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Uniform asymptotics of some Abel sums arising in coding theory

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Abstract

We derive uniform asymptotic expressions of some Abel sums appearing in some problems in coding theory and indicate the usefulness of these sums in other fields, like empirical processes, machine maintenance, analysis of algorithms, probabilistic number theory, queuing models, etc.

Keywords: Abel sums; Coding theory; Mellin transforms; W-function; Uniform asymptotics

1. Introduction

The following sums, recently studied by Szpankowski [28], appear in a number of applications:

$$S_{n,k} = \sum_{0 \leqslant j \leqslant n-k} \binom{n-k}{j} \left(\frac{j}{n}\right)^j \left(1 - \frac{j}{n}\right)^{n-j} \qquad (n \geqslant 0, 0 \leqslant k \leqslant n), \tag{1}$$

where 0^0 is interpreted as 1. (Note that our $S_{n,k}$ differs from his by 1.) For obvious reasons, sums of type (1) will be referred to as an *Abel sum*.

When $k = 0$, $S_{n,0} - 1$ is the so-called Ramanujan Q-function (cf. [1])

$$S_{n,0} - 1 = Q(n) = \sum_{1 \leqslant j \leqslant n} \frac{n!}{(n-j)! n^j} \tag{2}$$

(cf. [28] or [5]). A general form of this identity for $S_{n,k}$, $k \geqslant 1$, is given in (12).

The $Q(n)$ function was encountered in a number of problems in the analysis of algorithms and combinatorial probability: hashing schemes [18, 31], random mappings [22, 11], union-find algorithms [21], optimum caching [19], deadlock in multiprocessing systems [3], the birthday paradox [7, 8], and pseudo-random sequence [23].

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We describe yet another one in empirical process: $n/2+Q(n)/2$ is the expected value of the index j for which the maximum in

$$D_n^+ := \max_{1\leqslant j\leqslant n}\left(\frac{j}{n}-U_j\right) =: \frac{j^*}{n}-U_{j^*} \tag{3}$$

is reached, where $U_1 < U_2 < \cdots < U_n$ is an *ordered* sample of a random variable with uniform distribution in $(0,1)$; (see [27, Chapter 9]). Note that by conjugacy, the distribution of j^* is identical to that of the number of nonnegative elements among $\{j/n - U_j\}_{1\leqslant j\leqslant n}$; see, for instance, [30, p. 373].

For general k, $S_{n,k}-1$ was used to estimate the average worst-case probability of undetected error (over all systematic q-ary $[n,k]$ codes); see [24, 17, 28].

Szpankowski showed that, for $1\leqslant k = \mathrm{O}(1)$,

$$S_{n,k} = \frac{1}{4^k}\binom{2k}{k}\sqrt{\frac{\pi n}{2}} + \frac{1}{3} + \mathrm{O}(n^{-1/2}) \tag{4}$$

as $n\to\infty$. His approach proceeds along generating functions and the singularity analysis of Flajolet and Odlyzko [11] using an inductive argument.

In this paper, we give two approximate expressions for $S_{n,k}$ which completely characterize the asymptotic behaviors of $S_{n,k}$ for $1\leqslant k\leqslant n$, as $n\to\infty$. The first expression extends the domain of validity of (4) to $1\leqslant k = \mathrm{o}(n^{1/4})$. Our proof follows a similar line of generating functions but with an appeal to Mellin transforms (cf. [9]). This approach is computationally simpler. We then propose another uniform asymptotic expression for $S_{n,k}$ for $k\to\infty$ and $k\leqslant n$, as $n\to\infty$ using an elementary argument.

It should be noted that uniform asymptotic expressions are especially useful for practical purposes since in reality it is not obvious if the second parameter k is, say, $\mathrm{O}(\log n)$ or $\mathrm{O}(n^{1/100})$.

