles for magnetohydrodynamic channel flow problems have been discussed by Wenger [9], Smith [10, 11] and Sloan [12]. Here the use of the above formulation is illustrated in the context of such a problem.

The steady flow of a viscous, incompressible electrically conducting fluid in an insulated cylindrical pipe with cross-sectional area A and boundary ∂A is considered. The X, Y-plane is normal to the axis of the channel. There is a uniform pressure gradient K in the Z-direction and an applied magnetic field H_0 in the X-direction. The governing equations are [13]

$$\nabla^2 W + M\frac{\partial B}{\partial x} = -1, \tag{17}$$

$$\nabla^2 B + M\frac{\partial W}{\partial x} = 0, \tag{18}$$

$$W = B = 0 \qquad \text{on} \qquad \partial A, \tag{19}$$

where dimensionless variables and parameters have been introduced according to

$$\begin{gathered} W = \nu\rho W_z/[\alpha^2 K], \qquad B = H_z(\nu\rho/\sigma)^{1/2}/[\alpha^2 K], \\ (X, Y) = a(x, y), \\ M = \mu H_0 \alpha(\sigma/\nu\rho)^{1/2}, \end{gathered} \tag{20}$$

where W_z is the fluid velocity, H_z is the induced axial magnetic field, α is a representative length in the cross section of the pipe, and M is the Hartmann number. Further, ρ is the density, ν is the kinematic viscosity, μ is the magnetic permeability, and σ is the electrical conductivity of the fluid.

Equations (17), (18) may be written in the operator form

$$[T^*T + Q]\,\boldsymbol{\phi} = \mathbf{f} \qquad \text{in} \qquad A, \tag{21}$$

where

$$T = \begin{bmatrix} \text{grad} & 0 \\ 0 & 0 \end{bmatrix}, \qquad T^* = \begin{bmatrix} -\text{div} & 0 \\ 0 & 0 \end{bmatrix}, \tag{22), (23}$$

$$\boldsymbol{\phi} = \begin{bmatrix} w \\ B \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{24), (25}$$

$$Q = \begin{bmatrix} 0 & -M\dfrac{\partial}{\partial x} \\ M\dfrac{\partial}{\partial x} & \nabla^2 \end{bmatrix}, \tag{26}$$

while the boundary conditions become

$$\boldsymbol{\phi} = \mathbf{0} \qquad \text{on} \qquad \partial A. \tag{27}$$

Here, ϕ is treated as an element of the real vector Hilbert space H_φ with inner product defined by

$$\langle \phi, \Psi \rangle = \int_A (\phi^T \cdot \Psi)\, dA, \tag{28}$$

where ϕ^T denotes the transpose of ϕ. It is seen that

$$T: H_\phi \to \underline{H_\phi} \times \underline{H_\phi}\,, \qquad T^*: \underline{H_\phi} \times \underline{H_\phi} \to H_\phi\,, \qquad Q: H_\phi \to H_\phi\,. \tag{29}$$

The inner product of two elements $\underline{\phi}$, $\underline{\Psi} \in H_\phi \times H_\phi$ is defined by

$$(\underline{\phi}, \underline{\Psi}) = \int_A (\phi_1 \Psi_1 + \phi_2 \Psi_2)\, dA, \tag{30}$$

where

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \qquad \underline{\Psi} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \tag{31), (32}$$

Thus, if

$$\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix},$$

then

$$(\underline{\mathbf{u}}, T\phi) = \left(\underline{\mathbf{u}}, \begin{bmatrix} \operatorname{grad} w \\ 0 \end{bmatrix}\right) = \int_A \mathbf{u}_1 \operatorname{grad} w\, dA, \tag{33}$$

$$\langle T^* \underline{\mathbf{u}}, \phi \rangle = \left\langle \begin{bmatrix} -\operatorname{div} \mathbf{u}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} w \\ B \end{bmatrix} \right\rangle = -\int_A w \operatorname{div} \mathbf{u}_1\, dA, \tag{34}$$

whence, by Green's theorem in the plane,

$$(\underline{\mathbf{u}}, T\phi) = \langle T^* \underline{\mathbf{u}}, \phi \rangle + [\underline{\mathbf{u}}, \sigma_T \phi], \tag{35}$$

where the conjoint of $\underline{\mathbf{u}}$ and ϕ is given by

$$[\underline{\mathbf{u}}, \sigma_T \phi] = \oint_{\partial A} w\{-u_{12}\, dx + u_{11}\, dy\}, \tag{36}$$

where

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}.$$

The domain D_Q is taken as the collection of elements in H_ϕ which satisfy (18), possess the required derivatives in $A \cup \partial A$, and satisfy $B = 0$ on ∂A. It is assumed throughout that A and ∂A are of such a type as to permit the use of Green's theorem in the plane.

