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New sufficient conditions for bipancyclic bipartite graphs

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Abstract

We give here two sufficient conditions for a bipartite balanced graph of order $2n$ to be bipancyclic. The first one concerns graphs that satisfy a "bipartite Ore's condition", that is graphs such that any two nonadjacent vertices in both parts of the bipartition have degree sum at least n, and the second one is for bipartite balanced traceable graphs containing an hamiltonian path whose extremities are nonadjacent and have degree sum at least $n+1$.

Keywords: Graphs; Bipartite; Cycles; Pancyclic; Closure

1. Introduction and notations

We consider finite undirected graphs without loops or multiple edges. Given a graph G, we denote by $V(G), E(G)$, respectively, the sets of vertices and edges of G. For $A \subseteq V(G)$, $G[A]$ is the subgraph of G induced by A; for $x \in V(G)$, $N_A(x) = \{v \in A\colon\ vx \in E(G)\}$ and $d_A(x) = |N_A(x)|$; for $A = V(G)$, we often write $N(x)$ and $d(x)$. The notation $G \cup H$ means the disjoint union of the two graphs G and H (in particular $2G = G \cup G$), and $G + H$ the disjoint union of G and H plus all the edges between G and H. For any integer l, we denote by C_l a cycle of length l. If $C = c_1c_2\cdots c_lc_1$, $l \geqslant 3$, is a cycle (represented by the sequence of the vertices passed through), let $C[c_i, c_j]$ be the path $c_ic_{i+1}\cdots c_j$, and $C^-[c_i, c_j]$ the path $c_ic_{i-1}\cdots c_j$, where the indices are taken modulo l. For a subset S of $V(C)$, S^+ (S^-) denotes the set of the successors (predecessors) of S on C according to the orientation induced by the increasing subscripts. For two vertices u and v, a (u, v)-path is a path connecting u and v, and a hamiltonian (u, v)-path is a path connecting u and v containing all the vertices of $V(G)$. Given any (u, v)-path P and two vertices a and b of P, we will also write $P[a, b]$ for the subpath of P between a and b, including a and b.

The graph G is called *hamiltonian* if it contains a cycle through all the vertices of $V(G)$ and *pancyclic* if it contains cycles of every length between 3 and $|V(G)|$.

G is said to satisfy property P_k if any two nonadjacent vertices of G have degree sum at least k and the k-*closure* of G, $Cl_k(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pair remains.

A bipartite graph G with edge-set $E(G)$ will be denoted by $G = (V_1, V_2, E(G))$ where V_1 and V_2 are the two classes of the bipartition. Moreover G is said to be balanced if $|V_1| = |V_2|$.

Given a bipartite balanced graph $G = (V_1, V_2, E(G))$, we say, as above, that G is *hamiltonian* if it contains a cycle through all its vertices and *bipancyclic* if it contains cycles of every even length between 4 and $|V(G)|$.

Also G is said to satisfy property BP_k if any two nonadjacent vertices x and y with $x \in V_1$ and $y \in V_2$ have degree sum at least k and the k-*biclosure* of G, $BCl_k(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices that are not in the same part of the bipartition and whose degree sum is at least k until no such pair remains.

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For such a bipartite balanced graph G we define a *balanced independent* set of G as an independent subset S of $V(G)$ such that $|S \cap V_1| = |S \cap V_2|$. The *bipartite independence number* $\alpha_{\text{bip}}(G)$ of a balanced bipartite graph G is the order of a largest balanced independent set of G. We denote by $\bar{G}$ the complement of G with respect to $K_{|V_1|,|V_2|}$. If $G=(V_1,V_2,E(G))$ and $H=(V_1',V_2',E(H))$, then their disjoint union $G\cup H$ is the bipartite graph $(V_1\cup V_1', V_2\cup V_2', E(G)\cup E(H))$, and $G+H$ is the disjoint union of G and H plus all the edges between V_1 and V_2' and between V_1' and V_2. These last definitions are valid even if G and H are not balanced and they are used also in the "degenerated" case $V_1=\emptyset$ or $V_2=\emptyset$. Other notations and terminology can be found in [7].

In Section 2, at first we recall some well-known results concerning hamiltonicity and pancyclicity of graphs of order n in relation with property P_n (i.e. Ore's condition) and closures Cl_n and Cl_{n+1}. Those general results have a "bipartite version" for balanced bipartite graphs of order $2n$ considering property BP_{n+1} and biclosures BCl_{n+1} and BCl_{n+2}. We then give two new sufficient conditions for a bipartite balanced graph to be bipancyclic. The first one (Theorem 11) is obtained as a corollary of a characterization of bipartite balanced graphs that satisfy Property BP_k, $1\leqslant k\leqslant n+1$ (Theorem 10) and the second one concerns bipartite balanced graphs that are traceable with degree condition on both extremities of a hamiltonian path (Theorem 12).

In Sections 3, 4 and 5, we give the proofs of Theorems 10, 11 and 12, respectively.

2. Results

Let us first recall the well-known Ore and Bondy's results about property P_n.

Theorem 1 (Ore [15]). *Let G be a graph of order n satisfying property P_n. Then G is hamiltonian.*

Theorem 2 (Bondy [4]). *Let G be a graph of order n satisfying property P_n. Then G is either pancyclic or the bipartite complete graph $K_{n/2,n/2}$.*

As a generalization of Theorem 1, Bondy and Chvátal proved

Theorem 3 (Bondy and Chvátal [6]). *A graph G of order n is hamiltonian if and only if $Cl_n(G)$ is hamiltonian.*

There is no analogous result for pancyclicity but if we assume the closure to be complete, we obtain

Theorem 4 (Faudree et al. [9]). *Let G be a graph of order n such that $Cl_{n+1}(G)=K_n$. Then G is pancyclic.*

Considering now bipartite balanced graphs of order $2n$, we get the analogous results replacing property P_n by BP_{n+1}.

Theorem 5 (Moon and Moser [14]). *Let G be a bipartite balanced graph of order 2n satisfying property BP_{n+1}. Then G is hamiltonian.*

Theorem 6 (Bagga and Varma [3]). *Let G be a bipartite balanced graph of order 2n satisfying property BP_{n+1}. Then G is bipancyclic.*

Concerning biclosure, we also obtain analogous results to Theorems 3 and 4 as follows.

