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Journal of Functional Analysis 254 (2008) 2255–2274
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A new functional calculus for noncommuting operators

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Received 22 August 2007; accepted 18 December 2007

Available online 28 January 2008

Communicated by C. Villani

Abstract

In this paper we use the notion of slice monogenic functions [F. Colombo, I. Sabadini, D.C. Struppa, Slice monogenic functions, Israel J. Math., in press] to define a new functional calculus for an n-tuple T of not necessarily commuting operators. This calculus is different from the one discussed in [B. Jefferies, Spectral Properties of Noncommuting Operators, Lecture Notes in Math., vol. 1843, Springer-Verlag, Berlin, 2004] and it allows the explicit construction of the eigenvalue equation for the n-tuple T based on a new notion of spectrum for T. Our functional calculus is consistent with the Riesz–Dunford calculus in the case of a single operator.

Keywords: Slice monogenic functions; Functional calculus; Spectral theory; Noncommuting operators

1. Introduction

In a series of interesting papers, see e.g. [6–9] as well as in the book [5] and the references therein, the authors have developed a monogenic functional calculus, whose purpose is to deal with n-tuples of not necessarily commuting operators. The interest in this problem can be traced to the early works of Taylor, see for example [11,12], and it is of great physical interest. In this context, one has other approaches such as the Weyl calculus. A complete review of these theories can be found in [5] to which we refer the reader interested in the physical origin of the problem, in a variety of different approaches, and in their mutual relationships.

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.12.008

No matter how a functional calculus is developed, one will naturally require that it be consistent with the case in which only one operator is considered, and that it be sufficiently flexible to handle both commuting and noncommuting operators. In the case of a single operator, one expects to find the usual Riesz–Dunford calculus (see [4]). When several operators are considered, it is natural to look for functions defined on $\mathbb{R}^n$ with values in noncommuting algebras which may allow the formal treatment of noncommutativity. This set up is naturally within the scope of monogenic functions [1].

Let $\mathbb{R}_n$ be the real Clifford algebra, i.e. the real algebra generated by the n units $e_1, \ldots, e_n$ such that $e_i e_j + e_j e_i = -2\delta_{ij}$. An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ can be naturally identified with the element $x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}_n$. A differentiable function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is said to be monogenic if it is in the kernel of the Dirac operator $\partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_n \partial_{x_n}$, where ∂_{x_i} is shorthand for ∂/∂_{x_i}. The theory of such functions is fully developed in [1] (for the case of several Dirac operators see [3]), and the properties that such functions enjoy closely resemble those of holomorphic functions of a single complex variable. In particular, they can be represented by means of a Cauchy kernel. Specifically, let Σ_n denote the volume of the unit n-sphere in $\mathbb{R}^{n+1}$, let $\omega, x \in \mathbb{R}^{n+1}$ and $\omega \neq x$, then if

$$G(\omega, x) = \frac{1}{\Sigma_n} \frac{\bar{\omega} - \bar{x}}{|\omega - x|^{n+1}}$$

where $\bar{x} := x_0 - x_1 e_1 - \cdots - x_n e_n$, we have

$$\int_{\partial\Omega} G(\omega, x) n(\omega) f(\omega)\, d\mu(\omega) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with smooth boundary $\partial\Omega$ and exterior unit normal $n(\omega)$, and μ is the surface measure of $\partial\Omega$. Note that the kernel $G(\omega, x) = G_\omega(x)$ can be expanded, see [1], as

$$G_\omega(x) = \sum_{k \geqslant 0} \left(\sum_{(\ell_1, \ldots, \ell_k)} W_{\ell_1, \ldots, \ell_k}(\omega) V^{\ell_1, \ldots, \ell_k}(x) \right)$$

in the region $|x| < |\omega|$ where, for each $\omega \in \mathbb{R}^{n+1} \setminus \{0\}$, $W_{\ell_1, \ldots, \ell_k}(\omega) = (-1)^k \partial_{\omega_{\ell_1}} \ldots \partial_{\omega_{\ell_k}} G_\omega(0)$, $V^{\ell_1, \ldots, \ell_k}(x) = \frac{1}{k!} \sum_{j_1, \ldots, j_k} z_{j_1} \ldots z_{j_k}$, $z_j = x_j e_0 - x_0 e_j$, and the sum is taken over all different permutation of $\ell_1, \ldots, \ell_k$.