If

$$\phi_i = \begin{bmatrix} w_i \\ B_i \end{bmatrix} \in D_Q\,, \qquad i = 1, 2,$$

then

$$\begin{aligned}
&\langle \boldsymbol{\phi}_1, Q\boldsymbol{\phi}_2\rangle - \langle Q\boldsymbol{\phi}_1, \boldsymbol{\phi}_2\rangle \\
&= \left\langle \begin{bmatrix} w_1 \\ B_1 \end{bmatrix}, \begin{bmatrix} -M\dfrac{\partial B_2}{\partial x} \\ M\dfrac{\partial w_2}{\partial x} + \nabla^2 B_2 \end{bmatrix} \right\rangle - \left\langle \begin{bmatrix} -M\dfrac{\partial B_1}{\partial x} \\ M\dfrac{\partial w_1}{\partial x} + \nabla^2 B_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ B_2 \end{bmatrix} \right\rangle \\
&= \int_A \Big\{ -Mw_1 \frac{\partial B_2}{\partial x} + MB_1 \frac{\partial w_1}{\partial x} + B_1 \nabla^2 B_2 \\
&\quad + Mw_2 \frac{\partial B_1}{\partial x} - MB_2 \frac{\partial w_1}{\partial x} - B_2 \nabla^2 B_1 \Big\}\, dA;
\end{aligned}$$

that is, from (18),

$$\begin{aligned}
&\langle \boldsymbol{\phi}_1, Q\boldsymbol{\phi}_2\rangle - \langle Q\boldsymbol{\phi}_1, \boldsymbol{\phi}_2\rangle \\
&= \int_A \left[-Mw_1 \frac{\partial B_2}{\partial x} + Mw_2 \frac{\partial B_1}{\partial x} \right] dA \\
&= \int_A \left[-M \frac{\partial}{\partial x}(B_2 w_1) + MB_2 \frac{\partial w_1}{\partial x} + M \frac{\partial}{\partial x}(B_1 w_2) - MB_1 \frac{\partial w_2}{\partial x} \right] dA \\
&= M \left[\oint_{\partial A} [B_1 w_2 - B_2 w_1]\, dy + \int_A [-B_2 \nabla^2 B_1 + B_1 \nabla^2 B_2]\, dA \right] \\
&= M \Big\{ \oint_{\partial A} \Big[\Big(B_1 w_2 - B_2 w_1 - B_2 \frac{\partial B_1}{\partial x} + B_1 \frac{\partial B_2}{\partial x} \Big]\, dy \\
&\quad + \Big[B_2 \frac{\partial B_1}{\partial y} - B_1 \frac{\partial B_2}{\partial y} \Big)\, dx \Big] \Big\} \\
&= 0,
\end{aligned}$$

since $B_1 = B_2 = 0$ on ∂A. Thus, Q is a symmetric operator on D_Q. Further, in view of (18),

$$\begin{aligned}
\langle \boldsymbol{\phi}, Q\boldsymbol{\phi}\rangle &= \int_A \Big\{ -Mw \frac{\partial B}{\partial x} + MB \frac{\partial w}{\partial x} + B\nabla^2 B \Big\}\, dA \\
&= \int_A \Big\{ -Mw \frac{\partial B}{\partial x} \Big\}\, dA \\
&= -\int_{\partial A} MBw\, dy - \int_A \Big\{ \frac{\partial}{\partial x}\Big(B \frac{\partial B}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(B \frac{\partial B}{\partial y}\Big) \\
&\quad - \Big(\frac{\partial B}{\partial x}\Big)^2 - \Big(\frac{\partial B}{\partial y}\Big)^2 \Big\}\, dA \\
&= \oint_{\partial A} \Big[\Big\{ -MBw - B \frac{\partial B}{\partial x} \Big\}\, dy + B \frac{\partial B}{\partial y}\, dx \Big] + \int_A (\nabla B)^2\, dA \\
&= \int_A (\nabla B)^2\, dA \geqslant 0,
\end{aligned}$$

since $B = 0$ on ∂A, $\boldsymbol{\phi} \in D_Q$. Hence, Q is a positive operator on D_Q.

Result (7) may now be used to give

$$\left\{\int_A [2w_1 - (\nabla B_1)^2 - (\nabla w_1)^2]\, dA + 2\oint_{\partial A}\left[w_1 \frac{\partial w_1}{\partial x}\, dy - w_1 \frac{\partial w_1}{\partial y}\, dx\right]\right\} \leqslant \int_A w\, dA \leqslant \int_A [(\nabla B_2)^2 + \mathbf{U}_1 \cdot \mathbf{U}_1 + \mathbf{U}_2 \cdot \mathbf{U}_2]\, dA, \tag{37}$$

where

$$\underline{\mathbf{U}} = \begin{bmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{bmatrix} \in H_\Phi \times H_\Phi, \qquad \begin{bmatrix}w_i\\ B_i\end{bmatrix} \in D_Q, \qquad i = 1, 2,$$

and B_2, $\mathbf{U}_1$ are related according to

$$M \frac{\partial B_2}{\partial x} = -\{1 + \operatorname{div} \mathbf{U}_1\}. \tag{38}$$

The sharpest upper bound is obtained by taking $\mathbf{U}_2 = \mathbf{0}$. Thus, upper and lower bounds have been generated for the efflux of the conducting fluid through the insulated channel.

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