Theorem 7 (Bondy and Chvátal [6]). *A bipartite balanced graph G of order 2n is hamiltonian if and only if $BCl_{n+1}(G)$ is hamiltonian.*

Theorem 8 (Amar et al. [1]). *Let G be a bipartite balanced graph of order 2n such that $BCl_{n+2}(G)=K_{n,n}$. Then G is bipancyclic.*

In [9], Faudree et al. studied the structure of graphs of order n that satisfy P_k for some integer k, $1\leqslant k\leqslant n$ and obtained the following characterization.

Theorem 9 (Faudree et al. [9]). *Let G be a graph of order $n\geqslant 4$ that satisfies property P_k for some integer $k, 1\leqslant k\leqslant n$. Then $Cl_{k+1}(G)=K_n$ or G has one of following two forms*:

(i) *$k\geqslant n-2$ and G is isomorphic to $K_{k+2-n}+(K_r\cup K_{2n-k-2-r})$ or to $\bar{K}_{k+2-n}+(K_r\cup K_{2n-k-2-r})$ for some integer r with $1\leqslant r\leqslant 2n-k-3$.*
(ii) *k is even and G is isomorphic to A+C where A is any graph of order a with $0\leqslant a\leqslant k/2$ and C is any $(k/2-a)$-regular graph of order $n-a$.*

In this paper, we consider bipartite balanced graphs of order $2n$ that satisfy property BP_k for some integer k, $1 \leqslant k \leqslant n+1$, and show that such graphs whose $(k+1)$-biclosure is not complete have a structure belonging to one of the four cases described below.

Theorem 10. *Let $G=(V_1,V_2,E(G))$ be a bipartite balanced graph of order $2n$ satisfying property BP_k for some integer k, $1 \leqslant k \leqslant n+1$. Then $BCl_{k+1}(G)=K_{n,n}$ except in the following cases:*

1. *$k=n$ and G is isomorphic to $K_{a,a}\cup K_{n-a,n-a}$ for some a, $(n-1)/2 < a \leqslant n-1$.*
2. *$k=n+1$ and for some a, $1 \leqslant a \leqslant n-1$, G is isomorphic either to $K_{2,0}+(K_{a-1,a}\cup K_{n-a-1,n-a})$ or to $K_{1,1}+(K_{a-1,a-1}\cup K_{n-a,n-a})$ or to $\bar{K}_{1,1}+(K_{a-1,a-1}\cup K_{n-a,n-a})$.*
3. *k is even and G is isomorphic to $A+C$ where C is a $(k/2-a)$-regular balanced bipartite graph and A is a balanced bipartite graph on $2a$ vertices, $0 \leqslant a \leqslant k/2$.*
4. *There exists some positive integer $\gamma < k/2$ and disjoint subgraphs Γ_1 and Γ_2 of G satisfying $1 \leqslant |V(\Gamma_2)\cap V_1| \leqslant k-\gamma$ and $|V(\Gamma_2)\cap V_2| \leqslant \gamma$ such that G is isomorphic to $\Gamma_1+\Gamma_2$ and the vertices of Γ_1 satisfy the degree condition in G*

$$d_G(x)=\begin{cases}\gamma & \text{if } x\in V(\Gamma_1)\cap V_1,\\ k-\gamma & \text{if } x\in V(\Gamma_1)\cap V_2.\end{cases}$$

Using Theorem 10 and some results of Amar et al. [2], we then prove

Theorem 11. *If a bipartite balanced graph G of order $2n$, $n \geqslant 6$, satisfies property BP_n, then G is bipancyclic or isomorphic to $\bar{K}_{\gamma,n-\gamma}+\Gamma$ for some integer $\gamma \leqslant n/2$, where the bipartite graph Γ contains $\bar{K}_{n-\gamma,\gamma}$ as a spanning subgraph.*

Notice that Theorem 11 has Theorem 6 as a corollary.

We also obtain another sufficient condition for bipancyclicity as follows:

Theorem 12. *If a bipartite balanced graph $G=(V_1,V_2,E(G))$ on $2n$ vertices contains a hamiltonian path connecting two nonadjacent vertices $u\in V_1$ and $v\in V_2$ such that $d(u)+d(v)\geqslant n+1$, then G is bipancyclic.*

If $d(u)\geqslant (n+1)/2$, u is contained in a C_4 and for every k, $3\leqslant k\leqslant n$, there exists some C_{2k} that contains both u and v.

This last theorem is in fact the "balanced bipartite result" corresponding to the following one proved by Faudree et al. [10].

Theorem 13 (Faudree et al. [10]). *Let G be a graph of order n containing a hamiltonian (u,v)-path for a pair of nonadjacent vertices u and v such that $d_G(u)+d_G(v)\geqslant n$. Then G is pancyclic. If $d(u)\geqslant n/2$, u is contained in a C_3 and for every k, $4\leqslant k\leqslant n$, there exists some C_k that contains both u and v.*

3. Proof of Theorem 10

Suppose that $H=BCl_{k+1}(G)\neq K_{n,n}$. Then, by BP_k for G, $n\geqslant 2$ and the graph H satisfies the following property denoted by ($\bigstar$):

($\bigstar$) $d_H(x)+d_H(y)=k$ for every nonedge (xy) in H with x in V_1 and y in V_2.

Let A,B,C denote the subsets of vertices with degree in H, respectively, strictly greater than, strictly less than and equal to $k/2$. For $i=1,2$ put $A_i=V_i\cap A$, $B_i=V_i\cap B$, $C_i=V_i\cap C$ and a_i,b_i,c_i their respective cardinalities.

First of all, we notice that the bipartite subgraphs induced in H by A and B are complete since two nonadjacent vertices $x\in A_1$, $y\in A_2$ ($x\in B_1$, $y\in B_2$) have a degree-sum greater than k (less than k), respectively. Analogous arguments imply that the vertices of $A\cup B$ are adjacent in H to the vertices of C that are not in the same part of the bipartition. Consequently, H contains $(K_{a_1,a_2}\cup K_{b_1,b_2})+\bar{K}_{c_1,c_2}$ as a spanning subgraph. We deduce that if $C_1\neq\emptyset$ then, by the definition of C_1, we have $a_2+b_2\leqslant k/2$ and thus $c_2\geqslant n-k/2\geqslant n-(n+1)/2\geqslant \frac{1}{2}$, i.e. $c_2\geqslant 1$. In other words $C_2\neq\emptyset$. Analogously, $C_2\neq\emptyset$ implies $C_1\neq\emptyset$. Moreover if $C_1=C_2=\emptyset$ then at least one of B_1 and B_2 is not empty otherwise H would be complete.