Our general approach is also useful for uniform asymptotics of the following partial Abel sum

$$P_{n,k} := \frac{1}{n}\sum_{1\leqslant j\leqslant k}\frac{1}{j}\binom{n}{j-1}\left(\frac{j}{n}\right)^{j-1}\left(1-\frac{j}{n}\right)^{n-j} \quad (1\leqslant k\leqslant n), \tag{5}$$

which is the probability distribution of j^* in (3):

$$P_{n,k} = \mathrm{P}(j^* = k).$$

The Lambert W-function, or more precisely, the tree function $T(z) = -W(-z)$, plays a central rôle in our discussions:

$$T(z) = \sum_{j\geqslant 1}\frac{j^{j-1}}{j!}z^j \quad (|z|\leqslant \mathrm{e}^{-1}).$$

The function T satisfies $T(z) = z\mathrm{e}^{T(z)}$ and admits analytic continuation into the whole cut-plane $\mathbb{C}\backslash[\mathrm{e}^{-1},\infty)$. Properties together with a large number of applications of the

W-function were recently surveyed by Corless et al. [5]. Some other applications were stated in [10]. We add some other ones:

(1) Asymptotics of the Dickman function in the study of the distribution of integers free from large prime factors; see [15] for a survey of the subject. We note that the Dickman function also appears in the distribution of the largest cycle in random permutations and the largest degree of an irreducible factor in a random polynomial over a finite field; see [2, 14].

(2) Asymptotics of the coefficients of $1/\Gamma(z)$; see [6].

(3) The Borel distribution in probability theory is defined by

$$\mathrm{P}(X = j) = \frac{j^{j-1}}{j!}\,\mathrm{e}^{-j} \quad (j = 1, 2, \ldots).$$

Note that $\sum_{j\geqslant 1} \mathrm{P}(X = j) = 1$ or $T(\mathrm{e}^{-1}) = 1$. This distribution, together with its generalizations by Tanner, proved useful in many problems in queuing models, branching process, empirical process, etc., see [4] for a detailed account.

(4) Naor's distribution (cf. [16, p. 447]) is defined by

$$\mathrm{P}(X_n = j) = \frac{(n-1)!}{j!\,n^{n-j}}\,(n - j) \quad (j = 0, 1, \ldots, n - 1),$$

a distribution arising in some machine interference problems [1] (cf. [25]). One can also devise an urn-model interpretation of this law (cf. [16]) which in turn has applications to algorithmic analysis of some problems in the theory of markets; see [13]. The mean of X_n is easily seen to be $n - Q(n)$ and the variance $2n - Q^2(n) - Q(n)$, etc.

For completeness, we mention that the following sums:

$$D_n^*(m) := \frac{n!}{n^n}[z^n](B(z)(B(z) - 1)^{m-1}), \quad D_n(m) := \frac{n!}{n^n}[z^n]\frac{B(z)(B^m(z) - 1)}{B(z) - 1}$$

for $1 \leqslant m \leqslant n$, appear as solutions of some recurrences in multi-alphabet universal coding (cf. [29]), where $B(z) = 1/(1 - T(z))$. Uniform asymptotics of these sums can be obtained by appropriate application of the saddle-point method using the more convenient expressions (by Lagrange inversion formula)

$$D_n^*(m) = \frac{n!}{n^n}[z^{n-m+1}]\frac{\mathrm{e}^{nz}}{(1 - z)^{m-1}}, \quad D_n(m) = \frac{1}{n+1} + \frac{n!}{n^n}[z^n]\frac{\mathrm{e}^{nz}}{(1 - z)^{m-1}}.$$

Notation. Throughout this paper, the symbol $[z^n]f(z)$ represents the coefficient of z^n in the Taylor expansion of f. Following a number-theoretic convention, the Vinogradov symbol $\ll$ is used as a synonym of Landau's O(.) symbol. The symbols ε and ε' always denote arbitrarily small but fixed quantities whose values may vary from one

[1] The normal approximation in [25] is actually a Rayleigh distribution, as corrected by Salia and Shashiashvili [26]. See also [22, p. 153] for the number of cyclic points in a random mapping, which is nothing but $n - X_n$.

occurrence to another. All limits, including O, o, $\sim$, $\asymp$ and $\ll$, whenever unspecified, will be taken as $n \to \infty$.

2. Uniform asymptotics of $S_{n,k}$

Before the statement of each result, we will give a rough and heuristic derivation. Although these heuristics may be rigorously justified along the same line, we will instead adopt a different method of proof for more methodological interests.