Consider now an n-tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators acting on a Banach space X and let $R > (1 + \sqrt{2}) \| \sum_{j=1}^n T_j e_j \|$. If we formally replace z_j by T_j in the Cauchy kernel series, it can be shown (see [5, Lemma 4.7]) that

$$G_\omega(T) := \sum_{k \geqslant 0} \left(\sum_{(\ell_1, \ldots, \ell_k)} W_{\ell_1, \ldots, \ell_k}(\omega) V^{\ell_1, \ldots, \ell_k}(T) \right)$$

converges uniformly for all $\omega \in \mathbb{R}^{n+1}$ such that $|\omega| \geqslant R$. This fact leads to the following definition [5]: *the monogenic spectrum $\tilde{\gamma}(T)$ of the n-tuple T is the complement of the largest open set U in $\mathbb{R}^{n+1}$ in which the function $G_\omega(T)$ above is the restriction of a monogenic function with domain U.*

It is possible to show that if T is an n-tuple of noncommuting bounded linear operators satisfying suitable reality conditions on their joint spectrum (see [5]), then the function $\omega \to G_\omega(T)$ is the restriction to the region $|\omega| > (1+\sqrt{2})\|\sum_{j=1}^n T_j e_j\|$ of a monogenic function defined off $\mathbb{R}^n$.

Denote with the same symbol $G_\omega(T)$ its maximal monogenic extension. Under these hypotheses, if $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded open neighborhood of $\gamma(T)$ with smooth boundary and if f is a monogenic function defined in an open neighborhood of $\overline{\Omega}$, then one can show that the expression

$$f(T) := \int_{\partial\Omega} G_\omega(T) n(\omega) f(\omega)\, d\mu(\omega)$$

is well defined.

In the case of commuting bounded linear operators the spectrum can be determined in an explicit way in view of the following result [5].

Theorem 1.1. *Let $T = (T_1, \dots, T_n)$ be a n-tuple of commuting bounded linear operator acting on a Banach space X and suppose that the spectrum of T_j is real for all $j = 1, \dots, n$. Then $\gamma(T)$ is the complement in $\mathbb{R}^n$ of the set of all $\lambda \in \mathbb{R}^n$ for which the operator $\sum_{j=1}^n (\lambda_j \mathcal{I} - T_j)^2$ is invertible in $\mathcal{L}(X)$.*

We observe that the condition of invertibility of $\sum_{j=1}^n (\lambda_j \mathcal{I} - T_j)^2$ gives an eigenvalue equation which, at least in some cases, can be easily written. If for example n is odd, it is possible to write explicitly the Cauchy kernel as

$$G_\omega(T) = \frac{1}{\Sigma_n}\left(\omega_0^2 \mathcal{I} + \sum_{j=1}^n (\omega_j \mathcal{I} - T_j)^2\right)^{(-n-1)/2} \overline{(\omega\mathcal{I} - T)},$$

whose singularities lie on the set

$$\left\{(0, \omega_1, \dots, \omega_n) \in \mathbb{R}^{n+1} \,\middle|\, 0 \in \sigma\left(\sum_{j=1}^n (\omega_j \mathcal{I} - T_j)^2\right)\right\}.$$

In the case the operators T_j do not commute the term $\frac{1}{\Sigma_n}|\omega\mathcal{I} - T|^{-n-1}\overline{(\omega\mathcal{I} - T)}$ is not the sum of the Cauchy kernel series so that it is much more difficult to determine the spectrum.

In this paper we use a different approach to functional calculus. Specifically we replace the use of monogenic functions with the new concept, see [2], of slice-monogenic functions. These functions, whose definition we recall in Section 2, have the great advantage that polynomials as well as power series are special cases of slice-monogenic functions (this is in contrast to the usual definition of monogenic functions). The key observation which will make our approach successful is the fact that the sum of the so-called S-resolvent operator series (see (4)) is a function which can be utilized even in parts of the space where the series does not converge. This remark will make it possible to compute the spectrum also when the operators do not commute. In fact, the term $-(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})$ is the sum of the series (4) also when the operators T_j do not commute, and the singularity of the Cauchy kernel (the analogue of the maximal extension $G_\omega(T)$) is given by $(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I})v = 0$.

The outline of the paper is the following. In Section 2 we introduce the concept of slice-monogenic functions and we recall the properties we need to develop our functional calculus. In Section 3 we define the S-resolvent operator and we give the notion of S-spectrum, we show the S-resolvent equation and we develop the functional calculus for bounded operators. Finally in Section 4 we treat the case of unbounded operators.

2. Slice monogenic functions

In this section we collect the basic results on the theory of slice monogenic functions, developed by the authors in [2], to which we refer for the missing proofs in this section. Let $\mathbb{R}_n$ be the real Clifford algebra over n units $e_1, \dots, e_n$ such that $e_ie_j + e_je_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index and $e_A = e_{i_1}e_{i_2}\dots e_{i_r}$. An element $\underline{x} \in \mathbb{R}^n$ can be identified with a 1-vector in the Clifford algebra: $(x_1, x_2, \dots, x_n) \mapsto \underline{x} = x_1e_1 + \dots + x_ne_n$. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}_n$ is seen as a function $f(\underline{x})$ of $\underline{x}$.

An element in $\mathbb{R}_n^0 \oplus \mathbb{R}_n^1$ will be written as

$$x = x_0 + \underline{x} = x_0 + \sum_{j=1}^{n} x_je_j.$$

In the sequel the real part x_0 of x will be also denoted by $\mathrm{Re}[x]$.