Without loss of generality, one of the following three cases occurs:

Case 1: $C_1\neq\emptyset$ and $C_2\neq\emptyset$ (k is even).

By the definition of C_1 and C_2 we have $a_i + b_i \leqslant k/2$ and thus $c_i \geqslant n - k/2$ for $i = 1, 2$.

Subcase 1.1: $1 \leqslant k \leqslant n$.

Since, for $i=1,2$, $c_i \geqslant n-k/2 \geqslant k/2$, then necessarily $B=\emptyset$ (if not, every vertex in B_i would be adjacent to at least $k/2$ vertices in C_j, $j \neq i$). This implies that for every x in C_1 and y in C_2 we have $d_C(x)=k/2-a_2$ and $d_C(y)=k/2-a_1$. By considering the number of edges between C_1 and C_2 we obtain $(n-a_1)(k/2-a_2)=(n-a_2)(k/2-a_1)$, whence $a_1=a_2$ and $c_1=c_2$. So H is isomorphic to $K_{a,a}+C^*$, where C^* is a $(k/2-a)$-regular bipartite graph of order $2(n-a)$, $0 \leqslant a \leqslant k/2$.

Subcase 1.2: $k = n + 1$.

- If $B=\emptyset$ then, by similar argument, we obtain H isomorphic to $K_{a,a}+C^*$ where C^* is a $((n+1)/2-a)$-regular bipartite graph of order $2(n-a)$, $0 \leqslant a \leqslant (n+1)/2$.
- If $B_1 \neq \emptyset$ then $b_2+c_2 \leqslant (n-1)/2$. But since $c_2 \geqslant (n-1)/2$, then necessarily $b_2=0$, $c_2=(n-1)/2$ and $a_2=(n+1)/2$. By considering the degree of vertices of C_1 and B_1, we deduce that $E_H[C_1, C_2] = E_H[B_1, A_2] = \emptyset$.

Given y in A_2, y has degree $a_1 + c_1$ but also $k - (n-1)/2$ since y has no adjacency in B_1 and every vertex in B_1 has degree $(n-1)/2$, whence $d_H(y) = a_1 + c_1 = (n+3)/2$. Moreover, every vertex in C_2 has exactly $a_1 + b_1$ neighbors and so $a_1 + b_1 = (n+1)/2$, $c_1 = (n-1)/2$, $a_1 = 2$ and $b_1 = (n-3)/2$. The graph H is isomorphic to $K_{2,0} + (K_{(n-3)/2,(n-1)/2} \cup K_{(n-1)/2,(n+1)/2})$.

Case 2: $C=\emptyset$, $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$.

H contains the spanning subgraph $K_{a_1,a_2} \cup K_{b_1,b_2}$ with $a_i + b_i = n$ for $i = 1, 2$. Without loss of generality, assume that $a_1 + b_2 \leqslant n \leqslant a_2 + b_1$. By considering the degree of the vertices of H, we deduce that $1 \leqslant b_i < (n+1)/2$ and thus $(n-1)/2 < a_i \leqslant n-1$, $i = 1, 2$. If $a_2 + b_1 \geqslant n+2$ then $d_H(x) + d_H(y) \geqslant a_2 + b_1 \geqslant k + 1$ for every x in A_1 and y in B_2, whence y is adjacent to every vertex in A_1 and hence it has degree n, a contradiction. Therefore $n \leqslant a_2 + b_1 \leqslant n + 1$.

Subcase 2.1: $a_2 + b_1 = n$.

Then $a_1 = a_2 = a$ and $b_1 = b_2 = b$.

- If $k \leqslant n-1$ then $BCl_{k+1}(K_{a,a} \cup K_{n-a,n-a}) = K_{n,n}$ and thus $H = K_{n,n}$, a contradiction. Therefore $k \geqslant n$.
- If $k = n$ then $H = K_{a,a} \cup K_{n-a,n-a}$ (H cannot have additional edges, otherwise, by (★), we would get a vertex $x \in B$ with $d_H(x) = b + a = n > n/2$, a contradiction).
- If $k = n+1$ then $b \leqslant d_H(y) \leqslant b+1$ for every y in B_1. Otherwise, if $d_H(y) \geqslant b+2$ for some y in B_1 then, since $d_H(y) < k/2 \leqslant n$, there exists some x in A_2 nonadjacent to y and $d_H(x) + d_H(y) \geqslant a + b + 2 = k + 1$, a contradiction with (★).

Suppose now there exists some y in B_1 (or B_2) such that $d_H(y) = b$. Then, by (★), $d_H(x) = a + 1$ for every x in A_2 and thus necessarily $b \geqslant 2$ and x is adjacent to some y' in $B_1 - \{y\}$. Because of $d_H(y') < k/2 \leqslant n$ there is an x' in A_2 nonadjacent to y'. Then, by (★), $d_H(x')=n+1-d_H(y') \leqslant n+1-(b+1)=a$, a contradiction. We then have $d_H(y)=b+1$ for every y in B_1. Hence the vertices of B_1 are adjacent to the same vertex x of A_2, for otherwise if $y_1x_1 \in E(H)$ and $y_2x_2 \in E(H)$ with $x_1 \neq x_2$ then $d_H(y_1) + d_H(x_2) \geqslant b + 1 + a + 1 = k + 1$, a contradiction with (★) since $y_1x_2 \notin E(G)$. Analogously with B_2 instead of B_1 and so $H = K_{1,1} + (K_{a-1,a-1} \cup K_{n-a,n-a})$.

Subcase 2.2: $a_2 + b_1 = n + 1$.

Then H contains $K_{a,a+1} \cup K_{b,b-1}$ as a spanning subgraph, with $b \geqslant 2$ ($a=a_1$, $b=b_1$, $a_2=a+1$, $b_2=b-1$) and there are necessarily missing edges between A_1 and B_2 as between A_2 and B_1.

Let us choose x in A_1 and y in B_2 that are not adjacent. They satisfy $d_H(x)+d_H(y) \geqslant a+1+b=n+1$, so necessarily $k = n+1$, $d_H(x) = a+1$ and $d_H(y) = b$.

Assume there is some edge uv, $u \in A_1$, $v \in B_2$. Then $d_H(u) \geqslant a+2$ and every vertex in B_2 has a degree sum with vertex u greater than k, which implies by (★) that u is adjacent to every vertex in B_2. Symmetrically, $d_H(v) \geqslant b+1$ and v is adjacent to every vertex in A_1. We now observe that any two vertices $u' \in A_1$ and $v' \in B_2$ have degree sum greater than k and so are adjacent, a contradiction.

