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The singularity obstruction for group splittings ☆

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Abstract

We study an obstruction to splitting a finitely generated group G as an amalgamated free product or HNN extension over a given subgroup H and show that when the obstruction is "small" G splits over a related subgroup. Applications are given which generalise decomposition theorems from low dimensional topology. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A group G is said to split over a subgroup H if it can be decomposed as a non-trivial amalgamated free product $A *_H B$ or an HNN extension $A*_H$. There is an integer valued obstruction to splitting G over a given subgroup H, studied by Scott in [13], denoted $e(G, H)$. If G does split over H then $e(G, H) \geqslant 2$, however the converse is known to be false. Nonetheless in many situations where $e(G, H) \geqslant 2$ it is possible to show that G or some finite index subgroup of G splits over a subgroup related to H. Examples of this phenomenon include Stallings' characterisation of groups with more than one end [15], and the algebraic torus theorem [4]. In the former one starts with an arbitrary group G for which $e(G, 1) \geqslant 2$ and concludes that G splits over some finite subgroup. In the latter theorem one starts with an infinite cyclic subgroup $H < G$ for which $e(G, H) \geqslant 2$ and concludes that either G is an extension of a finite group by a triangle group or splits over some (possibly different) cyclic subgroup.

When $e(G, H) \geqslant 2$ there is another obstruction to splitting G over H, which we will call the singularity obstruction. It is not uniquely defined but depends on a choice of "proper

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H-almost invariant" subset $A \subset G$, (see below) so we will denote it by $\mathcal{S}_A(G, H)$. The singularity obstruction $\mathcal{S}_A(G, H)$ consists of a union of double cosets HFH for some subset $F \subset G$, and if it vanishes for some choice of A we obtain a splitting of G over H.

Scott's original approach to the problem of producing a splitting from a pair $H < G$ with $e(G, H) \geqslant 2$ included the method of passing to a finite index subgroup $G_1 < G$ which contains the subgroup H and avoids all the elements of $\mathcal{S}_A(G, H)$. The subset $G_1 \cap A$ can then be shown to be a proper H-almost invariant subset of G_1 with singularity obstruction given by $G_1 \cap \mathcal{S}_A(G, H)$. By construction this is empty so G_1 splits over H.

In this paper we take a somewhat different approach. Instead of trying to make the singularity obstruction empty we show that if $\mathcal{S}_A(G, H)$ is small in one of two technical senses then we obtain a splitting of G over a subgroup related to H. This result is more closely related to Stallings' theorem [15] and the algebraic torus theorem [4], and potential applications include generalisations of those results.

In order to define the singularity obstruction we recall the definition of an H-almost invariant set:

Let G be a group, and H a subgroup of G. A proper H-almost invariant subset of G is a subset A satisfying the following conditions:

(a) H is the left stabiliser of A.

(b) A is H-almost invariant, i.e., for any element $g \in G$ the symmetric difference $A + Ag$ is H-finite (contained in finitely many right cosets of H).

(c) A is H-proper, i.e., neither A nor $G \setminus A$ is H-finite.

According to Scott [13] if G is a finitely generated group and H is a subgroup of G then the positive integer $e(G, H)$ is at least 2 if and only if G contains a proper H-almost invariant set. According to Dunwoody [1] G splits over H if and only if it contains a proper H-almost invariant set A which also satisfies:

(d) For any element $g \in G$, at least one of the following intersections is empty: $A \cap gA$, $A \cap gA^*$, $A^* \cap gA$, $A^* \cap gA^*$. (Here * denotes the complement of a subset.)

We can now define the singularity obstruction $\mathcal{S}_A(G, H)$ given a proper H-almost invariant subset A. It will consist of precisely those elements of G for which condition (d) fails.

Definition 1. Let H be a subgroup of a group G and let A be a subset of G satisfying conditions (a)–(c). Define the singularity (or splitting) obstruction of the triple (G, H, A) to be the subset $\mathcal{S}_A(G, H) = \{g \in G \mid gA \cap A \neq \emptyset, gA^* \cap A \neq \emptyset, gA \cap A^* \neq \emptyset, gA^* \cap A^* \neq \emptyset\}$.

Clearly $\mathcal{S}_A(G, H) = \emptyset$ if and only if A satisfies condition (d), so the vanishing of the obstruction leads to a splitting of G over H. To see that $\mathcal{S}_A(G, H)$ is of the form HFH for some subset $F \subset G$ we note that the singularity obstruction is invariant under left and right multiplication by elements of the left stabiliser of A.

To a geometric group theorist these definitions seem a little mysterious, however there is now an elegant geometric interpretation of them, given by Sageev in his thesis [11]. To understand it we recall the geometric interpretation of a group splitting given by Bass and Serre [16]:

A finitely generated group G splits over the subgroup H if and only if G acts on a simplicial tree T with no global fixed points, with a single edge orbit and edge stabiliser H.

Starting with a subgroup $H < G$ and a proper H-almost invariant subset A satisfying condition (d) Dunwoody showed directly how to construct a tree on which G acts as above, and thus recovered a splitting of G over H. In his thesis Sageev showed how to generalise Dunwoody's construction starting with any proper H-almost invariant subset A. In place of a tree Sageev constructed a contractible cubical complex X on which G acts and which satisfies Gromov's CAT(0) condition of non-positive curvature. In Dunwoody's theory H stabilises an edge of the tree, and in Sageev's construction there is a natural family of codimension-1 "hyperplanes" stabilised by the conjugates of H. These are in bijective correspondence with the translates of the unordered pair $\{A, A^*\}$ by the action of G and the subgroup H stabilises the hyperplane corresponding to $\{A, A^*\}$. From now on we will usually fail to distinguish between the hyperplanes in the cube complex and the pairs $\{gA, gA^*\}$. Since G acts transitively on the set $\{gA \mid g \in G\}$ there is one orbit of hyperplanes, and Sageev showed that the action satisfied a non-triviality condition, which, as Gerasimov has shown [5], is equivalent to the statement that there is no global fixed point in X.

