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Small Extended Generalized Quadrangles

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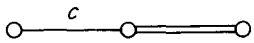
We consider extensions of generalized quadrangles with parameters (s, t), and establish lower bounds (in terms of s and t) for the number of points, sometimes under additional hypotheses. We also study the structure of geometries attaining these bounds, give several constructions and some uniqueness proofs, and examine the question of further extensions.

1. INTRODUCTION

An extended generalized quadrangle (EGQ) is a Buekenhout geometry with diagram



Alternatively, it is a connected point-block incidence structure with the property that, for any point P, the points collinear with P and the blocks incident with P form a generalized quadrangle. EGQs include both locally polar spaces of polar rank 2 (Buekenhout and Hubaut [6]) and 2-designs whose derived designs are generalized quadrangles (Thas [14]).

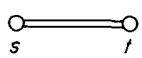
In this paper, we prove some lower bounds for the number of points of an EGQ (sometimes under additional hypotheses), and investigate EGQs which attain these bounds. We give several constructions, and show the uniqueness of some of the small examples. We also consider the question of further extensions.

2. PRELIMINARIES

An *incidence structure* consists of a set of points and a set of blocks with a relation of incidence between them. None of our incidence structures will have 'repeated blocks', and we will freely identify a block with the set of points incident with it.

The *point graph* of an incidence structure $\mathcal{G}$ has as vertices the points of $\mathcal{G}$, two vertices adjacent if there is a block containing both. The *residue* of $\mathcal{G}$ with respect to a point P consists of the points adjacent to P and the blocks incident with P; incidence is the same as in $\mathcal{G}$. If the point graph is connected and the residues with respect to all points are of the same type τ, we refer to $\mathcal{G}$ as an *extended* τ.

A *generalized quadrangle* $GQ(s, t)$ of order (s, t) $(s, t \geqslant 1)$ is an incidence structure (the blocks of which are called lines) satisfying:
(i) each point is on $t+1$ lines, and two distinct points are on at most one line;
(ii) each line contains $s+1$ points, and two distinct lines meet in at most one point;
(iii) if l is a line and P is a point with $P \notin l$, then there is a unique line containing P with meets l.
(With one small exception, we never deal with infinite or non-regular GQs.) A $GQ(s, t)$ is a Buekenhout geometry with diagram



A set of $st+1$ pairwise non-collinear points in a $GQ(s, t)$ is called an *ovoid*. We use the following result, the easy proof of which is omitted:

LEMMA 2.1. *If S is a set of points in a $GQ(s, t)$ which meets every line, then $|s| \geqslant st+1$, with equality iff S is an ovoid.*

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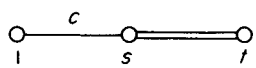
The order (s, t) of a GQ satisfies a number of restrictions. among them are the condition $s+t \mid st(st+1)$ and the inequalities $t \leqslant s^2$ (if $s>1$) and $s \leqslant t^2$ (if $t>1$).

For a general survey and list of the known GQs, with information on whether they contain ovoids, see Payne and Thas [10].

An *extended generalized quadrangle* (EGQ) is a connected incidence structure all of whose (point) residues are GQs. It is easy to see that all these GQs have the same order (s, t), and we speak of an $EGQ(s, t)$. Moreover, the following properties are easily deduced:

(i) every point is on $(t+1)(st+1)$ blocks;
(ii) every block contains $s+2$ points;
(iii) two distinct points are on 0 or $t+1$ blocks;
(iv) two distinct blocks meet in at most 2 points.

An $EGQ(s, t)$ is a Buekenhout geometry with diagram



where the three types of varieties are points, edges of the point graph, and blocks, with the extra condition 'no repeated lines'.

Applying property (iii) in the definition of a GQ gives the following useful result (Buekenhout [5]):

LEMMA 2.2. *Let P be a point and x a block of an $EGQ(s, t)$, with $P \notin x$. Then the number of points on x adjacent to P is even. Moreover, there is a pairing on these points, each pair lying on a unique block containing P.*

This lemma gives rise to a divergence between the cases s even and s odd, as we shall see.

Further extensions are defined analogously, and are Buekenhout geometries whose diagrams are obtained by adjoining additional c strokes on the left.

Throughout this paper, we use v to denote the number of points of an $EGQ(s, t)$.