Let us denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \dots + e_nx_n \mid x_1^2 + \dots + x_n^2 = 1\}.$$

The complex line $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by L_I.

An element belonging to L_I will be denoted by $u + Iv$, for $v, v \in \mathbb{R}$.

Observe that L_I, for every $I \in \mathbb{S}$, is a real subspace of $\mathbb{R}^{n+1}$ isomorphic to the complex plane.

Definition 2.1. Let $U \subseteq \mathbb{R}^{n+1}$ be a domain and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex line L_I. We say that f is a (left) slice-monogenic function if for every $I \in \mathbb{S}$

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right)f_I(u + Iv) = 0.$$

Analogously, it is possible to define a notion of right slice-monogenicity which gives a theory equivalent to the one left slice-monogenic functions. In the sequel, unless otherwise stated, we will consider monogenicity on the left and, for simplicity, sometimes we will denote by $\overline{\partial}_I$ the operator $\frac{1}{2}(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v})$ and we will refer to left slice monogenic functions as s-monogenic functions. We will also introduce a notion of I-derivative by means of the operator

$$\partial_I := \frac{1}{2}\left(\frac{\partial}{\partial u} - I\frac{\partial}{\partial v}\right).$$

Remark 2.2. The s-monogenic functions on $U \subseteq \mathbb{R}^{n+1}$ form a right module $\mathcal{M}(U)$ over $\mathbb{R}_n$.

Remark 2.3. For each $a_m \in \mathbb{R}_n$ the monomials $x \mapsto x^m a_m$ are left s-monogenic, while the monomials $x \mapsto a_m x^m$ are right s-monogenic. Thus also polynomials $\sum_{m=0}^{N} x^m a_m$ are left s-monogenic and any power series $\sum_{m=0}^{+\infty} x^m a_m$ is left s-monogenic in its domain of convergence. The function $R(x) = (x - y_0)^{-m}$, $m \in \mathbb{N}$, is s-monogenic (left and right) if and only if $y_0 \in \mathbb{R}$.

Definition 2.4. Let U be a domain in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Its s-derivative ∂_s is defined as

$$\partial_s(f) = \begin{cases} \partial_I(f)(x), & x = u + Iv,\ v \neq 0, \\ \partial_u f(u), & x = u \in \mathbb{R}. \end{cases} \tag{1}$$

Note that the definition of derivative is well posed because it is applied only to s-monogenic functions. Furthermore, any holomorphic function $f : \Delta(0, R) \to \mathbb{C}$ can be extended (uniquely, up to a choice of an order for the elements in the basis of $\mathbb{R}_n$) to an s-monogenic function $\tilde{f} : B(0, R) \to \mathbb{R}_n$.

A key fact is that any s-monogenic function can be developed into power series and also that it admits a Cauchy integral representation.

Proposition 2.5. *If $B = B(x_0, R) \subseteq \mathbb{R}^{n+1}$ is a ball centered in a real point x_0 with radius $R > 0$, then $f : B \to \mathbb{R}_n$ is s-monogenic if and only if it has a series expansion of the form*

$$f(x) = \sum_{m \geqslant 0} x^m \frac{1}{m!} \frac{\partial^m f}{\partial u^m}(x_0) \tag{2}$$

converging on B.

Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_x = \begin{cases} \frac{\underline{x}}{|\underline{x}|}, & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S}, & \text{otherwise}. \end{cases}$$

We have the following.

Theorem 2.6. *Let $B = B(0, R) \subseteq \mathbb{R}^{n+1}$ be a ball with center in 0 and radius $R > 0$ and let $f : B \to \mathbb{R}_n$ be an s-monogenic function. If $x \in B$ then*

$$f(x) = \frac{1}{2\pi} \int_{\partial \Delta_x(0,r)} (\zeta - x)^{-1}\, d\zeta_{I_x}\, f(\zeta),$$

where $\zeta \in L_{I_x} \cap B$, $d\zeta_{I_x} = -d\zeta\, I_x$ and $r > 0$ is such that

$$\overline{\Delta_x(0, r)} = \{ u + I_x v \mid u^2 + v^2 \leqslant r^2 \}$$

contains x and is contained in B.

Remark 2.7. An analogue statement holds for regular functions in an open ball centered in a real point x_0.

Remark 2.8. If f is an s-monogenic function on a domain U and $\gamma : [a, b] \to \mathbb{R}^{n+1}$ is a curve, then the integral $\int_\gamma f(\xi)\,d\xi$ is defined as $\int_a^b f(\gamma(t))\gamma'(t)\,dt$. In particular, the curve γ can have values on a complex plane L_I.

The key ingredient to define a functional calculus is what we call noncommutative Cauchy kernel series.