We then deduce $E_H(A_1, B_2) = \emptyset$ which implies $b < (n+1)/2 < a+1$.

If there exists y in B_1 with $d_H(y) = b-1$, then, by (★), every vertex in A_2 has degree equal to $k-(b-1) = a+2$, whence there are at least two vertices of B_1 that are in the neighborhood of A_2 and so have degree at least b. Again from (★), such vertices are adjacent to every vertex in A_2, and therefore, they would have degree $n \geqslant k/2$, which contradicts the definition of B. Hence $d_H(y) \geqslant b$ for every y in B_1.

Suppose that for every x in A_2 we have $d_H(x) \leqslant n-1$, that is x has (at least) one nonadjacency y_x in B_1. Thus $d_H(x) + d_H(y_x) = k = n+1$. Since $d_H(y_x) \geqslant b$, we deduce $d_H(x) \leqslant a+1$, and x has at most one neighbor in B_1.

If $d_H(x) = a$, x has no adjacency in B_1 and every vertex in B_1 has degree $k - a = b+1$, whence there is some $x' \in A_2$ with degree at least $a+1$ and so, from (★), adjacent to every vertex in B_1, a contradiction with $d_H(x') \leqslant n-1$. Therefore, for every x in A_2, we have $d_H(x) = a+1$. Consequently, $d_H(y_x) = b$ and y_x has exactly one neighbor in A_2; this holds for every vertex y_x in B_1 being nonadjacent to x, hence for every vertex in B_1 but the one which is adjacent to x. Since each vertex in A_2 has exactly one neighbor in B_1 and $|B_1| = b < a+1 = |A_2|$, there is a vertex y in B_1 which has at least two neighbors in A_2. Then $d_H(y) \geqslant b+1$ and it follows that y is adjacent to every vertex x in A_2. This implies $d_H(y) = n \geqslant k/2$, a contradiction. So there exists a vertex in A_2 with degree n.

We denote by S, the set of such vertices and by R its complement in A_2. It follows $R \neq \emptyset$, otherwise $d_H(y) = n \geqslant k/2$ for every y in B_1, a contradiction. Given an x in R, there is some y in B_1 nonadjacent to x. Then $d_H(x)+d_H(y)=n+1 \geqslant a+b-1+|S| = n+|S|-1$. So $1 \leqslant |S| \leqslant 2$.

Assume first $|S| = 1$ and $\{s\}$. We know (from $b \geqslant 2$ and $|R| = a > b-1$) that $|R|$ is at least 2 and every vertex in B_1 is adjacent to s, whence either $d_H(x) = a$ and $d_H(y) = b+1$, or $d_H(x) = a+1$ and $d_H(y) = b$. This is true for every vertex y of B_1 being nonadjacent to x.

- If $d_H(x) = a$ and $d_H(y) = b+1$ then y has exactly one neighbour x' in R. Let y' in B_1 nonadjacent to x'. Then $d_H(x') + d_H(y') = n+1$ and necessarily $d_H(x') = a+1$ and $d_H(y') = b$. Thus $d_H(x) + d_H(y') \neq k$, a contradiction with ($\bigstar$) and with the fact that x is not adjacent to y'.
- If $d_H(x) = a+1$ and $d_H(y) = b$, then y has no adjacency in R and then, given x' in R we have, by ($\bigstar$), $d_H(x') = n+1-d_H(y) = n+1-b = a+1$. Let z be the only neighbor of x in B_1, z is adjacent to s and x and cannot be adjacent to x', otherwise it would have degree sum greater than k with every vertex in A_2 and would be adjacent to every vertex in A_2. However we have $d_H(x') + d_H(z) \geqslant (a+1)+(b+1) > k$, a contradiction with ($\bigstar$).

  Therefore $|S| = 2$ and consequently $d_H(x) = a$, $d_H(y) = b+1$. This is valid for every $x \in R$ and every $y \in B_1$ being nonadjacent to x, hence for every $y \in B_1$. Thus $E_H(B_1, R) = \emptyset$.

So H is isomorphic to $K_{0,2} + (K_{a,a-1} \cup K_{n-a,n-a-1})$.

Case 3: $C = \emptyset$, $B_2 = \emptyset$, $B_1 \neq \emptyset$.

Then H contains $K_{a,n} \cup K_{n-a,0}$ as spanning subgraph, where $a = |A_1|$.

Suppose $|B_1| \geqslant 2$.

- If $k \leqslant n$ then, by definition of B_1, for any two vertices x_1 and x_2 in B_1 we have $d_H(x_1)+d_H(x_2) < k \leqslant n$. This implies that there exists some y in A_2 adjacent neither to x_1 nor to x_2. Then, by ($\bigstar$), $d_H(x_1) + d_H(y) = k = d_H(x_2) + d_H(y)$ and thus $d_H(x_1) = d_H(x_2)$.
- If $k = n+1$, assume that there exist two vertices x_1 and x_2 in B_1 for which $d_H(x_1) < d_H(x_2) < (n+1)/2$. Then $d_H(x_1)+d_H(x_2) \leqslant 2d_H(x_2)-1 < n$, also there exists y in A_2 adjacent neither to x_1 nor to x_2, in contradiction with ($\bigstar$) and $d_H(x_1) < d_H(x_2)$.

  Hence all the vertices of B_1 have the same degree γ in H and $\gamma < k/2$ from definition of B_1.

  If S denotes the vertices of A_2 with degree n and $\beta = |S|$, we clearly have $\beta \leqslant \gamma < k/2$.

There is a vertex of A_2 with degree at most $n-1$ and for every such vertex y there exists some x in B_1 nonadjacent to y. Then, by ($\bigstar$), $d_H(y) = k - d_H(x) = k - \gamma > k/2$.

Consequently, for some $\gamma < k/2$, $H = K_{a,\beta} + \Gamma_1$, $1 \leqslant a \leqslant k-\gamma$, $\beta \leqslant \gamma$ and

$$d_H(x) = \begin{cases} \gamma & \text{if } x \in V(\Gamma_1) \cap V_1, \\ k-\gamma & \text{if } x \in V(\Gamma_1) \cap V_2. \end{cases}$$

Remark 1. With the above notations, in the case when $\beta = \gamma$, we have $E(B_1, A_2 - S) = \emptyset$, $a = k - \gamma$ and H is isomorphic to $K_{k-\gamma,\gamma} + \bar{K}_{n-k+\gamma,n-\gamma}$, that is $\Gamma_1 = \bar{K}_{n-k+\gamma,n-\gamma}$.