Maximal cubes of dimension n in Sageev's construction correspond to subsets $\{g_1, \ldots, g_n\}$ in G such that the hyperplanes (g_iA, g_iA^*) all cross one another transversely, where two hyperplanes (g_1A, g_1A^*) and (g_2A, g_2A^*) cross one another transversely if and only if the four subsets $g_1A \cap g_2A, g_1A \cap g_2A^*, g_1A^* \cap g_2A, g_1A^* \cap g_2A^*$ are all non-empty. It follows that the cube complex has dimension 1, and is therefore a tree, if and only if the singularity obstruction vanishes. We obtain a splitting of the group when $\mathcal{S}_A(G, H) = \emptyset$ by applying the Bass–Serre theorem to this tree.

The task of finding a splitting of G now becomes the task of finding a subgroup $H < G$ and a proper H-almost invariant subset A such that $\mathcal{S}_A(G, H)$ is empty.

In this paper we will start with the assumption that we have a finitely generated group G, a subgroup H such that $e(G, H) \geqslant 2$, and a subset $A \subset G$ satisfying conditions (a)–(c) with non-empty splitting obstruction $\mathcal{S}_A(G, H) = HFH$. We will then show how to obtain a splitting of G given the assumption that HFH is "small":

Theorem A. *Let G be a finitely generated group with a subgroup H and a subset A satisfying conditions* (a)–(c)*, with non-empty splitting obstruction HFH for some subset F in G. If the subgroup $\langle HFH\rangle$ is a proper subgroup of G then G splits over a subgroup of $\langle HFH\rangle$.*

To state the second main theorem we need two definitions. Two subgroups H and K are said to be commensurable if the intersection $H \cap K$ has finite index in both H and K. The commensurator of a subgroup $H < G$ is the subgroup consisting of those elements $g \in G$ such that H and H^g are commensurable.

Theorem B. *Let G be a finitely generated group with a finitely generated subgroup H and a subset A satisfying conditions* (a)–(c)*, with non-empty splitting obstruction HFH*

for some finite subset F in G. If the subgroup $\langle HFH\rangle$ lies in the commensurator of H in G then G splits over a subgroup commensurable with H.

The subgroup $\langle HFH\rangle$ is *a priori* infinitely generated even when H is finitely generated and F is finite, however when F is non-empty we will show that it is equal to the subgroup $\langle H, F\rangle$. The requirement that F should be finite in Theorem B is no restriction, since for any finitely generated subgroup H the splitting obstruction can always be expressed as a finite union of double cosets (we give an outline argument for this later).

The proofs of Theorems A and B are given in Section 2, and proceed by analysing the topology of the corresponding cube complex. The conditions of Theorem A ensure the existence of a separating vertex in the cube complex, from which one can construct an essential action on a tree. The conditions of Theorem B ensure that the cube complex has more than one topological end, and one can then apply a result of Dunwoody [2] to obtain a splitting of the group.

In Section 3 we illustrate the theorems by considering the special case of infinite cyclic subgroups of surface groups, interpreting the conditions of the theorems in terms of properties of curves on surfaces, and showing how the induced splittings appear geometrically.

In Section 4 we consider applications of the main results, and in particular generalise a result of Shalen concerning π_1-injective immersions of surfaces in 3-manifolds to obtain:

Theorem C. *Let G be a group with a subgroup H and a subset A satisfying conditions* (a)–(c), *with singularity obstruction HFH for some finite subset F in G. If H is contained in a chain of proper finite index subgroups $G_{|F|+1} < G_{|F|} < \cdots < G_1 = G$ then for some i the subgroup G_i splits over a subgroup of $\langle HF_iH\rangle$ where F_i denotes the subset $F \cap G_i$.*

Again note that if HF_iH is non-empty then $\langle HF_iH\rangle = \langle H, F_i\rangle$. Theorem C is related to Scott's theorem in [13] which uses the stronger hypothesis that H is an intersection of finite index subgroups of G to draw the conclusion that some subgroup G_i splits over H, and to results in [7] concerning HNN extensions.

It should be noted that Theorem B was known to Dunwoody and Roller [3], and an outline proof of it appears in the recent paper by Scott and Swarup [14]. Their approach to constructing group splittings is somewhat different from the one given here, and relies on a combinatorial generalisation of the notion of intersection number for co-dimension 1 immersions rather than the more group theoretic singularity obstruction studied in this paper.

2. The main theorems

We will begin by showing that a CAT(0) cube complex X has a separating vertex if and only if the "transversality graph" associated to X by Roller in [10] is disconnected. We will then show how to use this to collapse the cube complex onto a tree, so that if G is a group acting on the complex then the collapse is equivariant. Assuming the action on the cube complex is essential and the transversality graph is disconnected we obtain an essential action of G on a tree. By the Bass–Serre theorem G will split over an edge stabiliser for this action.