Definition 2.9. Let $x = \mathrm{Re}[x] + \underline{x}$, $s = \mathrm{Re}[s] + \underline{s}$ be such that $sx \neq xs$. We will call noncommutative Cauchy kernel series the following expansion:

$$S^{-1}(s,x) := \sum_{n\geqslant 0} x^n s^{-1-n} \tag{3}$$

defined for $|x| < |s|$.

Theorem 2.10. *(See [2].) Let $x = \mathrm{Re}[x] + \underline{x}$, $s = \mathrm{Re}[s] + \underline{s}$ be such that $xs \neq sx$. Then*

$$\sum_{n\geqslant 0} x^n s^{-1-n} = -\big(x^2 - 2x\,\mathrm{Re}[s] + |s|^2\big)^{-1}(x - \bar{s})$$

for $|x| < |s|$.

We will call the expression $(x^2 - 2x\,\mathrm{Re}[s] + |s|^2)^{-1}(x - \bar{s})$, defined for $x^2 - 2x\,\mathrm{Re}[s] + |s|^2 \neq 0$, *noncommutative Cauchy kernel*. Therefore note that the noncommutative Cauchy kernel is defined on a set which is larger then the set $\{(x,s)\colon\ |x| < |s|\}$ where the noncommutative Cauchy kernel series is defined. Since $x = \bar{s}$ is a solution of $\bar{s}^2 - 2\bar{s}\,\mathrm{Re}[s] + |s|^2 = 0$, one may wonder if the factor $(x - \bar{s})$ can be simplified from the expression of the noncommutative Cauchy kernel. However, as shown in the next result, this is not possible and the noncommutative Cauchy kernel cannot be extended to a continuous function in $x = \bar{s}$. With an abuse of notation, we will denote the noncommutative Cauchy kernel series and the noncommutative Cauchy kernel with the same symbol $S^{-1}(s,x)$.

Theorem 2.11. *Let $S^{-1}(s,x)$ be the noncommutative Cauchy kernel with $xs \neq sx$. Then $S^{-1}(s,x)$ is irreducible and $\lim_{x\to\bar{s}} S^{-1}(s,x)$ does not exist.*

Proof. We prove that we cannot find a degree one polynomial $Q(x)$ such that

$$x^2 - 2x\,\mathrm{Re}[s] + |s|^2 = \big(s + x - 2\,\mathrm{Re}[s]\big)Q(x).$$

The existence of $Q(x)$ would allow the simplification

$$S^{-1}(s,x) = Q^{-1}(x)\big(s + x - 2\,\mathrm{Re}[s]\big)^{-1}\big(s + x - 2\,\mathrm{Re}[s]\big) = Q^{-1}(x).$$

We proceed as follows. First of all note that $Q(x)$ has to be a monic polynomial of degree one, so we set

$$Q(x) = x - r,$$

where $r = r_0 + \sum_{j=1}^{n} r_j e_j$. The equality

$$\bigl(s + x - 2\,\mathrm{Re}[s]\bigr)(x - r) = x^2 - 2x\,\mathrm{Re}[s] + |s|^2$$

gives

$$sx - sr - xr + 2r\,\mathrm{Re}[s] - |s|^2 = 0.$$

Solving for r, we get

$$r = \bigl(s + x - 2\,\mathrm{Re}[s]\bigr)^{-1}\bigl(sx - |s|^2\bigr),$$

which depends on x. Let us now prove that the limit does not exist. Let $\varepsilon = \varepsilon_0 + \sum_{j=1}^{n} \varepsilon_j e_j$, and consider

$$\begin{aligned} S^{-1}(s, \bar{s} + \varepsilon) &= \bigl((\bar{s} + \varepsilon)^2 - 2(\bar{s} + \varepsilon)\,\mathrm{Re}[s] + |s|^2\bigr)^{-1}\varepsilon \\ &= \bigl((\bar{s} + \varepsilon)^2 - 2(\bar{s} + \varepsilon)\,\mathrm{Re}[s] + |s|^2\bigr)^{-1}\varepsilon \\ &= \bigl(\bar{s}\varepsilon + \varepsilon\bar{s} + \varepsilon^2 - 2\varepsilon\,\mathrm{Re}[s]\bigr)^{-1}\varepsilon \\ &= \bigl(\varepsilon^{-1}\bigl(\bar{s}\varepsilon + \varepsilon\bar{s} + \varepsilon^2 - 2\varepsilon\,\mathrm{Re}[s]\bigr)\bigr)^{-1} \\ &= \bigl(\varepsilon^{-1}\bar{s}\varepsilon + \bar{s} + \varepsilon - 2\,\mathrm{Re}[s]\bigr)^{-1}. \end{aligned}$$

If we now let $\varepsilon \to 0$, we obtain that the term $\varepsilon^{-1}\bar{s}\varepsilon$ does not have a limit because

$$\varepsilon^{-1}\bar{s}\varepsilon = \frac{\bar{\varepsilon}}{|\varepsilon|^2}\bar{s}\varepsilon$$

contains scalar addends of type $\frac{\varepsilon_i \varepsilon_j s_\ell}{|\varepsilon|^2}$ with $i, j, \ell \in \{0, 1, 2, 3\}$ that do not have limit. □

The following result will be useful in the sequel and its proof follows by a simple computation.