We have now characterized the graph H but we need to go back to the initial graph G to achieve the proof of Theorem 10. Let us notice that a vertex which has at least one nonadjacency in H has exactly the same neighbors in G as in H but if it is adjacent to every possible vertex of H, it can have in H more neighbors than in G. Using this observation, we examine the different cases and subcases of the above proof.

In Subcases 1.1 and 1.2 when $B = \emptyset$, since H is isomorphic to $K_{a,a} + C^*$, where C^* is a $(k/2-a)$-regular bipartite graph of order $2(n-a)$, $0 \leqslant a \leqslant k/2$, we get $G = A^* + C^*$, where C^* is the same $(k/2-a)$-regular bipartite graph of order $2(n-a)$, $0 \leqslant a \leqslant k/2$, and A^* is a bipartite balanced graph of order $2a$. This is Case 3 of Theorem 10.

In Subcase 1.2 when $B \neq \emptyset$, H is isomorphic to $K_{2,0} + (K_{(n-3)/2,(n-1)/2} \cup K_{(n-1)/2,(n+1)/2})$, G is isomorphic to H and we are in Case 2 of Theorem 10 with $a = (n-1)/2$.

In Subcase 2.1

- If $k = n$ then G is isomorphic to $H = K_{a,a} \cup K_{n-a,n-a}$, $(n-1)/2 < a \leqslant n-1$ and this is Case 1 of Theorem 10.
- If $k=n+1$, $H=K_{1,1}+(K_{a-1,a-1} \cup K_{n-a,n-a})$, $(n-1)/2 < a \leqslant n-1$ and G is isomorphic to $K_{1,1}+(K_{a-1,a-1} \cup K_{n-a,n-a})$ or to $\bar{K}_{1,1}+(K_{a-1,a-1} \cup K_{n-a,n-a})$, that is Case 2 of Theorem 10.

In Subcase 2.2, G is isomorphic to $H = K_{0,2} + (K_{a,a-1} \cup K_{n-a,n-a-1})$, $(n-1)/2 < a \leqslant n-2$, that is also Case 2 of Theorem 10

In Case 3, for some $\gamma < k/2$, $H = K_{a,\beta} + \Gamma_1$, $1 \leqslant a \leqslant k-\gamma$, $\beta \leqslant \gamma$, with

$$d_H(x) = \begin{cases} \gamma & \text{if } x \in V(\Gamma_1) \cap V_1, \\ k-\gamma & \text{if } x \in V(\Gamma_1) \cap V_2. \end{cases}$$

and G is isomorphic to $\Gamma_1 + \Gamma_2$, where Γ_1 is the same as above and Γ_2 is a subgraph of $K_{a,\beta}$.

We then are in Case 4 of Theorem 10 which is now proved. □

Remark 2. The special case of Remark 1 when $\gamma = \beta$ corresponds to $\Gamma_1 = \bar{K}_{n-k+\gamma,n-\gamma}$ and Γ_2 is a bipartite subgraph of $K_{k-\gamma,\gamma}$. Notice that if $k = n, G$ is equal to $\bar{K}_{\gamma,n-\gamma} + \Gamma_2$ where Γ_2 is a bipartite subgraph of $K_{n-\gamma,\gamma}$, and if moreover $\Gamma_2 = \bar{K}_{n-\gamma,\gamma}$, we then obtain for G the graph $K_{\gamma,\gamma} \cup K_{n-\gamma,n-\gamma}$.

4. Proof of Theorem 11

Before proving Theorem 11 we first give a useful result due to Amar, Ordaz, Raspaud, that can be found in [2] (the fact that for G nonhamiltonian we have $d(x_1) = d(x_n) = d(y_1) = d(y_n) = \delta(G)$ is not stated explicitly as a result by itself but is a direct consequence of Claims 1–5 in the proof of Proposition 2 taking $p = 1$).

Theorem 14. *Let G be a balanced bipartite graph of order 2n, with minimum degree $\delta(G)$ and bipartite independence number $\alpha_{\text{bip}}(G)$. If $\alpha_{\text{bip}}(G) \leqslant 2\delta(G) - 2$, then G is hamiltonian except in the case $\alpha_{\text{bip}}(G) = 2\delta(G) - 2$ and G is either isomorphic to $3K_{p,p} + K_{1,1}$ or to $3K_{p,p} + \bar{K}_{1,1}$ or it contains a cycle $C = x_1 y_1 \ldots x_{n-1} y_{n-1} x_1$ of length $2n-2$ such that $G - C$ is an edge $x_n y_n$. Moreover, in this last case, if $G - \{x_n, y_n\} \neq K_{n-1,n-1}$, w.l.o.g. we can suppose that x_n is adjacent to y_{n-1} but not to y_1, and we get $d(x_1) = d(x_n) = d(y_1) = d(y_n) = \delta(G)$; if $G - \{x_n, y_n\} = K_{n-1,n-1}$, then it follows $\delta(G) = 1$ and in fact $d(x_n) = 1$ or $d(y_n) = 1$.*

Proof of Theorem 11. Let G be a bipartite balanced graph of order $2n$ satisfying property BP_n.

Claim 1. *If $BCl_{n+1}(G)$ is not equal to $K_{n,n}$, then one of the following occurs*:

1. *n is even and G is isomorphic to $A + C$ where C is a $(n/2 - a)$-regular balanced bipartite graph and A is a balanced bipartite graph on 2a vertices, $0 \leqslant a \leqslant n/2$.*
2. *There exist some positive integer $\gamma < \leqslant n/2$ and disjoint subgraphs Γ_1 and Γ_2 of G satisfying $1 \leqslant |V(\Gamma_2) \cap V_1| \leqslant n-\gamma$ and $|V(\Gamma_2) \cap V_2| \leqslant \gamma$ such that G is isomorphic to $\Gamma_1 + \Gamma_2$ and the vertices of Γ_1 satisfy the degree condition in G*

$$d_G(x) = \begin{cases} \gamma & \textit{if } x \in V(\Gamma_1) \cap V_1, \\ n-\gamma & \textit{if } x \in V(\Gamma_1) \cap V_2. \end{cases}$$

Proof. This claim is a direct consequence of Theorem 10 with $k = n$. The possible exception graph $K_{a,a} \cup K_{n-a,n-a}$, $(n-1)/2 < a \leqslant n-1$, obtained from Case 1 of Theorem 10, is in fact a subcase of the above Case 2 as noticed in Remark 2 of the preceding section (and since a can be equal to $n/2$, we assume $\gamma \leqslant n/2$ and not $\gamma < n/2$). □

Claim 2. $\alpha_{\text{bip}}(G) \leqslant 2\delta(G)$.

