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Note

The correct solution to Berlekamp's switching game ☆

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Abstract

We look at Berlekamp's switching game of $n \times n$ grids for $n \geqslant 10$. We show that the previous result for R_{10} was incorrect, and that in fact $R_{10} = 35$. We also show that $R_{11} = 43$ and $R_{12} = 54$, and give new lower bounds for R_{13} through R_{20}.

Keywords: Berlekamp; Lightbulb; Coding theory

While working at Bell Labs in the 1960s, Elwyn Berlekamp built a 10×10 grid of light bulbs. The grid had an array of 100 switches in the back to control each light bulb individually. It also had 20 switches in the front, one for every row and column. Flipping a row or column switch would invert the state of each bulb in the row or column. A simplistic game that can be played with such a grid is to arrange some initial pattern of lighted bulbs using the rear switches, and then try to turn off as many bulbs as possible using the row and column switches. The player is allowed to flip any of the switches as many times as they want. If the player is skilled, the grid will soon reach an irreducible state, that is, no possible sequence of row or column flips will further reduce the number of lighted bulbs. The problem posed by Berlekamp, roughly, was to find a largest such irreducible state.

More formally, consider a general $n \times n$ grid of light bulbs with the same setup. Let S be any state of the grid, that is, any arrangement of lighted bulbs obtained using the n^2 rear switches.

Definition 1. Let the weight of S be the number of lighted bulbs in the grid.

Definition 2. *Let* $f(S)$ be the minimum weight achieved by applying any sequence of row and column switches to S.

Definition 3. A state S is *reducible* if $f(S)$ is less than the weight of S. Conversely, a grid is *irreducible* if $f(S)$ equals the weight of S.

Definition 4. Let $R_n = \max\ f(S)$ over all possible states S.

In other words, finding R_n is essentially the same as finding an irreducible such that no irreducible exists with a larger weight. The precise goal of Berlekamp's game is to find R_n. Note that R_n can also be interpreted as the covering radius of the so-called lightbulb code of length n^2, as explained in [2]. When Berlekamp introduced his game, it was quickly seen that the problem of finding R_{10} cannot be solved by hand. Even a brute force computational approach will not work because there are 2^{100} possible

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Table 1
Previous results

n	1	2	3	4	5	6	7	8	9	10
R_n	0	1	2	4	7	11	16	22	27	*34*

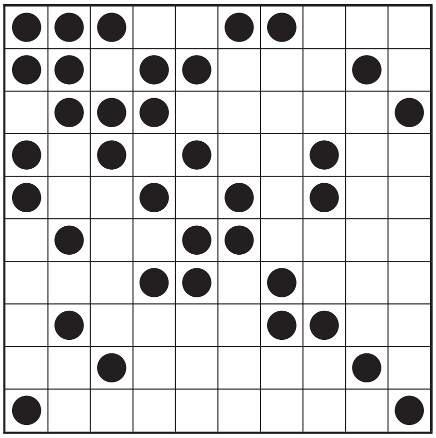

Fig. 1. $n = 10$, weight $= 35$.

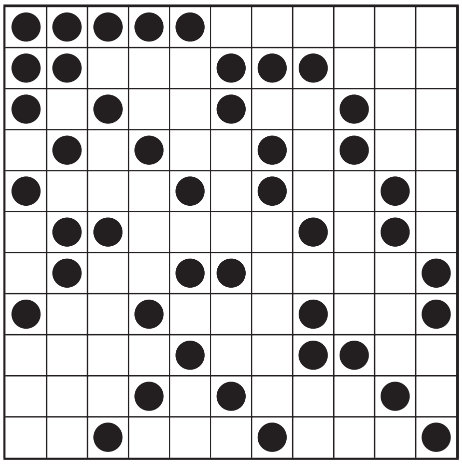

Fig. 2. $n = 11$, weight $= 43$.

initial states, a number far too big for any computer to handle. However, by eliminating many cases and using symmetry, Fishburn and Sloane found R_n for $n \leqslant 9$ by hand and found R_{10} with the aid of a computer. Their results are summarized in Table 1.