Proposition 2.12. *Let $s = \mathrm{Re}[s] + \underline{s}$. Then the following identity holds*:

$$s^2 - 2s\,\mathrm{Re}[s] + |s|^2 = 0.$$

3. Slice-monogenic functional calculus for bounded operators

In the sequel, we will consider a Banach space V over $\mathbb{R}$ (the case of complex Banach spaces can be discussed in a similar fashion) with norm $\|\cdot\|$. It is possible to endow V with an operation of multiplication by elements of $\mathbb{R}_n$ which gives a two-sided module over $\mathbb{R}_n$. We recall that a two-sided module V over $\mathbb{R}_n$ is called a Banach module over $\mathbb{R}_n$, if there exists a constant $C \geqslant 1$ such that $\|va| \leqslant C\|v\||a|$ and $\|av| \leqslant C|a|\|v\|$ for all $v \in V$ and $a \in \mathbb{R}_n$.

In the sequel, we will make use of the following notations.

- By V we denote a Banach space over $\mathbb{R}$ with norm $\|\cdot\|$.
- By V_n we denote the two-sided Banach module over $\mathbb{R}_n$ corresponding to $V \otimes \mathbb{R}_n$. An element in V_n is of the type $\sum_A v_A \otimes e_A$ (where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$, is a multi-index). The multiplications of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined as $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. We will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. We define $\|v\|^2_{V_n} = \sum_A \|v_A\|^2_V$.
- $\mathcal{B}(V)$ is the space of bounded $\mathbb{R}$-homomorphisms of the Banach space V to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$.
- Let $T_A \in \mathcal{B}(V)$. We define an operator $T = \sum_A T_A e_A$ and its action on $v = \sum v_B e_B \in V_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The operator T is a right-module homomorphism which is a bounded linear map on V_n. The set of all such bounded operators is denoted by $\mathcal{B}_n(V_n)$. We define $\|T\|^2_{\mathcal{B}_n(V_n)} = \sum_A \|T_A\|^2_{\mathcal{B}(V)}$.

3.1. The S-resolvent operator for bounded operators

Throughout the rest of this section, and unless otherwise specified, we will only consider operators of the form $T = T_0 + \sum_{j=1}^n e_j T_j$ where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, \dots, n$. The set of such operators in $\mathcal{B}_n(V_n)$ will be denoted by $\mathcal{B}_n^{0,1}(V_n)$.

Definition 3.1. Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $s = \mathrm{Re}[s] + \underline{s}$. We define the S-resolvent operator series as

$$S^{-1}(s, T) := \sum_{n \geqslant 0} T^n s^{-1-n} \tag{4}$$

for $\|T\| < |s|$.

Theorem 3.2. *Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $s = \mathrm{Re}[s] + \underline{s}$. Then*

$$\sum_{n \geqslant 0} T^n s^{-1-n} = -\left(T^2 - 2T\,\mathrm{Re}[s] + |s|^2 \mathcal{I}\right)^{-1} (T - \bar{s}\mathcal{I}), \tag{5}$$

for $\|T\| < |s|$.

Proof. In Theorem 2.10 the components of x and s are real numbers and therefore they obviously commute. When we formally replace x by operator T we cannot assume that $T_\mu T_\nu = T_\nu T_\mu$ and so we need to verify independently that (5) still holds. To this aim, we check that $-(T - \bar{s}\mathcal{I})^{-1}(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I})$ is the inverse of $\sum_{n \geqslant 0} T^n s^{-1-n}$. In what follows, we assume the convergence of the series to be in the norm of $\mathcal{B}_n(V_n)$:

$$-(T-\bar{s}\mathcal{I})^{-1}\big(T^2-2T\,\mathrm{Re}[s]+|s|^2\mathcal{I}\big)\sum_{n\geqslant 0}T^n s^{-1-n}=\mathcal{I},$$

so then we get

$$\big(-|s|^2\mathcal{I}-T^2+2T\,\mathrm{Re}[s]\big)\sum_{n\geqslant 0}T^n s^{-1-n}=T+\big(s-2\,\mathrm{Re}[s]\big)\mathcal{I}.$$

Observing that $-|s|^2\mathcal{I}-T^2+2T\,\mathrm{Re}[s]$ commutes with T^n we can write

$$\sum_{n\geqslant 0}T^n\big(-|s|^2-T^2+2T\,\mathrm{Re}[s]\big)s^{-1-n}=T+\big(s-2\,\mathrm{Re}[s]\big)\mathcal{I}.$$