Sageev showed that given a subgroup H in a finitely generated group G which admits a proper H-almost invariant set A there is a CAT(0) cube complex X (which depends for its construction on the choice of A) on which G acts. To obtain Theorem A we will show that the cube complex X has a separating vertex if and only if the singularity obstruction $\mathcal{S}_A(G, H)$ generates a proper subgroup of G. To do so we will analyse the family of "hyperplanes" in the complex. We start with a definition:

Let X be a CAT(0) cube complex and e be an edge in X. The geodesics through the midpoint of e at right angles to it form a totally geodesic codimension 1 subspace which we will call the hyperplane corresponding to e. If C is a cube containing e then the corresponding hyperplane intersects C in the Euclidean hyperplane orthogonal to and bisecting e, it therefore intersects only those edges parallel to C, and then only at their midpoints. It follows that the intersection of a hyperplane $\mathcal{H}$ in X with the cubes of X induces a cubical structure on the hyperplane, and since the hyperplane is totally geodesic this makes $\mathcal{H}$ into a CAT(0) cube complex in its own right. (For a more combinatorial description of this see [9].)

Now let $\mathcal{H}$, $\mathcal{J}$ be hyperplanes in X. We say that $\mathcal{H}$ and $\mathcal{J}$ are nested if they are disjoint or equal, and they are transverse if they are not nested. After Roller [10] we define a graph, called the transversality graph, with vertices the set X_H of hyperplanes of X, and an edge joining two vertices if and only if the corresponding hyperplanes are transverse.

Lemma 2. *Let X be a connected CAT(0) cube complex. Then the following are equivalent*:

(i) *X has a separating vertex.*

(ii) *There is a surjective function $f : X_H \to \{0, 1\}$ such that $f(\mathcal{H}) = f(\mathcal{J})$ whenever $\mathcal{H}$ is transverse to $\mathcal{J}$.*

(iii) *The transversality graph is disconnected.*

Proof. (i) ⇒ (ii): Suppose first that X has a separating vertex v, and let Y be a component of $X \setminus \{v\}$. Define $f(\mathcal{H}) = 1$ for any hyperplane $\mathcal{H}$ which intersects Y and $f(\mathcal{H}) = 0$ for any hyperplane contained in $X \setminus Y$. Since hyperplanes are connected and do not contain vertices of X, f is a well defined on X_H. Furthermore since any hyperplane is contained in either Y or its complement, if $\mathcal{H}$ is transverse to $\mathcal{J}$ then they must both lie in Y or both lie in $X \setminus Y$; it follows that $f(\mathcal{H}) = f(\mathcal{J})$ whenever $\mathcal{H}$ is transverse to $\mathcal{J}$.

(ii) ⇒ (i): Since any edge e intersects exactly one hyperplane $\mathcal{H}_e$ we may define a function (which, abusing notation, we also denote by f) from $X^{(1)}$ to $\mathbb{Z}_2$ by setting $f(e) = f(\mathcal{H}_e)$. Now consider a vertex v of X; its link is the simplicial complex whose 0-cells are the edges of X adjacent to v, 1-cells correspond to squares containing v and so on. It is clear that if two vertices in the link of a vertex v corresponding to hyperplanes $\mathcal{H}$ and $\mathcal{J}$ are joined by an edge then the hyperplanes $\mathcal{H}$ and $\mathcal{J}$ intersect in the corresponding square in the link of v, and so f takes the same value on the corresponding edges of X.

Hence our function f gives rise to a 0-cocycle on the link of v. We now see that if this function takes both values on some link then that link is disconnected and the vertex is local cut point. Since any CAT(0) cube complex is simply connected this will also be a global cut point, so it only remains to show that f must take both values in the neighbourhood of some vertex. If this were not the case then we could define a new function $g : X^{(0)} \to \mathbb{Z}_2$ by setting $g(v) = 0$ if and only if $f(e) = 0$ for some (and hence every) edge e adjacent to v. If two vertices v and w are adjacent along an edge e then $g(v) = f(e) = g(w)$ so g is a 0-cocycle on X, and, since X is connected, g is constant. But every hyperplane is adjacent to some vertex, and f itself is assumed to be not constant, and this is a contradiction.

(ii) $\Leftrightarrow$ (iii): Given a function $f : X_H \to \mathbb{Z}_2$ such that $f(\mathcal{H}) = f(\mathcal{J})$ whenever $\mathcal{H}$ is transverse to $\mathcal{J}$ we obtain a vertex colouring of the transversality graph which is constant on components. It follows that there is such a surjective function if and only if the transversality graph is disconnected. $\square$

The proof of Theorem A now proceeds in two steps, we will show that the hypothesis that the singularity obstruction generates a proper subgroup of G leads to the existence of a separating vertex in the corresponding cube complex X, and use this separating vertex to construct a tree on which G acts essentially. Assume for the moment that the first step has been accomplished. We will show how to construct the required action on a tree T using the existence of the separating vertices in X.

First remove the separating vertices from X and let $\{Y_i\}$ be the collection of components of the complement. Let the vertex set of T be the union of the set of separating vertices of X together with the set $\{Y_i\}$. Vertices of T of the first type will be called type A, and those of the second will be called type B. There is an edge joining a vertex v of type A to a vertex w of type B if and only if the separating vertex in X corresponding to v is adjacent to a hyperplane in w; there are no other edges. Since every vertex of type A is separating T is a tree.