3. THE CASE s EVEN

We consider first the minimal case where the EGQ is also a 2-design; we call this a *one-point extension*. The point graph of a one-point extension is complete. By (2.2), this can only happen if s is even. Now the number

$$v = 1 + (1+s)(1+st)$$

of points, and hence the number of blocks, are known; this yields a divisibility condition. Thas [14] has considered the known GQs:

THEOREM 3.1 (*Thas*). *If $\mathcal{G}$ is a one-point extension of a known $GQ(s, t)$, then $f(s, t)$ is one of the following*:

(i) $(q-1), q+1)$, *where q is an odd prime power*;
(ii) (2, 1), (2, 2), (2, 4), (4, 2), (4, 8), (6, 4), (8, 4), (8, 8), (8, 64), (10, 8), *or* (64, 512).

EXAMPLE 3.2. There exist one-point extensions of GQs of orders (2, 1), (2, 2) and (2, 4), each of which is unique.

CONSTRUCTION. Let $\mathcal{G}$ be a $GQ(2, t)$ and ∞ a new point. Let the points of the extension $\mathcal{G}'$ be those of $\mathcal{G}$ together with ∞. The blocks are of two types:
(i) if x is a line of $\mathcal{G}$, then $x \cup \{\infty\}$ is a block of $\mathcal{G}'$;
(ii) if x and y are intersecting lines of $\mathcal{G}$, then their symmetric difference $x \triangle y$ is a block of $\mathcal{G}'$.

UNIQUENESS. A one-point extension of a $GQ(2, t)$ must come from the above construction. For if blocks x and y meet in two points, then any three points of $x \triangle y$ lie in a unique block, and these blocks are either all equal or all different; but in the latter case, the three such blocks containing a point P of $x \triangle y$ form a triangle in the residue of P. Our claim now follows from the classification of GQs with $s = 2$ (see [10] or [11]). (An earlier existence proof is in Buekenhout [5].)

Note that the set of triples of points contained in blocks of the EGQ is a regular two-graph in each case (Seidel [12]). (Indeed, it is easy to show directly that this must be so, and then the uniqueness follows from similar results of Seidel for two-graphs.) With a regular two-graph is associated a distance-regular double cover of the complete graph; in each of these three cases, this double cover is the point graph of another EGQ, having twice as many points as the one-point extension.

EXAMPLE 3.3 (Thas [14]). There exist one-point extensions of the GQs $AS(q)$ of order $(q-1, q+1)$ for any odd prime power q.

Thas [14] has also considered further extensions, which are necessarily one-point extensions. We quote his result for the second extensions, and refer to his paper for further results:

THEOREM 3.4 (*Thas*). *Suppose that $\mathcal{G}$ is a connected incidence structure, the residues of which are one-point extensions of known $GQ(s, t)$s. Then $s = q-1$, $t = q+1$, where q is a prime power congruent to ± 1 (mod 6).*

We turn now to extensions with additional points.

LEMMA 3.5. *Let P and Q be non-adjacent points of an $EGQ(s, t)$ with s even. Then the number of points joined to both P and Q is at most $s(st+1)$. If equality holds, then the set of points joined to P but not Q is an ovoid in the residue of P.*

PROOF. Let S be the set of points joined to P but not Q. For each block x containing P, Q is non-adjacent to an even number of points of x, by Lemma 2.2; so x meets S. By 2.1, $|S| \geqslant st+1$, with equality iff it is an ovoid. Since $(s+1)(st+1)$ points are joined to P, the result follows.

COROLLARY 3.6. *Suppose that an $EGQ(s, t)$, with s even, is not a one-point extension. Then*

$$v \geqslant (s+2)(st+1)+2.$$

PROOF. Counting, P, Q, the residue of P, and the set of points joined to Q but not P, we have

$$v \geqslant 1+1+(s+1)(st+1)+(st+1).$$

We do not know any examples attaining this bound. (If it is met, then the set of points not equal or joined to P is a clique in the point graph; so the complement of the

point graph has valency $st+2$ and contains no triangles.) However, we can do better under an extra hypothesis:

THEOREM 3.7. *Suppose that an $EGQ(s, t)$, with s even, has the property that, if a point P is not on a block x, then some point of x is not adjacent to P. Then*

$$v \geqslant (s+2)(st+t+1).$$

If equality holds, then $t=1$ or $t=2$. When $t=1$, the point graph is the complement of a square lattice graph; when $t=2$, it is the complement of a triangular graph.