Since $S_{n,k}$ is essentially a Cauchy convolution, we have (cf. [28])

$$S_{n,k} = \frac{(n-k)!}{n^n}\,[z^n]B(z)B_k(z), \tag{6}$$

where

$$B(z) = \frac{1}{1-T(z)} = \sum_{j\geqslant 0} \frac{j^j}{j!}\, z^j \quad (|z| < \mathrm{e}^{-1})$$

and

$$B_k(z) = z^k B^{(k)}(z) = \sum_{j\geqslant k} \frac{j^j}{(j-k)!}\, z^j \quad (|z| < \mathrm{e}^{-1}).$$

It is well known that (cf. [5])

$$T(z) = 1 - \sqrt{2(1-\mathrm{e}z)} + \mathrm{O}((1-\mathrm{e}z)) \quad (z \to \mathrm{e}^{-1}, z \notin (\mathrm{e}^{-1}, \infty)).$$

Accordingly,

$$B(z) = \frac{1}{\sqrt{2(1-\mathrm{e}z)}} + \mathrm{O}(1) \quad (z \to \mathrm{e}^{-1}, z \notin [\mathrm{e}^{-1}, \infty)).$$

By (6) using Cauchy's integral formula, we have

$$\begin{aligned} S_{n,k} &= \frac{(n-k)!k!}{(2\pi\mathrm{i})^2 n^n} \oint z^{-n+k-1} B(z) \oint w^{-k-1} B(w+z)\,\mathrm{d}w\,\mathrm{d}z \\ &\approx \frac{(n-k)!k!}{2(2\pi\mathrm{i})^2 n^n} \oint z^{-n+k-1} (1-\mathrm{e}z)^{-1/2} \oint w^{-k-1} (1-\mathrm{e}z-\mathrm{e}w)^{-1/2}\,\mathrm{d}w\,\mathrm{d}z \\ &\approx \frac{(n-k)!k!}{2(2\pi\mathrm{i})^2 n^n} \oint z^{-n+k-1} \frac{\mathrm{e}^k}{(1-\mathrm{e}z)^{k+1}} \oint v^{-k-1} (1-v)^{-1/2}\,\mathrm{d}v\,\mathrm{d}z, \end{aligned}$$

where we used the change of variables $w = (1-\mathrm{e}z)v/\mathrm{e}$ in the penultimate line. But

$$\frac{1}{2\pi\mathrm{i}} \oint v^{-k-1}(1-v)^{-1/2}\,\mathrm{d}v = \binom{2k}{k} 4^{-k} \quad (k \geqslant 0),$$

from this and Stirling's formula, it follows that

$$\begin{aligned}S_{n,k} &\approx \frac{(n-k)!k!}{4\pi i n^n}\binom{2k}{k}4^{-k}\oint z^{-n+k-1}e^k(1-ez)^{-k-1}\,dz\\ &\sim \frac{(n-k)!}{2n^n}e^n n^k\binom{2k}{k}4^{-k}\\ &\sim \binom{2k}{k}4^{-k}\sqrt{\frac{\pi n}{2}}.\end{aligned}$$

A formal statement follows.

Theorem 1. *If* $1 \leqslant k = o(n^{1/4})$ *then*

$$\begin{aligned}S_{n,k} &= \frac{1}{4^k}\binom{2k}{k}\sqrt{\frac{\pi n}{2}} + \frac{1}{3} - (-1)^{\delta_{k,1}}\frac{\sqrt{\pi}\binom{2k}{k}(2k^2-4k+1)}{12(2k-1)4^k\sqrt{2n}}\\ &\quad + (-1)^{\delta_{k,1}}\frac{2(2k-1)}{135n} + O(k^3n^{-3/2})\end{aligned} \tag{7}$$

uniformly in k, where $\delta_{a,b}$ *is Kronecker's symbol.*

In particular, if $k \to \infty$ and $k = o(n^{1/4})$ then

$$S_{n,k} = \sqrt{\frac{n}{2k}}(1 + O(k^{-1})).$$

Note that, asymptotically, we can incorporate the third term on the right-hand side of (7) into the first by adding a slight perturbation to n:

$$S_{n,k} \sim \frac{1}{4^k}\binom{2k}{k}\sqrt{\frac{\pi}{2}\left(n - (-1)^{\delta_{k,1}}\frac{2k^2-4k+1}{6}\right)} + \frac{1}{3}.$$