Now expand the series as

$$\begin{aligned}\sum_{n\geqslant 0}&T^n\big(-|s|^2\mathcal{I}-T^2+2\,\mathrm{Re}[s]T\big)s^{-1-n}\\ &=\big(-|s|^2\mathcal{I}-T^2+2T\,\mathrm{Re}[s]\big)s^{-1}\\ &\quad+T^1\big(-|s|^2\mathcal{I}-T^2+2T\,\mathrm{Re}[s]\big)s^{-2}+T^2\big(-|s|^2\mathcal{I}-T^2+2T\,\mathrm{Re}[s]\big)s^{-3}+\cdots\\ &=-\big(|s|^2s^{-1}+T\big(-2s\,\mathrm{Re}[s]+|s|^2\big)s^{-2}+T^2\big(s^2-2s\,\mathrm{Re}[s]+|s|^2\big)s^{-3}+\cdots\big)\end{aligned}$$

and using Proposition 2.12, we get

$$\begin{aligned}\sum_{n\geqslant 0}&T^n\big(-|s|^2-T^2+2T\,\mathrm{Re}[s]\big)s^{-1-n}\\ &=-|s|^2s^{-1}\mathcal{I}+Ts^2s^{-2}=-|s|^2s^{-1}\mathcal{I}+T\\ &=-\bar{s}ss^{-1}\mathcal{I}+T=-\bar{s}\mathcal{I}+T=\big(s-2\,\mathrm{Re}[s]\big)\mathcal{I}+T.\qquad\square\end{aligned}$$

Proposition 3.3. *When $Ts\mathcal{I}=sT$, the operator $S^{-1}(s,T)$ equals $(s\mathcal{I}-T)^{-1}$ when the series* (4) *converges.*

Proof. It follows by direct computation. □

Definition 3.4 *(The S-spectrum and the S-resolvent set).* Let $T\in\mathcal{B}_n^{0,1}(V_n)$ and $s=\mathrm{Re}[s]+\underline{s}$. We define the S-spectrum $\sigma_{\mathrm{S}}(T)$ of T as follows:

$$\sigma_{\mathrm{S}}(T)=\big\{s\in\mathbb{R}^{n+1}\colon\ T^2-2\,\mathrm{Re}[s]T+|s|^2\mathcal{I}\ \text{is not invertible}\big\}.$$

An element in $\sigma_{\mathrm{S}}(T)$ will be called an S-eigenvalue.

The S-resolvent set $\rho_{\mathrm{S}}(T)$ is defined by

$$\rho_{\mathrm{S}}(T)=\mathbb{R}^{n+1}\setminus\sigma_{\mathrm{S}}(T).$$

Definition 3.5 *(The S-resolvent operator).* Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $s = \mathrm{Re}[s] + \underline{s} \in \rho_S(T)$. We define the S-resolvent operator as

$$S^{-1}(s,T) := -\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}). \tag{6}$$

Example 3.6 *(Pauli matrices).* As an example, we compute the S-spectrum of two Pauli matrices σ_3, σ_1 (compare with [5, Example 4.10]):

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let us consider the matrix $T = \sigma_3 e_1 + \sigma_1 e_2$ and let us compute $T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}$. We obtain the matrix

$$\begin{bmatrix} |s|^2 - 2 - 2\,\mathrm{Re}[s]e_1 & 2(e_1 - \mathrm{Re}[s])e_2 \\ -2(e_1 + \mathrm{Re}[s])e_2 & |s|^2 - 2 + 2\,\mathrm{Re}[s]e_1 \end{bmatrix}$$

whose S-spectrum is $\sigma_S(T) = \{0\} \cup \{s \in \mathbb{R}^3\colon\ \mathrm{Re}[s] = 0,\ |s| = 2\}$.

Theorem 3.7. *Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $s = \mathrm{Re}[s] + \underline{s} \in \rho_S(T)$. Let $S^{-1}(s,T)$ be the S-resolvent operator defined in* (6). *Then $S^{-1}(s,T)$ satisfies the* (*S-resolvent*) *equation*

$$S^{-1}(s,T)s - TS^{-1}(s,T) = \mathcal{I}. \tag{7}$$

Proof. Replacing (6) in the above equation we have

$$-\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I})s + T\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}) = \mathcal{I} \tag{8}$$

and applying $(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I})$ to both hands sides of (8), we get

$$\begin{aligned} &-(T - \bar{s}\mathcal{I})s + \big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)T\big(T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}) \\ &\quad = T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}. \end{aligned}$$

Since T and $T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}$ commute, we obtain the identity

$$-(T - \bar{s}\mathcal{I})s + T(T - \bar{s}\mathcal{I}) = T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}$$

which proves the statement. □

3.2. Properties of the spectrum and the functional calculus

Theorem 3.8 *(Structure of the S-spectrum). Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and let $p = \mathrm{Re}[p] + \underline{p}$ be an S-eigenvalue of T with $\underline{p} \neq 0$. Then all the elements of the sphere $s = \mathrm{Re}[s] + \underline{s}$ with $s_0 = p_0$ and $|\underline{s}| = |\underline{p}|$ are S-eigenvalues of T.*