(The hypothesis of (3.7) holds in all known EGQs which are not one-point extensions.)

PROOF. Let P be a point and x a block with $P \notin x$. By (2.2), P induces a pairing of the points on x joined to P, of which there are at most s. The number of pairs of points on x is $\frac{1}{2}(s+2)(s+1)$; each of these lies in t further blocks, each containing s points not on x. On the other hand, each point Q not on x induces at most $\frac{1}{2}s$ pairs on x, each pair lying in a unique block with Q. So there are at least $(s+2)(st+t)$ points not on x. Adding the $s+2$ points on x gives the result.

When equality holds, it is easy to check that the point graph of the EGQ is strongly regular. Its valency is $(s+1)(st+1)$; two adjacent points have $s(st+1-t)$ common neighbours, and two non-adjacent points have $s(st+1)$ common neighbours. This graph has an eigenvalue 1, so the complementary graph has least eigenvalue -2. Now applying Seidel's classification of such graphs [11], we find that the point graph is the complement of a cocktail party graph, a triangular graph, a square lattice graph, or one of seven exceptions (the Petersen, Clebsch, Shrikhande, Schläfli, and three Chang graphs).

The cocktail party graphs are trivially ruled out as their complements are disconnected, and the seven exceptions are excluded by *ad hoc* arguments. But in the other cases, consideration of parameters shows that $t=1$ for the square lattice graphs, and $t=2$ for the triangular graphs.

Finally we note that, when $t=1$, there is no further restriction (other than the parity of s), whereas for $t=2$, the inequality $s \leqslant t^2$ shows that $s=2$ or 4.

EXAMPLE 3.8. An $EGQ(s, 1)$ attaining the bound in (3.7) exists whenever $s=q-2$, where q is a power of 2. The example with $s=2$ is unique.

CONSTRUCTION. Let $\pi = PG(2, q)$, where q is a power of 2. Let P and Q be points of π, l the line PQ, C a hyperoval (a $(q+2)$-arc) containing P and Q, and T the group of all central collineations with centre P and axis containing Q. Then $|T| = q(q-1)$, and T fixes all lines on P and is sharply 2-transitive on the lines on Q other than l; so T is sharply transitive on pairs of affine points on fixed lines through P and not collinear with Q.

The EGQ is then formed by taking the point set to consist of the points of π not on l, and the block set to consist of the lines of π not containing P or Q and the images of $C \backslash \{P, Q\}$ under T.

To prove that the construction works, note that two points lie on a block iff the line joining them does not contain P or Q, so the point graph is the complement of a square lattice. Two adjacent points lie on a unique line and also in a unique translate of C (by the transitivity property mentioned above). Also, two blocks of the same type

through a point R have no further common point, whereas blocks of different types meet in one further point; so the residue of R is a grid.

Note that $PG(2, q)$ is not really required; all we need is a projective plane with an appropriate hyperoval and collineation group.

UNIQUENESS. Let A be any set. For a permutation g of A, let $B(g)$ be the graph of g, the set $\{(a, ag)\colon a \in A\}$ of $A \times A$. Any subset of $A \times A$ which is a transversal for both rows and columns of the square grid is of this form. An $EGQ(s, 1)$ attaining the bound of (3.7) consists of $2(s+1)(s+2)$ sets $B(g)$, where $|A| = s+2$. In the case $s = 2$, we have $2(s+1)(s+2) = (s+2)!$ and so the blocks of the EGQ are all possible sets $B(g)$.

EXAMPLE 3.9. There exists a unique $EGQ(4, 1)$ attaining the bound of (3.7); and it is unique.

CONSTRUCTION. In the notation of the uniqueness of proof of (3.8), let A be the projective line over $GF(5)$, and take as block set $\{B(g)\colon g \in PSL(2, 5)\}$. Since $PSL(2, 5) \times PSL(2, 5)$, acting on the two copies of A, is an automorphism group which is transitive on points, it suffices to consider the residue of (∞, ∞). The blocks of this residue are the sets $B(g)$, with g in the dihedral group of order 10. They fall into two families, corresponding to the cosets of the cyclic group of order 5; two blocks have just one further point in common if they lie in different families, and not otherwise. So the residue is a grid $GQ(4, 1)$.

This example cannot arise from any construction resembling (3.8), since there is no projective plane of order 6. Moreover, there are no further examples of this type. For the construction requires a 2-transitive permutation group, in which a 2-point stabiliser has order 2 and fixes just two points; and the only such groups are S_4 and $PSL(2, 5)$.