Proof. Suppose that G contains an independent set $\bar{K}_{\alpha,\alpha}$ with $\alpha > \delta(G)$, and let x be a vertex of degree $\delta(G)$. Then for each y nonadjacent to x, $d_G(y) \geqslant n - \delta > n - \alpha$. So there are at least $n - \delta$ vertices of degree at least $n - \alpha + 1$, and thus at least one of them is in $\bar{K}_{\alpha,\alpha}$ and has a neighbor in $\bar{K}_{\alpha,\alpha}$, a contradiction. □

Claim 3. G is *hamiltonian or it is isomorphic to $\Gamma + \bar{K}_{\gamma,n-\gamma}$ where $\Gamma \supseteq \bar{K}_{n-\gamma,\gamma}$ with $1 \leqslant \gamma \leqslant n/2$.*

Proof. Suppose that G is not hamiltonian. Then by Theorem 14 and Claim 2 two cases can occur:

Case 1: $\alpha_{\rm bip}(G) = 2\delta(G) - 2$.

If G is isomorphic to $3K_{p,p}+K_{1,1}$ or $3K_{p,p}+\bar{K}_{1,1}$ then, by BP_n, we get $n=3p+1=4$ which contradicts $n \geqslant 6$. Therefore, by Theorem 14, G contains a cycle $C = x_1y_1\cdots x_{n-1}y_{n-1}x_1$ such that $G - C$ is an edge x_ny_n with $x_ny_{n-1} \in E(G)$ and $x_ny_1 \notin E(G)$. If $G - \{x_n, y_n\} = K_{n-1,n-1}$, then it follows that G is isomorphic to $\Gamma + \bar{K}_{1,n-1}$, with $\Gamma \supseteq \bar{K}_{n-1,1}$, i.e. Claim 3 for $\gamma = 1$. If $G - \{x_n, y_n\} \neq K_{n-1,n-1}$, then, by Theorem 14, $d(x_1) = d(x_n) = d(y_1) = d(y_n) = \delta = \delta(G)$. Since $x_1y_n \notin E(G)$ and $x_{n-1}y_n \notin E(G)$, using BP_n, we get $2\delta = d(x_1) + d(y_n) \geqslant n$ and thus $\delta \geqslant n/2$. If we assume $\delta > n/2$ then for every $x \in V_1$, $y \in V_2$, $d(x)+d(y) \geqslant 2\delta \geqslant n+1$, thus G satisfies BP_{n+1} and is hamiltonian by Theorem 5, a contradiction. Therefore, $\delta(G)=n/2$. Since G is not hamiltonian, we have necessarily $(N_C^+(x_n) \cup N_C^-(x_n)) \cap N_C(y_n) = \emptyset$. This implies that $N_C(x_n) = \{y_{n/2+1}, \ldots, y_{n-1}\}$ and $N_C(y_n) = \{x_2, \ldots, x_{n/2}\}$. Let $i \in \{n/2+1, \ldots, n-1\}$. It is easy to check that x_i is not adjacent to y_1 otherwise we obtain a hamiltonian cycle $C' = x_ny_nx_2C[x_2, x_i]x_iy_1C^-[y_1, y_i]y_ix_n$. Using a similar argument, we get that x_i is not adjacent to $y_{n/2}$. Since $d(x_i) \geqslant \delta = n/2$ and $|\{y_{n/2+1}, \ldots, y_{n-1}\}| = n/2 - 1$, there exists $k \in \{2, \ldots, n/2-1\}$ such that x_iy_k is an edge of G. Then $C' = x_iy_kC^-[y_k, y_i]y_ix_ny_nx_{k+1}C[x_{k+1}, x_i]x_i$ is a hamiltonian cycle, a contradiction.

Case 2: $\alpha_{\rm bip}(G) = 2\delta(G)$.

If $BCl_{n+1}(G) = K_{n,n}$, we know that G would be hamiltonian because of Theorem 7, a contradiction. Thus $BCl_{n+1}(G) \neq K_{n,n}$ and Claim 1 can be applied.

Subcase 2.1: If G has form (1) in Claim 1, then $\delta(G) = n/2$ and a balanced independent set of cardinality $\alpha_{\rm bip}(G) = 2\delta(G) = n$ is necessarily a subset of A or of C. In the first case, because of $\alpha_{\rm bip}(G) = n$, we see that $a = n/2$, and C and A are isomorphic to $\bar{K}_{n/2,n/2}$, i.e., we have Claim 3 with $\gamma = n/2$ and $\Gamma = \bar{K}_{n/2,n/2}$; so we only consider the second case when C contains an induced subgraph $\bar{K}_{n/2,n/2}$. Since C is $(n/2 - a)$-regular, every $x \in \bar{K}_{n/2,n/2}$ has $(n/2 - a)$ neighbors in $C' = C - \bar{K}_{n/2,n/2}$. Since $|C'| = 2((n - a) - n/2) = 2(n/2 - a)$, every x in one of the two vertex-classes (the first or the second) of $\bar{K}_{n/2,n/2}$ is adjacent to every y in the other vertex-class (the second or the first, respectively) of C', hence it follows $C = \bar{K}_{n/2,n/2} + C'$. Since for every y in C', $n/2 - a = d_C(y) \geqslant n/2$ we get $d(y) \geqslant n/2$, $|A| = a = 0$, and G is isomorphic to $C = \bar{K}_{n/2,n/2} + \Gamma$, with $\Gamma = C' = \bar{K}_{n/2,n/2}$, i.e. we have landed at Claim 3 with $\gamma = n/2$. (By Remark 2, G is also isomorphic to $2K_{n/2,n/2}$.)

Subcase 2.2: If G has form (2) in Claim 1, the structure of G depends on the integer $\gamma \leqslant n/2$.

We first suppose that we are in the case when $\gamma < n/2$.