Proof. It is immediate and is left to the reader. □

Definition 3.9. Let $S^n = \{x \in \mathbb{R}^{n+1}\colon\ |x| = 1\}$ denote the unit sphere of $\mathbb{R}^{n+1}$. For any set $A \subseteq \mathbb{R}^{n+1}$, let us define the *circularization of* A as set

$$circ(A) := \bigcup_{u+vI \in A} u + vS^n.$$

Definition 3.10. Let $T = T_0 + \sum_{j=1}^n e_j T_j \in \mathcal{B}_n^{0,1}(V_n)$. Let $U \subset \mathbb{R}^{n+1}$ be an open set such that:

(i) $\partial(U \cap L_I)$ is union of a finite number of closed rectifiable Jordan curves for every $I \in \mathbb{S}$,
(ii) U contains the circularization of the S-spectrum $\sigma_{\mathrm{S}}(T)$.

A function f is said to be locally s-monogenic on $\sigma_{\mathrm{S}}(T)$ if there exists an open set $U \subset \mathbb{R}^{n+1}$, as above, on which f is s-monogenic.

We will denote by $\mathcal{M}_{\sigma_{\mathrm{S}}(T)}$ the set of locally s-monogenic functions on $\sigma_{\mathrm{S}}(T)$.

Remark 3.11. Note that any open set U containing the circularization of the S-spectrum contains open balls with center in x_0 for all $x_0 \in \sigma_{\mathrm{S}}(T) \cap \mathbb{R}$. Moreover, by Theorem 3.8, if the $(n-1)$-sphere $\sigma = \{s \in \mathbb{R}^{n+1}\colon\ \mathrm{Re}[s] = s_0,\ |\underline{s}| = r\}$ belongs to $\sigma_{\mathrm{S}}(T)$, then U must contain an open annular domain with center in $s_0 \in \mathbb{R}$. In fact, set $m = \min_{s \in circ(\sigma)} \mathrm{dist}(s, \partial U)$. Then for any $R < m$ the annular domain $\{x \in \mathbb{R}^{n+1} \mid r - R < |x - s_0| < r + R\}$ is contained in U.

Theorem 3.12. *Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $f \in \mathcal{M}_{\sigma_{\mathrm{S}}(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be an open set as in Definition* 3.10 *and let $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then the integral*

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\, f(s) \tag{9}$$

does not depend on the choice of the imaginary unit I and on the open set U.

Proof. We first note that the integral (9) does not depend on the choice of U by the Cauchy theorem applied on the plane L_I, see [2]. We now show the independence of the choice of $I \in \mathbb{S}$. Note that since the S-spectrum is bounded (because it is contained in the ball $\{s \in \mathbb{R}^{n+1}\colon\ |s| \leqslant \|T\|\}$) we can choose a finite number of open balls $B_1, \ldots, B_\nu$ and of open annular domains $A_1, \ldots, A_\mu$, $\nu, \mu \in \mathbb{N}$, containing the S-spectrum of T and contained in U. We observe that thanks to the Cauchy theorem we can write:

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\, f(s)$$
$$= \frac{1}{2\pi} \sum_{i=1}^{\nu} \int_{\partial(B_i \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s) + \frac{1}{2\pi} \sum_{i=1}^{\mu} \int_{\partial(A_i \cap L_I)} S^{-1}(s, T)\, ds_I\, f(s), \tag{10}$$

where the right-hand side does not depend on the choice of the B_i's and A_i's. Since f admits series expansion on the B_i's for Taylor theorem and on A_i's by the Laurent theorem (see [2]), we can integrate term by term. Let us now choose another imaginary unit $I' \in \mathbb{S}$, $I \neq I'$, and let us write the analogue of (10) on $L_{I'}$. The S-spectrum contains either real points or, by Theorem 3.8, $(n-1)$-spheres of the type $\{s \in \mathbb{R}^{n+1}\colon \operatorname{Re}[s] = s_0,\ |\underline{s}| = r\}$. Every complex line $L_I = \mathbb{R} + I\mathbb{R}$ contains all the real points belonging to the S-spectrum. Let $\{s \in \mathbb{R}^{n+1}\colon s = s_0 + It,\ |t| = r\}$ be an $(n-1)$-sphere in the S-spectrum. The two points of the sphere lying on the complex line L_I are $s_0 \pm rI$ so, on the plane, they have coordinates $(s_0, \pm r)$. The coordinates of the two intersection points on a different complex line $L_{I'}$ are still $(s_0, \pm r)$, so the right-hand side of (10) does not depend on the choice of $I \in \mathbb{S}$. Thus

$$\frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I\,f(s) = \frac{1}{2\pi}\int_{\partial U_{I'}} S^{-1}(s,T)\,ds_{I'}\,f(s). \qquad \square$$

We give a result that motivates the functional calculus.