UNIQUENESS. Any 36-point $EGQ(4, 1)$ arises from a set S of 60 permutations of a 6-set A such that, for any $a, b \in A$,

$$\{B(g)\colon ag = b\}$$

forms a 5×5 grid. This property is unaffected by left or right multiplication by any permutation; so we may assume that S contains the identity permutation 1. We have to show that $S = PSL(2, 5)$.

Step 1. For each $a \in A$, S contains a cyclic group $P(a)$ of order 5 fixing a, together with a non-trivial coset of $P(a)$ in its normaliser.

For, in the residue of (a, a), we have a net with four parallel classes, consisting of the lines of the residue together with the rows and columns not containing a of the 6×6 grid. This net has deficiency 2, and so, by Bruck [3], can be completed to an affine plane. The uniqueness of the affine plane of order 5 shows that the lines are two cosets of a cyclic group $P(a)$ or order 5. Since $1 \in S$, one coset is $P(a)$.

Step 2. The elements of order 5 in S are all those of a group G permutation isomorphic to $PSL(2, 5)$.

For any subgroup $P(a)$ of order 5 lies in a unique subgroup isomorphic to $PSL(2, 5)$. For $b \neq a$, S contains a unique non-trivial element t of the normaliser of $P(a)$ fixing b, and a unique non-trivial element u of the normaliser of $P(b)$ fixing a. But S contains only two elements fixing a and b, and one of them is 1; so $t = u$. Moreover, there are only two subgroups of order 5 normalised by t and fixing b, and one of them is excluded since it contains such elements agreeing in three places with elements of $P(a)$.

Thus $P(a)$ determines $P(b)$. In the known example, $P(a)$ and $P(b)$ lie in the same subgroup $PSL(2, 5)$; so this holds in general.

Step 3. For all n, the product of any n elements of order 5 in S lies in S.

We prove this by induction on n; the claim is clear for $n \leqslant 1$. Suppose that $n \geqslant 2$ and that the claim holds for products of fewer than n elements; let $g_1, \ldots, g_n \in S$ have order 5. Since $g_2 \cdots g_n \in S$, the set $Sg_n^{-1} \cdots g_2^{-1}$ satisfies the same conditions as S (including containing 1) and contains $(g_3 \cdots g_n)(g_n^{-1} \cdots g_2^{-1}) = g_2^{-1}$; so it contains the same elements of order 5 as S. So $g_1 \in Sg_n^{-1} \cdots g_2^{-1}$, whence $g_1 \cdots g_n \in S$.

Now, since G is generated by its elements of order 5, we have $G \subseteq S$, and so (by comparing orders) $G = S$.

EXAMPLE 3.10. There is an $EGQ(2, 2)$ attaining the bound of (3.7); and it is unique.

The point set consists of all 2-subsets of an 8-set; blocks are all sets of 4 pairwise disjoint 2-sets. The residue of a point is the analogously defined structure whose points are the 2-subsets of a 6-set; this is a well known representation of the $GQ(2, 2)$. Uniqueness is immediate from the facts that a block is a 4-coclique in the triangular graph $T(8)$ and that the numbers of blocks and 4-cocliques are equal (viz. 105).

Cameron and Hughes [7] have recently shown the non-existence of an $EGQ(4, 2)$ meeting the bound in (3.7).

Regarding further extensions, the following results are known:

EXAMPLE 3.11. The $EGQ(2, 1)$ of (3.8) can be extended infinitely often.

The nth extension has as blocks all sets $B(g)$, where $|A| = n + 4$.

EXAMPLE 3.12. The $EGQ(2, 2)$ of (3.10) can be extended infinitely often.

The points of the nth extension are the 2-subsets of a $(2n + 8)$-set; the blocks are all sets of $n + 4$ pairwise disjoint 2-sets.

4. THE CASE s ODD

LEMMA 4.1. *Let P and Q be adjacent points of an $EGQ(s, t)$ with s odd. Then the number of points joined to both P and Q is at most $s(st + 1)$. Equality implies that, if S is the set of points joined to P but not Q, then $\{Q\} \cup S$ is an ovoid in the residue of P.*

The proof is almost identical to that of (3.5).