Now, since hyperplanes are connected, and no hyperplane of X contains a vertex of X, any hyperplane intersects (and is contained in) exactly one component Y_i. Furthermore any two hyperplanes which intersect in X must lie in the same component Y_i and intersect there. It follows that the components Y_i are in bijective correspondence with components of the transversality graph of X. This correspondence is easily seen to be a G-map, so the stabiliser of a vertex of type B is just the stabiliser of the corresponding component of the transversality graph. So we have:

Lemma 3 (The collapsing lemma). *Let X be a CAT(0) cube complex with disconnected transversality graph, and let G be a group acting essentially on X with one orbit of hyperplanes. Then there is an essential action of G on a tree T with every edge stabiliser a subgroup of the stabiliser of a component of the transversality graph.*

Proof. We have already constructed the tree T above. The action of the group G on the cube complex X permutes both the separating vertices of X and the components of its transversality graph, and preserves adjacency relations between them, so we obtain an action of G on T. The action respects the decomposition of the vertex set into vertices of type A and type B, and so has no edge inversions. It follows that if there is a fixed point for the action then there is a fixed vertex. Hence the action will be essential if and only if there are no vertices fixed by the entire group. If a vertex of type A is fixed by G then so is the corresponding separating vertex of X, and this contradicts the fact that the action of G on X is essential. If a vertex of type B is fixed by G then G preserves the corresponding component of the transversality graph. But G acts transitively on hyperplanes so in this case there can be only one component in the transversality graph, which is again a contradiction. □

Our next proposition describes the stabilisers of hyperplanes and of components of the transversality graph for a CAT(0) cube complex on which a group acts with one orbit of hyperplanes.

Proposition 4. *Let (G, H, A) be a triple with splitting obstruction $\mathcal{S}_A(G, H) = HFH$, and let X be the associated CAT(0) cube complex. Then the stabiliser $\overline{H}$ of the hyperplane $\{A, A^*\}$ contains H as a subgroup of index at most 2 and if HFH is non-empty then the subgroup $\langle HFH\rangle$ contains $\overline{H}$. The stabiliser of the component of the transversality graph containing $\{A, A^*\}$ is $\overline{H}$ if HFH is empty and $\langle HFH\rangle = \langle H, F\rangle$, otherwise.*

Proof. It is obvious that the stabiliser of the pair $\{A, A^*\}$ contains the stabiliser of A as a subgroup of index at most 2, so $\overline{H}$ contains H as a subgroup of index at most 2. If HFH is non-empty then we may choose an element $k \in HFH$ so that the four intersections $A \cap kA$, $A \cap kA^*$, $A^* \cap kA$, $A^* \cap kA^*$ are all non-empty. Now for any element $g \in \overline{H}$ we have either $gA = A$ and $gA^* = A^*$ or $gA = A^*$ and $gA^* = A$. In either case the four intersections $A \cap kgA$, $A \cap kgA^*$, $A^* \cap kgA$, $A^* \cap kgA^*$ are all non-empty so kg lies in HFH. It follows that $g \in \langle HFH\rangle$ as required.

If HFH is empty then there are no hyperplanes transverse to $\{A, A^*\}$ so the transversality graph is totally disconnected and the stabiliser of the component containing $\{A, A^*\}$ is $\overline{H}$. If HFH is non-empty let $\mathcal{H}$ denote the hyperplane $\{A, A^*\}$. A hyperplane $g\mathcal{H} \neq \mathcal{H}$ lies in the same component of the transversality graph as $\mathcal{H}$ if and only if there is a sequence $e = g_0, g_1, \ldots, g_n = g$ with $g_{i+1}\mathcal{H}$ transverse to $g_i\mathcal{H}$ for $i = 2, \ldots, n$, i.e., if and only if $g_i^{-1}g_{i+1} \in HFH$, if and only if $g_{i+1} \in g_i HFH$. Since we start with $g_1 = e \in H$ it follows that $g\mathcal{H}$ is in the same component of the transversality graph as $\mathcal{H}$ if and only if $g \in \langle HFH\rangle$. Now an element $g \in G$ preserves the component of the transversality graph containing $\mathcal{H}$ if and only if $g\mathcal{H}$ lies in the same component as $\mathcal{H}$, i.e., if and only if $g \in \overline{H}$ or $g \in \langle HFH\rangle$. Since $\overline{H} < \langle HFH\rangle$ we see that the stabiliser is precisely $\langle HFH\rangle$ as required.

Finally since $H < \overline{H} < \langle HFH\rangle$ we have $\langle HFH\rangle = \langle H, F\rangle$. □

Proposition 5. *Let (G, H, A) be a triple with splitting obstruction $\mathcal{S}_A(G, H) = HFH$, and let X be the associated CAT(0) cube complex. The following are equivalent*:

(i) *The transversality graph of X is disconnected.*

(ii) *The subgroup* $\langle HFH\rangle$ *is a proper subgroup of* G.
(iii) *The subgroup* $\langle HFH\rangle$ *is an infinite index subgroup of* G.

Proof. (i) $\Leftrightarrow$ (ii) If HFH is empty then both statements are true since the transversality graph is totally disconnected and there is a bijection between components of the transversality graph and cosets of $\overline{H}$ since G acts transitively on hyperplanes. If HFH is non-empty then by Proposition 4 the subgroup $\langle H, F\rangle$ is the stabiliser of the component of the transversality graph containing $\{A, A^*\}$ and again, since G acts transitively on the set of components of the transversality graph there is a bijection between this set and the set of left cosets of the stabiliser.