Our approach to this problem was to guess a value, p, for R_n, and then use a computer to generate all $n \times n$ states of weight p that could possibly be irreducible, up to certain symmetries. The program either returns one or more $n \times n$ irreducible states with p lighted bulbs, proving that $R_n \geqslant p$, or the program declares that no irreducible states of weight p exist. If no irreducible states of weight p exist, then $R_n < p$ (see [1] for a proof). In order to eliminate symmetry, we consider all row and column sequences subject to the following. These sequences have n entries each of which corresponds to the number of lighted bulbs in that given row or column. The sum of the entries of the sequence equals p. Also, every entry is less than or equal to $n/2$, because if a row or column had more than $n/2$ lighted bulbs then the state could be reduced by inverting that row or column. We eliminate states with symmetry under interchanging rows or columns by requiring all row and column sequences to be non-increasing. We can also easily eliminate states that are symmetric under transposition.

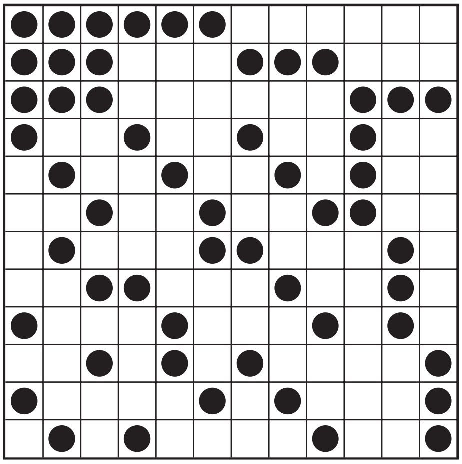

Fig. 3. $n = 12$, weight $= 54$.

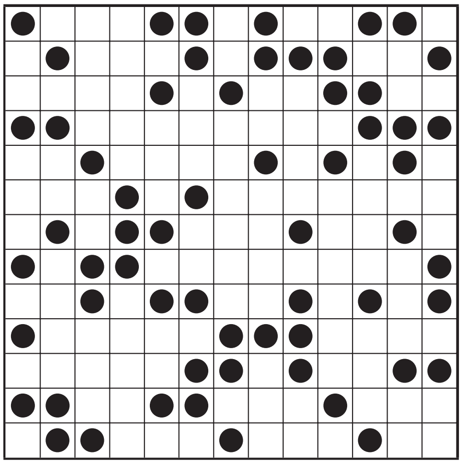

Fig. 4. $n = 13$, weight $= 60$.

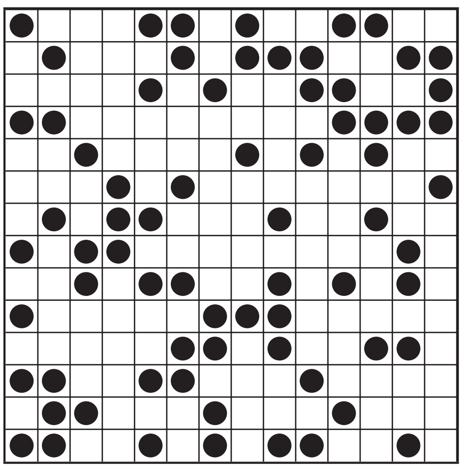

Fig. 5. $n = 14$, weight $= 71$.

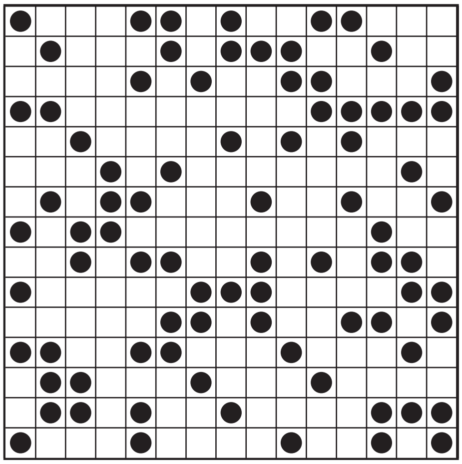

Fig. 6. $n = 15$, weight $= 82$.