When k becomes large, the singularity of $B_k(z)$ is much "heavier" than that of $B(z)$. We may therefore expand $B(z)$ at $z = r$ and compute the corresponding residues:

$$\begin{aligned}S_{n,k} &\approx B(r) + B'(r)\left(\left(1-\frac{k}{n}\right)\rho_1^{n-1} - r\right)\\ &\quad + \frac{B''(r)}{2}\left(\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)\rho_2^{n-2} - 2r\rho_1^{n-1}\left(1-\frac{k}{n}\right) + r^2\right),\end{aligned}$$

where, for simplicity, $\rho_1 = 1 - 1/n$ and $\rho_2 = 1 - 2/n$. Thus a good choice for r is

$$r = \left(1-\frac{k}{n}\right)\rho_1^{n-1} = \left(1-\frac{k}{n}\right)\left(1-\frac{1}{n}\right)^{n-1}, \tag{8}$$

so that the second term disappears:

$$\begin{aligned} S_{n,k} &\approx B(r) + \frac{B''(r)}{2}\left(1-\frac{k}{n}\right)\left(\left(1-\frac{k+1}{n}\right)\rho_2^{n-2} - \left(1-\frac{k}{n}\right)\rho_1^{2n-2}\right) \\ &= B(r) + \mathrm{O}(B''(r)(1-k/n)kn^{-2}). \end{aligned}$$

Note that when $k=n$, $r=0$ and the above "$\approx$" is actually an identity.

Theorem 2. *If $k\to\infty$ and $k\leqslant n$ then*

$$S_{n,k} = B(r) + \mathrm{O}(B''(r)(1-k/n)kn^{-2}) \tag{9}$$

uniformly in k, where r is defined in (8).

The asymptotic nature of (9) will be clearer from the following corollaries.

Corollary 1. *If $k\to\infty$ and $k=\mathrm{O}(n^{1/3})$ then*

$$S_{n,k} = \sqrt{\frac{n}{2k-1}}(1+\mathrm{O}(k^{-1})).$$

Corollary 2. *For $n^{1/3} \ll k = \mathrm{o}(n)$,*

$$S_{n,k} = \sqrt{\frac{n}{2k-1}} + \frac{1}{3} + \mathrm{O}\left(\frac{\sqrt{n}}{k^{3/2}} + \frac{\sqrt{k}}{\sqrt{n}}\right).$$

Corollary 3. *If $\varepsilon n \leqslant k \leqslant (1-\varepsilon')n$ then*

$$S_{n,k} = B(r)\left(1+\mathrm{O}\left(\frac{1}{\varepsilon n} + \frac{\varepsilon'}{n}\right)\right).$$

Corollary 4. *If $n-k=\mathrm{o}(n)$ and $k\leqslant n$ then*

$$S_{n,k} = 1 + \frac{n-k}{n}\mathrm{e}^{-1}\left(1+\mathrm{O}\left(\frac{n-k}{n}\right)\right).$$

It should be noted that, from a computational point of view, the B function is easily computed by the relation $B(r)=1/(1+W(-r))$, where W is a standard function in Maple (*LambertW* in Maple V, R5). The approximation of $S_{n,k}$ by $B(r)$ is very precise even for small values of n, when k becomes slightly large. See Tables 1 and 2 for numerical examples.

An application of our results is that Massey's bound (cf. [24]) for the average worst-case probability of undetected error is asymptotically equivalent to Kløve's one (cf. [17]) as $n\to\infty$ and $k\to\infty$, $k\leqslant n$; see [28].

Table 1
Absolute and relative errors when approximating $S_{n,k}$ by $B(r)$ for $n=10$

k	1	2	3	4	5	6	7	8	9	10
$B(r)-S_{n,k}$	1.086	0.277	0.126	0.068	0.040	0.024	0.014	0.008	0.003	0
$(B(r)-S_{n,k})/S_{n,k}$	0.466	0.152	0.080	0.049	0.031	0.020	0.013	0.007	0.003	0

Table 2
Absolute errors of approximating $S_{n,k}$ by $B(r)$ for $n=20$

k	1	2	3	4	5	6	7	8	9	10
$B(r)-S_{n,k}$	1.603	0.436	0.213	0.128	0.085	0.059	0.044	0.033	0.0252	0.0196
k	11	12	13	14	15	16	17	18	19	20
$B(r)-S_{n,k}$	0.015	0.012	0.009	0.007	0.005	0.0038	0.0027	0.0016	0.0007	0

3. Proof of the theorems

Before proving the theorems, we briefly discuss some elementary properties of $S_{n,k}$ and $B_k(z)$.