(ii) $\Leftrightarrow$ (iii) Both statements are true if HFH is empty and in any case it is obvious that (iii) implies (ii). Suppose now that HFH is non-empty and $\langle HFH\rangle$ is a proper subgroup of G so by the equivalence of (i) and (ii) the transversality graph is disconnected. By the collapsing lemma there is an essential action of G on a tree T with vertices the separating vertices of X together with the components of the transversality graph of X, and as already noted there is a bijection between this latter set and cosets of $\langle HFH\rangle$ in G. If $\langle HFH\rangle$ has finite index in G then there are only finitely many such components, and G has a bounded orbit in its action on T, which contradicts essentiality. □

Theorem A. *Let* G *be a finitely generated group with a subgroup* H, *a subset* A *satisfying conditions* (a)–(c), *with non-empty splitting obstruction* HFH *for some subset* F *in* G. *If the subgroup* $\langle HFH\rangle$ *is a proper subgroup of* G *then* G *splits over a subgroup of* $\langle HFH\rangle$.

Proof. By Proposition 5 the Sageev cubing has a separating vertex, so by Lemmas 2 and 3 we obtain the required essential action of G on a tree T. Edge stabilisers all lie in stabilisers of components of the transversality graph, and according to Proposition 4 these are all conjugate into $\langle HFH\rangle$. It follows that G splits over a conjugate of a subgroup of $\langle HFH\rangle$ and hence over a subgroup of $\langle HFH\rangle$ as required. □

Remark 6. The subgroup of $\langle HFH\rangle$ over which G splits is the stabiliser of an edge in the tree, which is necessarily the intersection of the two vertex stabilisers corresponding to the end points of the edge. One of the vertices has stabiliser conjugate to the subgroup $\langle HFH\rangle$ as remarked above, since it is the stabiliser of a vertex of type B. In the statement of the theorem we have not bothered to remark that the other vertex stabiliser is the stabiliser of a vertex of type A so is the stabiliser of a separating vertex in the complex. Hence we can arrange that the splitting subgroup is actually the intersection of $\langle HFH\rangle$ with the stabiliser of a separating vertex in the cube complex.

We now turn our attention to Theorem B. Again we interpret the stated condition on the singularity obstruction in terms of the topology of the cube complex, and then apply a result of Dunwoody's to obtain the required splitting of the group.

Proposition 7. *Let (G, H, A) be a triple with G and H both finitely generated and with splitting obstruction $\mathcal{S}_A(G, H) = HFH$, and let X be the associated CAT(0) cube complex. Then the following are equivalent*:

(i) *The transversality graph of the cubing is locally finite.*

(ii) *Each hyperplane in the cubing is compact.*

(iii) *The splitting obstruction HFH lies in the commensurator* $\mathrm{Comm}_G H$ *of H in G.*

Proof. (i) $\Leftrightarrow$ (ii) Note that since G acts transitively on hyperplanes the transversality graph is locally finite if and only if some vertex of the graph has finite valency. As remarked above the hyperplane $\mathcal{H}$ is a CAT(0) cube complex whose cell decomposition is given by the intersection of $\mathcal{H}$ with cells of X which it intersects and the hyperplanes of $\mathcal{H}$ are in bijective correspondence with the hyperplanes of X which are transverse to it [9]. It follows that the transversality graph for the cube complex $\mathcal{H}$ is the full subgraph of the transversality graph for X spanned by those vertices adjacent to $\mathcal{H}$. This is finite if and only if the transversality graph of X is locally finite. Now the maximal cells in any CAT(0) cube complex are defined uniquely by the hyperplanes they intersect so maximal cubes are in bijective correspondence with maximal complete subgraphs of the transversality graph. The equivalence of (i) and (ii) is immediate.

(i) $\Rightarrow$ (iii) Note that H is always a subgroup of its own commensurator, $\mathrm{Comm}_G H$, and so the splitting obstruction HFH is a subset of $\mathrm{Comm}_G H$ if and only if $F \subseteq \mathrm{Comm}_G H$. Since H preserves the hyperplane $\{A, A^*\}$ it permutes the hyperplanes transverse to it, and if the transversality graph is locally finite then there are only finitely many of these, so H has a finite index subgroup H_0 which preserves all of them. It follows that the stabiliser of any hyperplane transverse to $\{A, A^*\}$ contains a finite index subgroup of H. Transversality is a symmetric relation so this shows that stabilisers of transverse hyperplanes are commensurable, and so for any $g \in HFH$, the subgroups H and H^g are commensurable.

(iii) $\Rightarrow$ (i) Since H is finitely generated we may add the generators of H to the generating set for G to ensure that the coboundary of the set A is connected in the Cayley graph of G. It follows that translates (gA, gA^*) and (A, A^*) are transverse if and only if their coboundaries intersect. Since the coboundaries are H-finite this ensures that the singularity obstruction is H-finite, so we may assume that F is finite. For each $f \in F$ define $H_f = H \cap H^f$. Since each f lies in the commensurator of H, H_f is a finite index subgroup of both H and H^f, and as F is finite the intersection of these subgroups, which we shall denote H_0, has finite index in H and hence in H^f. For each f choose a left transversal T_f to H_0 in H^f.