THEOREM 4.2. *For any $EGQ(s, t)$ with s odd, we have*

$$v \geqslant (s + 2)(st + 1).$$

If equality holds, then the point set is the disjoint union of $s + 2$ 'groups' of size $st + 1$, and each block is a transversal for the 'groups'. Thus the point graph is complete multipartite, and the EGQ is a group divisible design with $\lambda = t + 1$. The groups not containing P form a partition of the residue of P into ovoids.

PROOF. By (4.1), if P and Q are joined, there are at least st points joined to Q but not P, and so at least $1 + (s + 1(st + 1) + st$ points altogether.

If equality holds, then for any point P, there are st points not joined to P; and these points are joined to all points in the residue of P, so no two of them are joined to one another. Thus P together with these points form a set of $st+1$ pairwise non-adjacent points, all having the same set of common neighbours. Thus the point graph is complete multipartite, with $s+2$ 'groups' of size $st+1$.

Finally, as no two points in any group are joined and there are $s+2$ 'groups', each block must contain exactly one point from each 'group'.

REMARK. The existence of a partition of a GQ into ovoids is very restrictive. Of the known GQs with s odd, the only ones to have such a partition are those with $s=1$ (the complete bipartite graphs), $t=1$ (grids), or $t=s+2$.

EXAMPLE 4.3 (Shult [13]). For any $t \geqslant 1$, there is a unique $EGQ(1, t)$. It attains the bound of (4.2). Its blocks are all transversals to three disjoint $(t+1)$-sets.

In fact, this remains true (suitably modified) with no assumptions of finiteness or regularity.

EXAMPLE 4.4. An $EGQ(s, 1)$ attaining the bound in (4.2) exists whenever $s=q-1$, where q is a power of 2. The examples with $s=1$ and $s=3$ are unique.

CONSTRUCTION. Let $\pi = PG(2, q)$, where q is a power of 2. Let P be a point of π, C a hyperoval containing P, and T the group (of order q^2) of all elations with centre P. The points of the EGQ are the points of π different from P; the blocks are the lines not containing P, and the translates of $C\backslash\{P\}$ under T. (The 'groups' are the lines containing P.) The proof is similar to that for (3.8).

UNIQUENESS. For $s=1$ this is trivial (or follows from (4.3)). Consider the case $s=3$. We show first that, if x, y, z are blocks with $|x \cap y| = |x \cap z| = 2$, then $|y \cap z| = 1$. This is clear if there is a point $P \in x \cap y \cap z$, by considering the residue of P; so suppose not. Then the points lying in more than one of x, y, z are in distinct groups; since there are only five groups, $|y \cap z| \leqslant 1$. But, if P is the point of y whose group contains no point of $x \cap y$ or $x \cap z$, then P lies in two blocks meeting x in two points each; these blocks must be y and z.

Now easy counting shows that, if $|x \cap y| = |x \cap z| = 1$, then $|y \cap z| = 1$. So the blocks fall into two families, two blocks meeting in one point iff they belong to the same family.

Take a new point ∞, and let 'lines' be the sets $G \cup \{\infty\}$ (where G is a group) together with the blocks of one family. The resulting point–line structure is a projective plane of order 4, and the blocks of the other family (with ∞ adjoined) are hyperovals. The uniqueness now follows from the uniqueness of $PG(2, 4)$ and well known properties of its hyperovals.

We consider further extensions.

EXAMPLE 4.5 (Shult [13]). The $EGQ(1, t)$ of (4.3) can be infinitely often extended.

The point set of the nth extension consists of $n+3$ disjoint $(t+1)$-sets; the blocks are all the transversals.

THEOREM 4.6. *Let $\mathcal{G}$ be a connected geometry, the residues of which are $EGQ(s, 1)$s*

(with s odd) attaining the bound of (4.2). *Let* $w = v + d = (s+1)(s+2) + d$ *be the number of points of* $\mathcal{G}$. *Then one of the following holds*:
(i) $d = s+1$, $s \equiv \pm 1$ *(mod* 6), $s \neq 5$;
(ii) $s = 3$, $d = 1$;
(iii) $s = d^2 - d - 1$, $d = 3$ *or* 6.

PROOF. Set $q = s+1$, and let Γ be the point graph of $\mathcal{G}$. We have $p_{21}^1 \leq q-1$ and $p_{11}^2 \geq q^2+1$. (Here, for example, p_{11}^2 denotes the number of common neighbours of two points at distance 2; we do not assume its constancy.) So the number of points at distance 2 from a point P is at most

$$q(q+1)(q-1)/(q^2+1) < q,$$

and hence at most $q-1$.