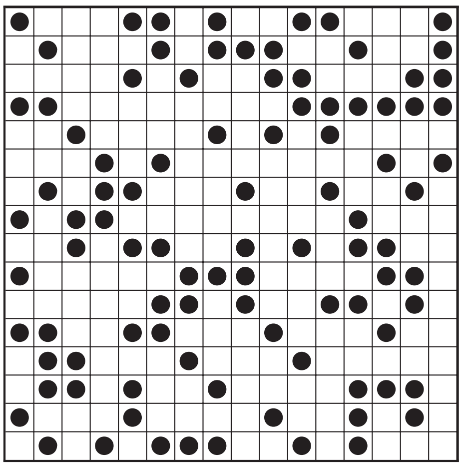

Fig. 7. $n = 16$, weight $= 94$.

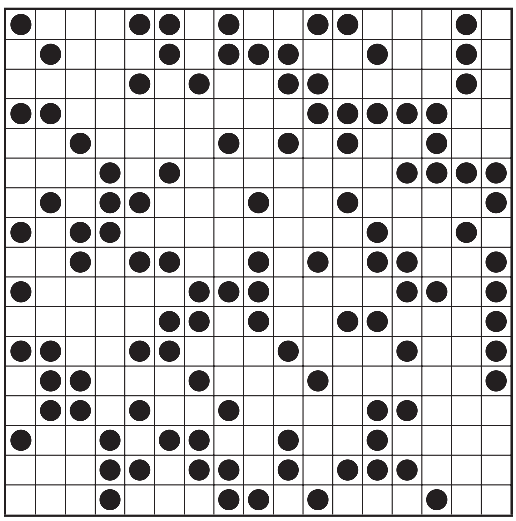

Fig. 8. $n = 17$, weight $= 106$.

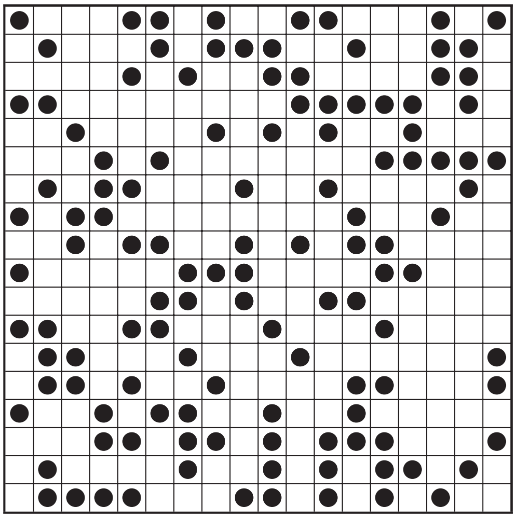

Fig. 9. $n = 18$, weight $= 120$.

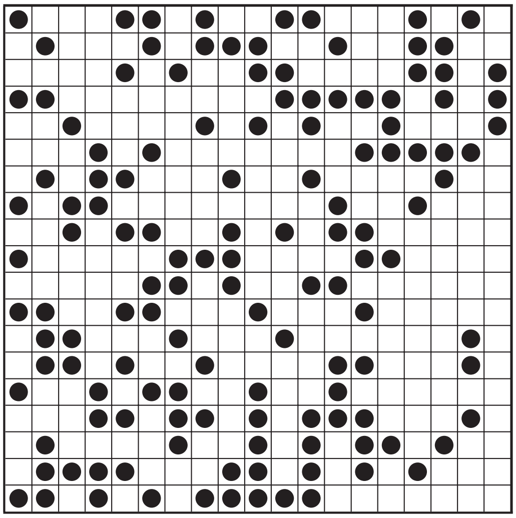

Fig. 10. $n = 19$, weight $= 132$.