Now the set HF is a disjoint union of the sets $ff^{-1}Hf$ as f ranges over F, and each of these sets may be rewritten as fT_fH_0. Letting S denote the union over F of the subsets fT_f we see that HF is contained in SH_0 and hence $HFH \subseteq SH$. It follows that any g for which $\{gA, gA^*\}$ is transverse to $\{A, A^*\}$ can be written in the form sh for some $s \in S$ and $h \in H$. But since H preserves the hyperplane $\{A, A^*\}$, there are at most $|S|$ distinct hyperplanes transverse to $\{A, A^*\}$. As G acts transitively on the hyperplanes the transversality graph is locally finite. $\square$

Theorem B. *Let G be a finitely generated group with a finitely generated subgroup H, and a subset A satisfying conditions* (a)–(c), *with non-empty splitting obstruction HFH for some finite subset F in G. If the subgroup $\langle HFH\rangle$ lies in the commensurator of H in G then G splits over a subgroup commensurable with H.*

Proof. By Proposition 7 the hyperplane $\{A, A^*\}$ is compact so it intersects only finitely many edges. It follows that the 1-skeleton of the cube complex is separated by these edges, and, since the action is essential there are translates of this cut lying arbitrarily far from it on either side (this was Sageev's original definition of essentiality in [11]). It follows that the 1-skeleton of the cube complex is a graph with more than one end. According to Dunwoody [2] given any group G acting with unbounded orbits on a graph Γ with more than one end we obtain a splitting of the group G over the stabiliser of some finite subset of the edges of Γ. Applying this result to the action of G on the 1-skeleton of the cube complex X we obtain a splitting of G over a subgroup commensurable with the stabiliser of an edge and hence commensurable with the corresponding hyperplane stabiliser. By Proposition 4 this is (a conjugate of) at most an index 2 extension of H. □

The result in [2] used above can be viewed as a generalisation of Stallings' theorem concerning finitely generated groups with more than one end. In a more recent paper [8]. I give a new geometric proof of Stallings' theorem which can be adapted to prove Dunwoody's result as well using methods similar to those used here.

3. Example

In this section we will illustrate Theorems A and B by considering the special case of an infinite cyclic subgroup of the fundamental group of a closed orientable surface Σ of genus at least 2. It is technically convenient, but not necessary, to equip Σ with a metric of constant negative curvature so that its universal cover is the hyperbolic plane $\mathbb{H}^2$ and the fundamental group G of Σ acts by orientation preserving isometries on $\mathbb{H}^2$. Let $H = \langle h\rangle$ be a maximal cyclic subgroup of G; we will first illustrate how to construct a proper H-almost invariant subset $A \subset G$ from this action, and then show how to interpret the conditions of Theorems A and B in terms of it.

The subgroup H acts freely and properly discontinuously on the universal cover $\mathbb{H}^2$ of Σ and there is a geodesic line ℓ in $\mathbb{H}^2$ along which h translates the plane. This line cuts $\mathbb{H}^2$ into two infinite components $\ell^{\pm}$, and since G acts cocompactly on $\mathbb{H}^2$, given any point $p \in \mathbb{H}^2$ there are points in the orbit of p on both sides of the line.

Choose a point p in $\mathbb{H}^2$ such that none of the points $g(p)$ lie on ℓ for any $g \in G$, and let $A = \{g \in G \mid g(p) \in \ell^+\}$. Since the action of G preserves orientation on $\mathbb{H}^2$ no element preserving ℓ switches sides, so the stabiliser of ℓ is exactly the stabiliser of ℓ^+ and is equal to the left stabiliser of the subset A. By construction the stabiliser of ℓ in G contains H, and since G acts freely and properly discontinuously the stabiliser is infinite cyclic. Since H is a maximal infinite cyclic subgroup it is the line stabiliser, and therefore the left stabiliser of A. We claim that A is a proper H-almost invariant set.

First note that for any element $x \in G$ the set $A + Ax$ consists of those elements $g \in G$ such that exactly one of the elements g or gx^{-1} lies in A, i.e., $g \in A + Ax$ if and only if every path in $\mathbb{H}^2$ joining $g(p)$ to $gx^{-1}(p)$ crosses the line ℓ. Equivalently $g \in A + Ax$ if and only if the line $g^{-1}(\ell)$ crosses every path from p to $x^{-1}(p)$. Now pick a path from p to $x^{-1}(p)$; since G acts properly discontinuously on $\mathbb{H}^2$ and discretely on the translates of ℓ there are only finitely many translates $g^{-1}(\ell)$ which intersect this path and so g^{-1} lies in one of finitely many left cosets of H. It follows that g is contained in finitely many right cosets of H, so $A + Ax$ is H-finite for each $x \in G$.

If A is H-finite then the set $\{g(p) \mid g \in A\}$ is contained in the set $\{h^n x_i(p) \mid n \in \mathbb{Z}\, x_i \in S\}$ for some finite set $S \subset G$. It follows that every translate of p lying in ℓ^+ lies in a bounded neighbourhood of the line ℓ, where the bound is given by the maximum distance from the points $x_i(p)$ to ℓ. A simple inspection shows that there are points arbitrarily far from ℓ in ℓ^+ and since G acts co-compactly on the plane $\mathbb{H}^2$ every point in $\mathbb{H}^2$ is uniformly close to the orbit of p and this is a contradiction. The same argument shows that A^* is also not H-finite, so A is a proper H-almost invariant set.