If P, Q, R satisfy $d(P, Q) = 2$, $d(Q, R) = 1$, then P and Q have at least q^2 common neighbours, and Q and R at least q^2; so P and R have at least $q^2 + 1 + q^2 - q^2 - q > 0$, and $d(P, R) \leq 2$. Thus Γ has diameter at most 2, and $d \leq q$.

Since $d < q+2$, if P is a point and x a block with $P \notin x$, then x contains a point Q joined to P. In the residue of Q, all but one point of x are joined to P; and, in fact, for all but one point R of $x \backslash \{Q\}$, there is a unique block containing P, Q, R, and another point S of $x \backslash \{Q\}$. This allows two possible types of antiflags (x, P):
(i) those having a unique point of x lying on no such block;
(ii) those for which the pairs $\{Q, R\}$ such that no block contains P, Q, and R form a 1-factor (a partition of x).

If (x, P) has type (i), then the triples of points of x lying on a block with P form a Steiner triple system of order $q+1$, so that $q \equiv 0$ or 2 (mod 6). On the other hand, if (x, P) has type (ii), then there is a Steiner triple system of order $q+3$—take the triples as above, together with those obtained by adjoining a new point ∞ to each pair in the distinguished 1-factor—so that $q \equiv 0$ or 4 (mod 6).

Next, we show that d divides q. A block not containing P has at most one point not joined to P. So two points at distance 2 from P are at distance 2 from one another, and (as in (4.2)) Γ is complete multipartite with block size d. But its restriction to the residue of P contains a complete multipartite graph with block size q, establishing the claim.

Set $q = de$. For any point P, there are

$$2d^2e^2(d^2e^2 + d + 2)/(de + 2)$$

antiflags (x, P), of which

$$2(d-1)d^2e^2$$

are of type (i) and the remaining

$$2d^3e^2(de+1)(e-1)/(de+2)$$

are of type (ii). Thus:

$$d = 1 \text{ iff all antiflags are of type (ii);}$$

$$e = 1 \text{ iff all antiflags are of type (i).}$$

For $e = 1$, we get the stated congruence on s from the Steiner triple system. To show non-existence for $s = 5$, let x be a block, G a 'group', with $G \cap x = \{P\}$. For each point $Q \in G \backslash \{P\}$, the blocks containing Q and meeting x in three points define a Steiner triple system on $x \backslash \{P\}$; and any three points of $x \backslash \{P\}$ form a triple in just one of the

five systems (as P varies over $G\backslash\{P\}$). So the triples from a 7-set are partitioned into five Steiner triple systems. The impossibility of such a partition (due to Cayley) is well known. (Unfortunately, no other values of s can be excluded by this argument.)

To finish, we need one further observation. We build a new incidence structure D, the points of which are the 'groups' (of size d) of the complete multipartite graph Γ, and whose collinear triples are the triples of points such that no representatives ae contained in a block of $\mathcal{G}$; in other words, a line of D has the form $X \cup Y$, where X is a 'group', and Y is a 'group' (of size q) for the complete multipartite point graph of the residue of a point of X. Then D is a $2\text{-}(de^2+e+1,\ e+1,\ 1)$ design.

If $d=1$, then D is a projective plane of order $e=q$. The divisibility condition from the antiflag count above gives that $e+2$ divides 24, whence $e=q=2$, 4, 6, 10 or 22. The Bruck–Ryser theorem [4] excludes 6 and 22; the condition $q \equiv 0$ or 4 (mod 6) excludes 2; and the computer result of Lam *et al.* [8] excludes 10, since a block of the EGQ is a hyperoval in the plane.

Finally, suppose that d and e are both greater than 1, so that $q \equiv 0 \pmod 6$. We have that $de+2$ divides $8(d+2)$; and, from counting blocks of D, $e+1$ divides $d(d-1)$. If $q \equiv 0 \pmod 4$, then $de+2 \equiv 2 \pmod 4$, and so $de+2$ dividies $2(d+2)$. Since $e>1$, this forces $de+2=2(d+2)$, giving $d=2$, $e=3$, a contradiction. So $q \equiv 2 \pmod 4$, whence $q=6 \pmod{12}$. Now $de+2$ divides $8(d+2)$, and the quotient is less than 8. Taking each value in turn, there are only finitely many possibilities for d and e, and all except $d=3$, $e=2$ and $d=6$, $e=5$ are excluded by divisibility.