By induction, we have

$$B_k(z) = \frac{T^k(z)\Pi_k(T(z))}{(1-T(z))^{2k+1}} \quad (k \geqslant 0), \tag{10}$$

where $\Pi_0(v) = \Pi_1(v) = 1$ and

$$\Pi_k(v) = (3k-2-(k-1)v)\Pi_{k-1}(v) + (1-v)\Pi'_{k-1}(v) \quad (k \geqslant 2).$$

From this last recurrence, it follows that

$$\Pi_k(1) = \frac{(2k)!}{2^k k!}, \quad \Pi'_k(1) = -\frac{k-1}{3}\Pi_k(1), \quad \Pi''_k(1) = \frac{k(k-1)(k-2)}{9(k-1/2)}\Pi_k(1),$$
$$\Pi'''_k(1) = \frac{(k-2)(k-3)(5k^2+1)}{135(k-1/2)}\Pi_k(1)$$

and, in general,

$$\begin{aligned} &\Pi_k^{(m)}(1) = (2k-1-m)\Pi_{k-1}^{(m)}(1) - m(k-1)\Pi_{k-1}^{(m-1)}(1) \quad (m \geqslant 1),\\ &\Pi_k^{(m)}(1) \ll k^m \Pi_k(1) \quad (m \geqslant 1). \end{aligned} \tag{11}$$

Also the exponential generating function of $\Pi_k(v)$ satisfies

$$\sum_{k\geqslant 0} \frac{\Pi_k(v)}{k!} u^k = \frac{1-v}{1-T(u\mathrm{e}^{-v}(1-v)^2 + v\mathrm{e}^{-v})}.$$

If we write

$$\Pi_k(v) = \sum_{0\leqslant j<k} \pi_{k,j} v^j,$$

then $S_{n,k}$ satisfies

$$S_{n,k} = \sum_{k\leqslant j\leqslant n} \frac{(n-k)!j}{(n-j)!n^{j+1}} \sum_{0\leqslant \ell<k} \pi_{k,\ell} \binom{k+1+j-\ell}{2k+1}. \tag{12}$$

This generalizes (2). In particular, we have

$$\begin{aligned}
S_{n,1} &= \sum_{1\leqslant j\leqslant n} \frac{(n-1)!j}{(n-j)!n^{j+1}} \binom{j+2}{3},\\
S_{n,2} &= \sum_{1\leqslant j\leqslant n} \frac{(n-2)!j}{(n-j)!n^{j+1}} \left(4\binom{j+3}{5} - \binom{j+2}{5}\right),\\
S_{n,3} &= \sum_{1\leqslant j\leqslant n} \frac{(n-3)!j}{(n-j)!n^{j+1}} \left(27\binom{j+4}{7} - 14\binom{j+3}{7} + 2\binom{j+2}{7}\right).
\end{aligned} \tag{13}$$

The proof of (12) follows from (6), (10) and the formal identity

$$\begin{aligned}
\sum_{j\geqslant 1} a_j T^j(z) &= \sum_{j\geqslant 1} a_j j \sum_{m\geqslant j} \frac{m^{m-j-1}}{(m-j)!} z^j\\
&= \sum_{m\geqslant 1} \frac{m^{m-1}}{m!} z^m \sum_{1\leqslant j\leqslant m} \frac{j a_j m!}{m^j (m-j)!}
\end{aligned}$$

[by Lagrange inversion formula]; see [5].

Still more complicated identities for $S_{n,k}$ can be derived by integration by parts:

$$B(z)B^{(k)}(z)\,\mathrm{d}z = \mathrm{d}(B(z)B^{(k-1)}(z)) - z^{-1}B^2(z)(B(z)-1)B^{(k-1)}(z)\,\mathrm{d}z,$$

giving

$$S_{n,k} = S_{n,k-1} - \frac{(n-k)!}{n^n}[z^n](B^2(z)(B(z)-1)B_{k-1}(z)).$$

Thus for $k\geqslant 1$

$$\begin{aligned}
S_{n,k} &= S_{n,k-1}\\
&\quad - \frac{1}{n-k+1} \sum_{1\leqslant j\leqslant n-k+1} \binom{n-k+1}{j} \left(\frac{j}{n}\right)^j \left(1-\frac{j}{n}\right)^{n-j} Q(j) S_{n-j,k-1},
\end{aligned}$$

since $Q(n) = (n!/n^n)[z^n]B(z)(B(z)-1)$. In particular, we have

$$S_{n,1} = \frac{1}{2}S_{n,0} = S_{n,0} - \frac{1}{n} \sum_{1\leqslant j\leqslant n} \binom{n}{j} \left(\frac{j}{n}\right)^j \left(1-\frac{j}{n}\right)^{n-j} Q(j) S_{n-j,0},$$

where the first identity follows from an integration by parts. For a closely related recurrence, see [20].