We know that G is isomorphic to $\Gamma_1 + \Gamma_2$ with $d_G(x) = \gamma$ if $x \in V(\Gamma_1) \cap V_1$ and $d_G(x) = n - \gamma$ if $x \in V(\Gamma_1) \cap V_2$. Let us recall the exact structure of G that was obtained from Case 3 of the proof of Theorem 10 (since we have assumed $\gamma < n/2$). We have $V_1 = A_1 \cup B_1$ where A_1 corresponds to the vertices of V_1 with degree (in the biclosure) more than $n/2$, $|A_1| = a \geqslant 1$ and B_1 consists of vertices of degree γ. The set $V_2 = A_2$ has all its vertices of degree (in the biclosure) more than $n/2$ and contains a subset S of cardinality β whose vertices have degree n while the other vertices have degree $n - \gamma$. The graph Γ_1 corresponds to the bipartite subgraph induced by $(B_1, V_2 - S)$ and Γ_2 is a subgraph of $K_{a,\beta}$ with the same vertex set. These properties imply $\beta \leqslant \gamma$, $\delta(G) = \gamma$ and there is a balanced independent set of cardinality $\alpha_{\rm bip}(G) = 2\delta(G) = 2\gamma$ which is a subset of $V(\Gamma_1)$. Also we have $n \geqslant a + \gamma$ and we will distinguish two cases corresponding to equality or strict inequality in this formula.

- If $n = a + \gamma$, we get $\beta = \gamma$ (namely, because every $y \in A_2 - S$ is adjacent to all vertices of A_1 and has degree $d_G(y) = n - \gamma$, and since $|A_1| = a = n - \gamma$, it follows that the $n - a = \gamma \geqslant 1$ vertices of B_1 having degree γ can be adjacent only to vertices of S and therefore, $\beta \geqslant \gamma$); so we are in the case of Remark 2, that is $G = \bar{K}_{\gamma,n-\gamma} + \Gamma$ where Γ contains $\bar{K}_{n-\gamma,\gamma}$ as a spanning subgraph. This case corresponds to the exception graph of Claim 3 and is clearly not hamiltonian.
- If $n > a + \gamma$, consider a balanced independent set (W_1, W_2), $W_1 \subseteq V_1 \cap V(\Gamma_1)$, $W_2 \subseteq V_2 \cap V(\Gamma_1)$.

Every vertex y in W_2 satisfies $d_G(y) = n - \gamma$ and so is adjacent to the $n - a - \gamma$ vertices in $B_1 - W_1$ which is not empty by our assumption. Moreover, every vertex x in $B_1 - W_1$ has degree γ and consequently has no neighbors out of W_2. We then deduce that S is empty, i.e. $\beta = 0$, since vertices of $B_1 - W_1$ should be adjacent to every vertex in S, and Γ_1 is isomorphic to $K_{n-a-\gamma,\gamma} \cup \Gamma_0$, where Γ_0 is induced by $(W_1, V_2 - W_2)$. Therefore G is isomorphic to $K_{a,0} + (K_{n-a-\gamma,\gamma} \cup \Gamma_0)$.

On the other hand, let us consider the bipartite balanced graph G' with $2(a+\gamma)$ vertices equal to $K_{a,0} + (K_{0,2\gamma+a-n} \cup \Gamma_0)$, i.e. the subgraph of G obtained by suppressing $B_1 - W_1$ in V_1 and a subset T in W_2 with $n - a - \gamma$ vertices, which is possible since $\gamma \geqslant n - a - \gamma$ can be verified.

The degree in G' of the vertices of $V(\Gamma_0) \cap V_1$ and $V(\Gamma_0) \cap V_2$ is still equal to γ and $n - \gamma$, respectively, and every vertex in $V(\Gamma_0) \cap V_1$ has at least one nonadjacency in $V(\Gamma_0) \cap V_2$. Using this remark together with $n > a + \gamma$, it is easy to check that the graph $BCl_{a+\gamma+1}(G')$ is the complete bipartite graph $K_{a+\gamma,a+\gamma}$, and then, by Theorem 7, the graph G' is hamiltonian.

We can easily extend a hamiltonian cycle of G' to a hamiltonian cycle of G, replacing the edge uv, $u \in A_1$ and $v \in W_2 - T$ (that necessarily exists if we assume that $T \neq W_2$, i.e. $\gamma > n - a - \gamma$) by $uwPtv$ where $w \in N_G(u) \cap T$, $t \in N_G(v) \cap (B_1 - W_1)$

and P is a path from w to t containing all the vertices of the bipartite subgraph of G induced by $(B_1 - W_1, T)$. So G is hamiltonian, a contradiction. Now we have to consider the case $\gamma = n - a - \gamma$, i.e. $T = W_2$. Then it can be easily proved that every $x \in B_1 - W_1$ is adjacent to every $y \in W_2$ and nonadjacent to every $y \in V_2 - W_2$ and that every $y \in V_2 - W_2$ is adjacent to every $x \in A_1 \cup W_1$. This implies that the subgraph of G induced by $(B_1 - W_1, V_2 - W_2)$ is isomorphic to $\bar{K}_{n-a-\gamma,n-\gamma} = \bar{K}_{\gamma,n-\gamma}$ and, if Γ denotes the subgraph of G induced by $(A_1 \cup W_1, W_2)$, that G is isomorphic to $\Gamma + \bar{K}_{\gamma,n-\gamma}$ and $\Gamma \supseteq \bar{K}_{n-\gamma,\gamma}$. Thus we have gotten the assertion of Claim 3 and the case when $\gamma < n/2$ is finished.

If we consider now the case when $\gamma = n/2$, we can also obtain easily that G is isomorphic to $\bar{K}_{n/2,n/2} + \Gamma$ with $\Gamma \supseteq \bar{K}_{n/2,n/2}$ as a spanning subgraph and Claim 3 is now proved. □

Claim 4. *If G is hamiltonian then it is bipancyclic.*

Proof. Let $I = \{(i,j)/x_i y_j \notin E(G)\}$ and $m = |E(G)|$. Then $d(x_i) + d(y_j) \geqslant n$ for each $(i,j) \in I$.