EXAMPLE 4.7. Examples of the situation described in (4.6) are known for $s=1$ and for $s=3$. For $s=1$, see (4.5). The 20-point $EGQ(3,1)$ can be extended twice. The second extension (due to D. R. Hughes) can be described as follows. Let P and Q be points of the Witt design $\mathcal{D}$ (the unique $5\text{-}(24,8,1)$ design). The points of $\mathcal{G}$ are those of $\mathcal{D}$ different from P and Q; the blocks of $\mathcal{G}$ are those of $\mathcal{D}$ containing exactly one of P and Q.

As in the case s even, we turn now to EGQs having more than the minimum number of points. However, rather than establishing an inconclusive result like (3.6), we turn to the analogue of (3.7).

THEOREM 4.8. *Suppose that an $EGQ(s,t)$, with s odd and $s>1$, has the property that, whenever a point P is not on a block x, there is more than one point of x not joined to P. Then*

$$v \geqslant (s+2)(s+1)st/(s-1)+(s+2).$$

If equality holds, then $s=3$, $t=1$ or 9, and the EGQ is a locally polar space of polar rank 2.

PROOF. A similar argument to (3.7) gives the inequality, noting that P is joined to at most $s-1$ points of x.

When equality holds, it is easy to check that the point graph of the EGQ is strongly regular: its valency is $(s+1)(st+1)$; two adjacent vertices have $s(st-2t+1)$ common neighbours, and two non-adjacent vertices have $(s-1)(st+1)$ common neighbours. This graph has an eigenvalue of multiplicity

$$(s+1)+2st(s+1)(s+2)/(st+3)(s-1),$$

which must be an integer; so $st+3$ divides $6(s+1)(s+2)$. Since

$$st+3=(s+1)t-(t-3),$$

$st+3$ divides $6(t-3)(s+2)$, and hence $st+3$ divides $6(2t-3s-9)$. We distinguish three cases:
(i) $2t>3s+9$, $st+3\leqslant 12t-18s-54$;
(ii) $2t<3s+9$, $st+3\leqslant 18s-12t+54$;
(iii) $2t=3s+9$.

In addition, from the number of points of the EGQ, we see that $s-1$ divides $6t$; and, from the necessary condition for a GQ, we know that $s+t$ divides $st(st+1)$, $t\leqslant s^2$, and $s\leqslant t^2$ (if $t>1$). In case (i) we have $s\leqslant 11$, and hence $t\leqslant 121$. In case (ii), $t\leqslant(18s+51)/(s+12)\leqslant 7$, and hence $s\leqslant 43$. In case (iii), $s-1$ divides $9s+27$, so $s-1$ divides 36. So there are only finitely many possible orders (s,t), and the divisibility conditions and inequalities eliminate all cases but $s=3$, $t=1$, 3 or 9.

Finally, by hypothesis, $s=3$ inplies that, if $P\notin x$, then P is joined to exactly two points of x. Hence $\mathscr{G}$ is a locally polar space of polar rank 2. (For more on locally polar spaces, see Buekenhout and Hubaut [6].) The non-existence of an $EGQ(3,3)$ satisfying our hypotheses has been shown recently by Blokhuis and Wilbrink [2].

Example 4.9. $EGQ(3,t)$s satisfying the hypothesis and attaining the bound of (4.8) exist for $t=1$ and for $t=9$. (For $t=1$, the points are the 35 partitions of $\{1,\ldots,8\}$ into two 4-sets, the blocks the 3-subsets of $\{1,\ldots,8\}$; a point and block are incident if the 3-set is contained in a part of the partition. For $t=9$, the points and blocks are the vertices and 5-cliques of McLaghlin's graph [9].) Both examples are unique. (The uniqueness proofs, like the non-existence proof for $t=3$, are given in the context of Zara graphs, for which we refer also to [1] and [15]. We are grateful to the referee for pointing out this connection. The overlap between EGQs and Zara graphs is larger than we have indicated, but it is not our purpose to discuss this here.)

Both examples are known to have further extensions. In each case, there is a one-point extension which is a regular two-graph, and another extension with twice as many points whose point graph is the associated double cover of the complete graph. (See Seidel [10] for more information on two-graphs.)

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