From the relation $S_{n,1}=S_{n,0}/2$ and (13), we have the identity

$$\frac{1}{3}\sum_{1\leqslant j\leqslant n}\frac{(n-1)!j^2}{(n-j)!n^{j+1}}(j+1)(j+2)=\sum_{0\leqslant j\leqslant n}\frac{n!}{(n-j)!n^j}$$

for $n\geqslant 1$.

3.1. Proof of Theorem 1

From (6), we have, by a change of variables,

$$\begin{aligned}S_{n,k}&=\frac{(n-k)!}{n^n}\frac{1}{2\pi\mathrm{i}}\oint_{|z|=\rho} z^{-n+k-1}B(z)B^{(k)}(z)\,\mathrm{d}z \quad (0<\rho<\mathrm{e}^{-1})\\&=\frac{(n-k)!\mathrm{e}^n}{2\pi\mathrm{i}n^n}\int_{c-\mathrm{i}\pi}^{c+\mathrm{i}\pi}\mathrm{e}^{n\tau}B(\mathrm{e}^{-1-\tau})B_k(\mathrm{e}^{-1-\tau})\,\mathrm{d}\tau \quad (c>0).\end{aligned}$$

Analytic continuation of $B_k(z)$ follows from that of $B(z)$ which in turn is obtained from that of $T(z)$; see [12] or [5]. We now make explicit the local behavior of $B_k(\mathrm{e}^{-1-\tau})$ as $\tau\to 0$. From the Mellin inversion formula (cf. [9])

$$\mathrm{e}^{-w}=\frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\Gamma(s)w^{-s}\,\mathrm{d}s \quad (\Re w>0, a>0),$$

it follows, by absolute convergence, that

$$B_k(\mathrm{e}^{-1-\tau})=\frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\Gamma(s)\tau^{-s}Y_k(s)\,\mathrm{d}s \quad \left(a>k+\frac{1}{2}\right),$$

where

$$Y_k(s)=\sum_{j\geqslant k}\frac{j^j\mathrm{e}^{-j}}{(j-k)!j^s} \quad \left(\Re s>k+\frac{1}{2}\right).$$

The singularities of $Y_k(s)$ will be determined by the asymptotic behavior of $j^j\mathrm{e}^{-j}/(j-k)!$ as $j\to\infty$. By Stirling's formula

$$\begin{aligned}\frac{j^j\mathrm{e}^{-j}}{(j-k)!j^s}=\frac{j^{k-s-1/2}}{\sqrt{2\pi}}\Bigg(1-\frac{6k^2-6k+1}{12j}\\+\frac{36k^4-120j^3+120j^2-26j+1}{288j^2}+\mathrm{O}(k^6j^{-3})\Bigg),\end{aligned}$$

it follows that $Y_k(s)$ admits meromorphic continuation into the whole plane with simple poles at $s=k+1/2, k-1/2,\ldots$. The corresponding residues are given by the coefficients in the above expansion. Note that $\Gamma(s)$ has simple poles at $s=0,-1,-2,\ldots$. By standard arguments of Mellin transform, we deduce that, for $k \geqslant 2$,

$$\begin{aligned}B_k(\mathrm{e}^{-1-\tau}) = {} & \frac{\Gamma(k+1/2)}{\sqrt{2\pi}}\tau^{-k-1/2} - \frac{6k^2-6k+1}{12\sqrt{2\pi}}\Gamma(k-1/2)\tau^{-k+1/2} \\ & + \frac{36k^4-120j^3+120j^2-26j+1}{288\sqrt{2\pi}}\Gamma(k-3/2)\tau^{-k+3/2} \\ & + \mathrm{O}(k^6\Gamma(k-5/2)\tau^{-k+5/2})\end{aligned}$$

and

$$B_1(\mathrm{e}^{-1-\tau}) = (2\tau)^{-3/2} - \frac{1}{12}(2\tau)^{-1/2} - \frac{4}{135} + \mathrm{O}(\tau^{1/2}).$$

These expressions hold a priori as $\tau \to 0$ in $\Re\tau > 0$. But it is easily seen using (10) that it is still valid as $\tau \to 0$ in the cut-plane $\mathbb{C}\backslash(-\infty, 0]$.