Hence

$$\sum_{(i,j)\in I} (d(x_i) + d(y_j)) \geqslant n(n^2 - m) \Leftrightarrow \sum_{i=1}^{n} d(x_i)(n - d(x_i)) + \sum_{j=1}^{n} d(y_j)(n - d(y_j)) \geqslant n(n^2 - m)$$

$$\Leftrightarrow \sum_{i=1}^{n} d^2(x_i) + \sum_{j=1}^{n} d^2(y_j) - 3nm + n^3 \leqslant 0.$$

By the Cauchy Schwarz inequality we have

$$\left(\sum_{i=1}^{n} d(x_i)\right)^2 \leqslant n \sum_{i=1}^{n} d^2(x_i) \quad \text{and} \quad \left(\sum_{i=1}^{n} d(y_i)\right)^2 \leqslant n \sum_{i=1}^{n} d^2(y_i).$$

This observation and the fact that $\sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} d(y_i) = m$, imply $\sum_{i=1}^{n} d^2(x_i) \geqslant m^2/n$ and $\sum_{i=1}^{n} d^2(y_i) \geqslant m^2/n$. Using these minorations, we then obtain

$$\frac{2m^2}{n} - 3mn + n^3 \leqslant 0.$$

Therefore $m \geqslant n^2/2$. We know from the following theorem of Schmeichel and Mitchem [13] that G is bipancyclic as soon as $m > n^2/2$. If $m = n^2/2$, then, by property BP_n, we can show that G is $n/2$-regular which can be proved to be impossible. Hence G is bipancyclic.

Theorem 15 (Mitchem and Schmeichel [13]). *Let G be a hamiltonian bipartite balanced graph of order $2n$ and size m. If $m > n^2/2$, then G is bipancyclic.*

Claim 4 is now proved and also Theorem 11 which is a direct consequence of Claims 1–4. □

5. Proof of Theorem 12

Let us first recall the following result of Schmeichel and Mitchem that appears in the proof of Lemma 1 of [12].

Theorem 16. *Let G be a bipartite graph containing a hamiltonian cycle $C = x_1 y_1 \cdots x_n y_n x_1$. If $d(x_1) + d(y_n) \geqslant n + 2$ then for every k, $2 \leqslant k \leqslant n$, G contains a cycle C_{2k} of one of the following forms*:

(1) $x_1 y_p x_{p+1} y_{p+1} \cdots x_{p+k-1} y_n x_1$ *for some* p, $1 \leqslant p \leqslant n - k + 1$,
(2) $x_1 y_p x_{p+1} y_{p+1} \cdots x_n y_n x_{k+p-n} y_{k+p-n} \cdots y_1 x_1$ *for some* p, $n - k + 2 \leqslant p \leqslant n - 1$.

Proof of Theorem 12. Let $P = x_1 y_1 x_2 y_2 \cdots x_n y_n$ be a hamiltonian path of G such that $x_1 = u$, $x_2 = v$.

We have assumed that $d(u) = d(x_1) \geqslant (n+1)/2$. Consequently, there exists some $i \leqslant n - 2$ such that y_i and y_{i+1} are adjacent to x_1, and some j, $2 \leqslant j \leqslant n - 1$, such that $x_j y_n \in E(G)$ and $y_j x_1 \in E(G)$. So G contains a C_4 containing u and a C_{2n}.

We now consider cycles of length $2k$, $3 \leqslant k \leqslant n - 1$.

By the degree assumption, $|\{i/x_1 y_i \in E(G) \text{ and } y_n x_{i+1} \in E(G)\}| \geqslant 2$. Let

$$d = \max\{i/x_1 y_i \in E(G) \text{ and } y_n x_{i+1} \in E(G)\}$$

and $W = P[x_{d+1}, y_n]$

Without loss of generality, we assume that $d \geqslant (n+1)/2$. (Otherwise we can consider the path P^{-1} instead of P.)

We define a bipartite balanced graph H on $2d$ vertices by

$$V(H) = \{x_1, \dots, x_d\} \cup \{y_1, \dots, y_d\}$$

and

$$E(H) = \{x_i y_i / 1 \leqslant i \leqslant d\} \cup \{y_i x_{i+1} / 1 \leqslant i \leqslant d-1\} \cup \{x_1 y_i \in E(G) / 1 \leqslant i \leqslant d\}$$
$$\cup \{y_d x_i / 1 \leqslant i \leqslant d \text{ and } y_n x_i \in E(G)\}.$$

For any k, $2 \leqslant k \leqslant d-1$, we define a bipartite graph H_k and an integer t_k as follows: if $y_n x_d \in E(G)$ then $H_k = H$ and $t_k = 0$. If $y_n x_d \notin E(G)$, then $H_k = H - \{x_1 y_{d-k+1}\}$ and $t_k = 1$ when $x_1 y_{d-k+1} \in E(G)$, or $H_k = H$ and $t_k = 0$ when $x_1 y_{d-k+1} \notin E(G)$.

Then $d_{H_k}(x_1) = d(x_1) - d_W(x_1) - t_k$ and $d_{H_k}(y_d) \geqslant d(y_n) - d_W(y_n) + 1 + t_k$ and thus $d_{H_k}(x_1) + d_{H_K}(y_d) \geqslant n + 2 - d_W(x_1) - d_W(y_n)$. By definition of d we have $d_W(x_1) + d_W(y_n) \leqslant n - d$. This implies $d_{H_k}(x_1) + d_{H_K}(y_d) \geqslant d + 2$ and H_k is pancyclic from Theorem 16 and so contains a cycle C_{2k} of form (1) or (2) described in Theorem 16.

If C_{2k} contains an edge $y_d x_i$ with $i \neq d$ and $i \neq 1$ then in G we put

$$C_{2(k+1)} = [C_{2k} - \{y_d x_i\}] \cup \{y_d x_{d+1}, x_{d+1} y_n, y_n x_i\}$$

and

$$C_{2(k+n-d)} = [C_{2k} - \{y_d x_i\}] \cup \{y_n x_i\} \cup P[y_d, y_n].$$

If C_{2k} contains $x_1 y_d$ and $x_d y_d$, then, by Theorem 16, $C_{2k} = x_1 y_d x_d y_{d-1} \cdots x_{d-k} y_{d-k+1} x_1$. Since $y_{d-k+1} x_1 \in E(H_k)$ and by the definition of H_k, we know that $y_n x_d \in E(G)$. In G we put

$$C_{2(k+1)} = [C_{2k} - \{y_d x_d\}] \cup \{y_d x_{d+1}, x_{d+1} y_n, y_n x_d\}$$

and

$$C_{2(k+n-d)} = [C_{2k} - \{y_d x_d\}] \cup \{y_n x_d\} \cup P[y_d, y_n].$$

Thus G contains a cycle C_{2m} for every m, $3 \leqslant m \leqslant d$ and $n - d + 2 \leqslant m \leqslant n - 1$, which contains both u and v. Moreover, $n - d + 2 \leqslant d + 1$, since $d \geqslant (n+1)/2$.

Hence G is bipancyclic and every C_{2m}, $3 \leqslant m \leqslant n$, contains both u and v. □

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