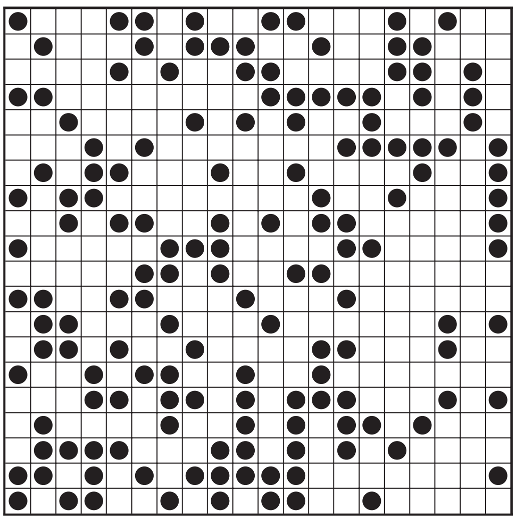

Fig. 11. $n = 20$, weight $= 148$.

Table 2
New results

n	10	11	12	13	14	15	16	17	18	19	20
R_n	35	43	54	$\geqslant 60$	$\geqslant 71$	$\geqslant 82$	$\geqslant 94$	$\geqslant 106$	$\geqslant 120$	$\geqslant 132$	$\geqslant 148$

The program takes one of these sequences, assigns it to the columns, and constructs states row by row. After constructing each row, it checks whether the partial state can be reduced simply by knowing what some of the rows look like. We ran the program for $n \leqslant 9$ and confirmed the previous results. Running the program for $n = 10$, $p = 35$ yields an irreducible state of weight 35 shown in Fig. 1. This shows that the result $R_{10} = 34$ found in [1] is incorrect. Running the program for $n = 10$, $p = 36$ returns no irreducible states. Thus we give the following,

Theorem 5. $R_{10} = 35$.

We used the same method for $n = 11, 12$ to obtain the following results:

Theorem 6. $R_{11} = 43$.

Theorem 7. $R_{12} = 54$.

The maximal irreducible states we obtained are shown in Figs. 2 and 3. We also used this program to prove results about $n = 13, 14$, but obtaining definite values for R_{13} and R_{14} would take months or even years with our current algorithm.

The problem of proving lower bounds on R_n is much simpler than finding the actual value. If we find an irreducible state with p bulbs on, we have proven that $R_n \geqslant p$. Our approach to finding lower bounds was to take an $(n-1) \times (n-1)$ irreducible matrix, and use it to construct unique irreducible $n \times n$ states which contain the input state. We have 12×12 irreducible states from our first program, and we used those as inputs to find 13×13 irreducible states. We then used these 13×13 states as inputs to find 14×14 irreducible states. Because the time it takes to complete the program increases exponentially with n, we tried fewer cases as n increased, so the likelihood of our lower bounds being equal to R_n decreases as n increases. With the use of this program, we give our final results.

Theorem 8. $R_{13} \geqslant 60$, $R_{14} \geqslant 71$, $R_{15} \geqslant 82$, $R_{16} \geqslant 94$, $R_{17} \geqslant 106$, $R_{18} \geqslant 120$, $R_{19} \geqslant 132$, $R_{20} \geqslant 148$.

Proof. The states in Figs. 4–11 are irreducible. $\square$

Our results are summarized in Table 2.

References

[1] P.C. Fishburn, N.J.A. Sloane, The solution to Berlekamps switching game, Discrete Math. 74 (1989) 263–290.
[2] N.J.A. Sloane, Unsolved problems related to the covering radius of codes, in: T.M. Cover, B. Gopinath (Eds.), Open Problems in Communication and Computation, Springer, New York, 1987, pp. 51–56.