In a similar manner, we have

$$B(\mathrm{e}^{-1-\tau}) = (2\tau)^{-1/2} + \frac{1}{3} + \frac{1}{12}(2\tau)^{1/2} + \frac{4\tau}{135} + \mathrm{O}(\tau^{3/2})$$

as $\tau \to 0$ and $\tau \notin \mathbb{C}\backslash(-\infty, 0]$. By arguments similar to the singularity analysis, we obtain, for $k \geqslant 2$,

$$\begin{aligned}S_{n,k} = {} & \frac{(n-k)!}{n^n}\mathrm{e}^n n^k \left(\frac{\Gamma(k+1/2)}{2\sqrt{\pi}\Gamma(k+1)} + \frac{n^{-1/2}}{3\sqrt{2\pi}} - \frac{\Gamma(k-1/2)(k-1)(3k-1)}{12\sqrt{\pi}\Gamma(k)}n^{-1}\right. \\ & \left. - \frac{90k^2-106k+23}{540\sqrt{2\pi}}n^{-3/2} + \mathrm{O}(k^{7/2}n^{-2})\right)\end{aligned}$$

and for $k = 1$

$$S_{n,1} = \frac{n!\mathrm{e}^n}{n^n}\left(\frac{1}{4} - \frac{1}{3\sqrt{2\pi n}} - \frac{23}{540\sqrt{2\pi}n} + \frac{23}{6048\sqrt{2\pi}n^{3/2}} + \mathrm{O}(n^{-2})\right).$$

From the asymptotic formula

$$\frac{(n-k)!}{n^n}\mathrm{e}^n n^k = \sqrt{2\pi n}\left(1 + \frac{6k^2-6k+1}{12n} + \mathrm{O}(k^4 n^{-2})\right)$$

for $k = \mathrm{o}(\sqrt{n})$, the result (7) follows. $\square$

We note that (7) can also be derived by (10) and estimate (11) using singularity analysis. This is the approach used by Szpankowski [28].

3.2. Proof of Theorem 2

Recall that $r = (1-k/n)(1-1/n)^{n-1}$. We observe first that the terms in $S_{n,k}$ are decreasing functions of j and that for $n-k = \mathrm{o}(n)$ and $k \leqslant n$, the sum definition (1) of $S_{n,k}$ is itself an asymptotic expansion:

$$S_{n,k} = 1 + \frac{n-k}{n}\left(1-\frac{1}{n}\right)^{n-1} + \mathrm{O}\left(\frac{(n-k)^2}{n^2}\right)$$
$$= 1 + r + \mathrm{O}\left(\frac{r}{n}\right).$$

This last expression is asymptotically equivalent to (9). It remains to prove (9) for the case $k \to \infty$ and $k \leqslant (1-\varepsilon)n$. The proof is divided into two parts: we first show that for all n and k

$$S_{n,k} \leqslant B(r), \tag{14}$$

then we show that

$$S_{n,k} \geqslant B(r) + \mathrm{O}\left(B''(r)\frac{k}{n(n-k)}\right). \tag{15}$$

The error term on the right-hand side is asymptotically equivalent to that in (9) when $k \to \infty$ and $k \leqslant (1-\varepsilon)n$, thus completing the proof.

Consider first (14). Write

$$S_{n,k} = \sum_{0 \leqslant j \leqslant n-k} \frac{j^j}{j!}\left(1-\frac{k}{n}\right)^j \left(1-\frac{1}{n-k}\right)\cdots\left(1-\frac{j-1}{n-k}\right)\left(1-\frac{j}{n}\right)^{n-j}.$$

We need only prove that

$$\left(1-\frac{1}{n-k}\right)\cdots\left(1-\frac{j-1}{n-k}\right)\left(1-\frac{j}{n}\right)^{n-j} \leqslant \left(1-\frac{1}{n}\right)^{j(n-1)}.$$

By monotonicity, it suffices to show that

$$b_{n,j} := \left(1-\frac{1}{n-1}\right)\cdots\left(1-\frac{j-1}{n-1}\right)\left(1-\frac{j}{n}\right)^{n-j}\left(1-\frac{1}{n}\right)^{-j(n-1)} \leqslant 1.$$

Now

$$\frac{b_{n,j}}{b_{n,j+1}} = \left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n-j}\right)^{n-j} \geqslant 1.$$

Inequality (14) follows from the fact that $b_{n,0} = 1$.