Now consider the splitting obstruction $\mathcal{S}_A(G, H)$. Note that if a translate $g\ell$ is disjoint from ℓ then the two lines cut the plane $\mathbb{H}^2$ into exactly three components, and it follows that at least one of the subsets $A \cap g^{-1}A$, $A \cap g^{-1}A^*$, $A^* \cap g^{-1}A$, $A^* \cap g^{-1}A^*$ must be empty (exactly which is empty depends on which of the four subsets $\ell^\pm \cap g\ell^\pm$ is empty). On the other hand, since ℓ is a geodesic, if ℓ is not equal to $g\ell$ but intersects it then the intersection is a single point and the two lines cut $\mathbb{H}^2$ into four infinite pieces. Each of the components must contain at least one translate of p and so the four subsets $A \cap g^{-1}A$, $A \cap g^{-1}A^*$, $A^* \cap g^{-1}A$, $A^* \cap g^{-1}A^*$ are all non-empty in this case. It follows easily that an element $g \in G$ lies in the splitting obstruction $\mathcal{S}_A(G, H)$ if and only if the lines ℓ and $g\ell$ cross (i.e., they intersect but are not equal).

Since H acts co-compactly on ℓ it has a compact fundamental domain and since G acts properly discontinuously on $\mathbb{H}^2$ there are only finitely many lines $g\ell$ which cross the fundamental domain. Hence there is a finite set $F \subset G$ such that the only lines crossing ℓ in the fundamental domain are the lines in $F\ell$. A translate $g\ell$ lies in the set $F\ell$ if and only if $g \in FH$ and a line $g'\ell$ crosses ℓ if and only if $h^n g'\ell$ crosses the fundamental domain for some n, and so $g'\ell$ crosses ℓ if and only if $g' \in HFH$.

Adapting the argument in Proposition 4 it is easy to see that the subgroup $\langle HFH\rangle$ is the stabiliser of a component of the union of translates of ℓ under the action of G, so $\langle HFH\rangle$ is a proper subgroup of G if and only if the set of lines $\{g\ell \mid g \in G\}$ is disconnected; in this case we can obtain a splitting geometrically as follows.

Let N denote the closure of an ε neighbourhood of $G\ell$ in $\mathbb{H}^2$. The components of N are in bijective correspondence with the left cosets of $\langle HFH\rangle$ and choosing ε to be sufficiently small we may ensure that any element $g \in G$ taking N to intersect itself must take a translate of ℓ in N to intersect another such translate. The component N_0 stabilised by $\langle HFH\rangle$ then projects to a compact embedded subsurface in Σ. If any component of the complement of the image is a disc then this disc lifts to a union of discs in the universal cover. Any of these discs which are incident with N_0 must have their entire boundary in N_0 otherwise some other component of N will intersect the boundary of N_0, which is a contradiction. So we may expand N_0 to include these discs and obtain a new surface which projects to a subsurface of Σ and such that the inclusion of this subsurface is π_1-injective. If the inclusion is surjective then the stabiliser of N_0 must surject on G which it does not do, so the image of N_0 is a proper π_1-injective subsurface which carries the subgroup $\langle HFH\rangle$. Choosing any component of the boundary of this subsurface, Van Kampen's theorem yields a splitting of G over the corresponding cyclic subgroup as an amalgamated free product or an HNN extension.

One situation where HFH is guaranteed to be a proper subgroup of G is when its rank is less than or equal to $2g-1$. Since $\langle HFH\rangle = \langle H, F\rangle$ and H is cyclic, this is guaranteed when $|F| \leqslant 2g-2$. But we have seen that $|F|$ measures the number of self intersection points of the curve, so we get examples whenever we choose a curve with self intersection number at most $2g-2$. An alternative way to view this is that a curve with self intersection number less than $2g-2$ cannot be a filling curve, which can be easily established using an Euler characteristic argument.

To apply Theorem B we need to know that the splitting obstruction HFH lies in the commensurator of the cyclic subgroup H. Since Σ is a surface of genus $g \geqslant 2$ and H is a maximal cyclic subgroup, it is its own commensurator. The assumption that the splitting obstruction $\mathcal{S}_A(G,H)$ lies in the commensurator is therefore equivalent to the assertion that it is empty, i.e., that the image of ℓ in Σ is an embedded curve. Applying Van Kampen's theorem we obtain a splitting of G over H as required. To obtain a more interesting example we choose an element $k \in G$ which has infinite order but does not generate a maximal cyclic subgroup, for example $k = h^2$. We can then deform the geodesic line ℓ to a quasi-geodesic ℓ' with stabiliser exactly $K = \langle k\rangle$, and repeat the construction above to obtain a proper K-almost invariant subset $B \subset G$. Now the commensurator of K is precisely H, so if the singularity obstruction $\mathcal{S}_B(G,K)$ lies in the commensurator of K it lies entirely in H. Since ℓ is a geodesic, any element $g \in G$ which takes ℓ to cross itself must also take ℓ' to cross itself, since crossing is determined by the linking of the end points of the geodesic and its translate, and the quasi-geodesic shares the same end points. It follows that $\mathcal{S}_A(G,H) \subset \mathcal{S}_B(G,K) \subset H$, and again we see that the image of ℓ in Σ is an embedded curve yielding a splitting of G over H, which, as required, is commensurable with $\langle k\rangle$.

4. Applications

As an application of Theorem A we have the following generalisation of a result of Shalen's concerning surface subgroups of the fundamental group of an aspherical 3-manifold. Briefly it asserts that if a subgroup H with $e(G,H) \geqslant 2$ is contained in sufficiently many finite index subgroups of G then G has a finite index subgroup which splits. This is analogous to Shalen's theorem which asserts that if an immersed incompressible surface in a 3-manifold lifts by degree 1 to sufficiently many finite covers of the 3-manifold then the 3-manifold has a finite cover which contains an embedded incompressible surface. The connection is spelt out in [7].