For the proof of (15), we first show that

$$S_{n,k} = \sum_{0 \leqslant j \leqslant (n-k)/2} \frac{j^j(n-k)\cdots(n-k-j+1)}{j!\,n^j}\left(1-\frac{j}{n}\right)^{n-j}$$
$$+ \mathrm{O}\left(\frac{n}{k\sqrt{n-k}}\mathrm{e}^{-k(n-k)/(2n)}\right).$$

For,

$$\sum_{j>(n-k)/2} \frac{j^j(n-k)\cdots(n-k-j+1)}{j!n^j}\left(1-\frac{j}{n}\right)^{n-j}$$

$$\leqslant \sum_{j>(n-k)/2} \frac{j^j}{j!}\left(1-\frac{k}{n}\right)^j \mathrm{e}^{1-j}\left(1-\frac{j-1}{n-1}\right)^{-n+j}\left(1-\frac{j}{n}\right)^{n-j}$$

$$\leqslant \frac{e}{\sqrt{2\pi}} \sum_{j>(n-k)/2} j^{-1/2}\left(1-\frac{k}{n}\right)^j\left(1-\frac{j}{n}\right)^{n-j}$$

$$\ll \int_{(n-k)/2}^{\infty} x^{-1/2}\,\mathrm{e}^{-kx/n}\,\mathrm{d}x \ll \frac{n}{k\sqrt{n-k}}\,\mathrm{e}^{-k(n-k)/(2n)}.$$

Now

$$\left(1-\frac{j}{n}\right)\prod_{0\leqslant \ell<j}\left(1-\frac{\ell}{n-k}\right) \geqslant \mathrm{e}^{-j+j/(2n)-Y},$$

where

$$Y = \sum_{\ell\geqslant 1} \frac{j^{\ell+1}}{\ell(\ell+1)}((n-k)^{-\ell}-n^{-\ell})$$

$$\leqslant \frac{kj^2}{n(n-k)}\sum_{\ell\geqslant 1}\frac{1}{\ell}\left(\frac{j}{n-k}\right)^{\ell-1}$$

$$\leqslant 2\log 2\frac{kj^2}{n(n-k)}$$

for $j\leqslant(n-k)/2$.
On the other hand,

$$\left(1-\frac{1}{n}\right)^{j(n-1)} \leqslant \mathrm{e}^{-j+j/(2n)}.$$

Accordingly,

$$\left(1-\frac{j}{n}\right)\prod_{0\leqslant \ell<j}\left(1-\frac{\ell}{n-k}\right) \geqslant \left(1-\frac{1}{n}\right)^{j(n-1)}\left(1-\frac{2\log 2}{n(n-k)}kj^2\right).$$

Thus,

$$S_{n,k} \geqslant \sum_{0\leqslant j\leqslant(n-k)/2} \frac{j^j}{j!}r^j\left(1-\frac{2\log 2}{n(n-k)}kj^2\right) + \mathrm{O}\left(\frac{n}{k\sqrt{n-k}}\,\mathrm{e}^{-k(n-k)/(2n)}\right).$$

It is easily seen that

$$\sum_{0\leqslant j\leqslant (n-k)/2} \frac{j^j}{j!} r^j = B(r) + \mathrm{O}\left(\frac{n}{k\sqrt{n-k}}\,\mathrm{e}^{-k(n-k)/(2n)}\right)$$

and that this error term is absorbed by that in (9) when $k\to\infty$ and $k\leqslant(1-\varepsilon)n$.
This proves (15) and completes the proof of Theorem 2. □

The corollaries follow from the estimates

$$B(z) = \frac{1}{\sqrt{1-\mathrm{e}z}} + \frac{1}{3} + \mathrm{O}(\sqrt{1-\mathrm{e}z}),$$
$$B''(z) \ll (1-\mathrm{e}z)^{-5/2},$$

as $z\to\mathrm{e}^{-1}$, and

$$r = \mathrm{e}^{-1}\left(1 - \frac{2k-1}{2n} + \mathrm{O}(kn^{-2})\right).$$

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