Theorem C. *Let G be a group with a subgroup H and a subset A satisfying conditions* (a)–(c)*, with singularity obstruction HFH for some finite subset F in G. If H is contained in a chain of proper finite index subgroups $G_{|F|+1} \subset G_{|F|} \subset \cdots \subset G_1 = G$ then for some i the subgroup G_i splits over a subgroup of $\langle H, F_i\rangle$ where F_i denotes the subset $F \cap G_i$.*

Proof. For each i let A_i denote the intersection of A with G_i. Since H is contained in all the subgroups G_i and each G_i has finite index in G, the triple (G_i, H, A_i) satisfies conditions (a)–(c). It is easy to see that the splitting obstruction $\mathcal{S}_{A_i}(G_i, H)$ takes the form HF_iH, where $F_i = F \cap G_i$. If $F_i = F_{i+1}$ for some i, then the subgroup $\langle H, F_i\rangle$ is a proper subgroup of G_i, since it lives inside G_{i+1}, and we can apply Theorem A to obtain a splitting of G_i over a subgroup of $\langle H, F_i\rangle$. If on the other hand F_{i+1} is always a proper subset of F_i, then $F_{|F|+1}$ is the empty set, and $G_{|F|+1}$ splits over H. □

As an application of Theorem B we have:

Proposition 8. *Let G be a group with a subgroup H and a subset A satisfying conditions* (a)–(c)*, and suppose that $AH = A$. If there is no subgroup H' of infinite index in H and subset B in G for which the triple (G, H', B) satisfies conditions* (a)–(c) *then G splits over a subgroup commensurable with H.*

Proof. According to Kropholler [6] the fact that A is right invariant under H implies that for any element $g \in \mathcal{S}_A(G, H)$, the subgroup $H \cap H^g$ admits a subset $B \subset G$ satisfying conditions (a)–(c). It follows from the hypotheses that for any such g the subgroup $H \cap H^g$ has finite index in H. Since the splitting obstruction is closed under inversion the subgroup $H \cap H^{g^{-1}}$ also has finite index in H and so any element in $\mathcal{S}_A(G, H)$ lies in the commensurator of H in G. Now by Theorem B the group G splits over a subgroup commensurable with H. □

This is a special case of:

Conjecture 9. If (G, H, A) is a triple satisfying conditions (a)–(c) and also $A = AH$, then G splits over a subgroup commensurable with a subgroup of H.

This conjecture has been verified in the case when H is polycyclic [3], or a quasi-convex subgroup of a word hyperbolic group [12]. Proposition 8 was also noted independently in [14]. It immediately gives:

Corollary 10. *Let G be the fundamental group of a closed aspherical* 3*-manifold and H a subgroup of G isomorphic to the fundamental group of a closed surface. If there is a subset $A \subset G$ satisfying conditions* (a)–(c) *and also $AH = A$ then G splits over a subgroup commensurable with H.*

Proof. A subgroup H' of infinite index in H is free, and an aspherical open 3-manifold with free fundamental group has one end, so the quotient of the universal cover of the 3-

manifold M by a subgroup of infinite index in H has one end. Since the action of G on the universal cover is co-compact the number of ends of this quotient is the same as the number of ends of the quotient of the Cayley graph of G under the action of H', and so according to Scott [13] equal to $e(G, H')$. Since $e(G, H') = 1$ there is no subset B such that the triple (G, H', B) satisfies conditions (a)–(c). It follows from Proposition 8 that the group G splits over a subgroup commensurable with H. □

It is worth remarking in passing that this corollary can be generalised to the situation of co-dimension 1 subgroups of Poincaré duality groups.

Finally we consider the situation of pairs of subgroups $H, K < G$ stabilising subsets A, B in G, respectively, both satisfying conditions (a)–(c) for their corresponding stabilisers. As Sageev remarks in his thesis [11], it is possible to use the subsets A and B together with their complements and all the translates of these sets to construct a cube complex X on which G acts with two orbits of hyperplanes. In general we expect neither system to give rise to a splitting of G, however the arguments in Section 2 may be adapted with little effort to show the following:

Theorem D. *Let G be a finitely generated group and suppose that G has two subsets A and B, with stabilisers H and K respectively, satisfying conditions* (a)–(c) *for their respective stabilisers and with splitting obstructions HFH and $KF'K$, respectively. Assume furthermore that for any element $g \in G$ at least one of the four intersections $A \cap gB, A \cap gB^*, A^* \cap gB, A^* \cap gB^*$ is empty. Then G splits over subgroups of $\langle HFH\rangle$ and of $\langle KF'K\rangle$.*

The result is analogous to the observation that if an irreducible 3-manifold contains disjoint immersed incompressible surfaces then it contains an embedded incompressible surface. In the case of 3-manifolds the proof proceeds by taking a regular neighbourhood of one of the surfaces and examining its boundary components. Standard surgery techniques and applications of Dehn's lemma yield an incompressible surface since the components of the complement of these bounding surfaces cannot all be handlebodies, containing as they do immersed incompressible surfaces. Another way to state the 3-manifold result is that any immersed incompressible surface in a non-Haken 3-manifold must have complement a union of handlebodies.

To prove Theorem D one shows that as in Theorem A the hypotheses ensure that the corresponding cube complex has a separating vertex and then uses that fact to build a Bass–Serre tree as before. To check that the edge stabilisers lie in $\langle HFH\rangle$ or $\langle KF'K\rangle$ one notices that these are precisely the stabilisers of components of the transversality graph.

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