Theorem 3.13. *Let $x = \operatorname{Re}[x] + \underline{x}$, $a = \operatorname{Re}[a] + \underline{a} \in \mathbb{R}^{n+1}$, $m \in \mathbb{N}$ and consider the monomial $x^m a$. Consider $T \in \mathcal{B}_n^{0,1}(V_n)$, let $U \subset \mathbb{R}^{n+1}$ be an open set as in Definition* 3.10*, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then*

$$T^m a = \frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I\,s^m a. \tag{11}$$

Proof. Let us consider the power series expansion for the operator $S^{-1}(s,T)$ and a circle C_r centered in the origin and of radius $r > \|T\|$. We have:

$$\frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I\,s^m a = \frac{1}{2\pi}\sum_{n\geqslant 0} T^n \int_{C_r} s^{-1-n+m}\,ds_I\,a = T^m a, \tag{12}$$

since

$$\int_{C_r} ds_I\,s^{-n-1+m} = 0 \quad \text{if } n \neq m, \qquad \int_{C_r} ds_I\,s^{-n-1+m} = 2\pi \quad \text{if } n = m. \tag{13}$$

The Cauchy theorem shows that the above integrals are not affected if we replace C_r by ∂U_I. $\square$

Theorem 3.14 *(Compactness of S-spectrum). Let $T \in \mathcal{B}_n^{0,1}(V_n)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set. Moreover $\sigma_S(T)$ is contained in $\{s \in \mathbb{R}^{n+1}\colon |s| \leqslant \|T\|\}$.*

Proof. Let $U \subset \mathbb{R}^{n+1}$ be an open set as in Definition 3.10, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then

$$\frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I\,s^m = T^m.$$

In particular, for $m = 0$, we have

$$\frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I = \mathcal{I},$$

where $\mathcal{I}$ denotes the identity operator, which shows that $\sigma_S(T)$ is a nonempty set, otherwise the integral would be zero by the vector-valued version of Cauchy's theorem. We now show that the S-spectrum is bounded. The series $\sum_{n\geqslant 0} T^n s^{-1-n}$ converges if and only if $\|T\| < |s|$ so the S-spectrum is contained in the set $\{s \in \mathbb{R}^{n+1}\colon\ |s| \leqslant \|T\|\}$, which is bounded and closed because the complement of $\sigma_S(T)$ is open. Indeed, the function $g : s \mapsto T^2 - 2\,\mathrm{Re}[s]T + |s|^2\mathcal{I}$ is trivially continuous and, by [10, Theorem 10.12], the set $\mathcal{U}(V_n)$ of all invertible elements of $\mathcal{B}_n(V_n)$ is an open set in $\mathcal{B}_n(V_n)$. Therefore $g^{-1}(\mathcal{U}(V_n)) = \rho_S(T)$ is an open set in $\mathbb{R}^{n+1}$. □

The preceding discussion allows to give the following definition.

Definition 3.15. Let $T \in \mathcal{B}_n^{0,1}(V_n)$ and $f \in \mathcal{M}_{\sigma_S(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be an open set as in Definition 3.10, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. We define

$$f(T) = \frac{1}{2\pi}\int_{\partial U_I} S^{-1}(s,T)\,ds_I\, f(s). \tag{14}$$

Remark 3.16. To compare our new functional calculus with the existing versions which the reader can find in the literature (see e.g. [5] and its references) we will now consider the subset $\mathcal{B}_n^1(V_n) \subset \mathcal{B}_n^{0,1}(V_n)$ whose elements are operators of the form $T = \sum_{j=1}^n T_j e_j$ where T_j are linear operators acting on the Banach space V. When $n = 1$, this corresponds to considering a single operator T_1 and $T = T_1 e_1$. To compute the S-spectrum we have to consider the S-eigenvalue equation. Since in this case, the variable $s \in \mathbb{C}$ commutes with T, the S-eigenvalue equation reduces to the classical eigenvalue equation. Finally, since the theory of s-monogenic functions coincides with the theory of holomorphic functions in one complex variables, our calculus reduces to the Riesz–Dunford calculus. Let $f(z)$ be any function holomorphic on the spectrum of T_1. The Riesz–Dunford calculus allows to compute $f(T_1)$. In our case, to get exactly the function $f(T_1)$ we need to consider $\tilde{f}(z) = f(-ze_1) = f(x_1 - e_1 x_0)$ where we have denoted $z = x_0 + e_1 x_1$. This is not surprising, since we are considering not the given operator T_1 as in the classical case, but its tensor with the imaginary unit e_1. The two calculi are therefore equivalent up to this identification.

More generally, the following result holds.

Theorem 3.17. *Let $T \in \mathcal{B}_n^1(V_n)$ and $f \in \mathcal{M}_{\sigma_S(T)}$. The slice-monogenic functional calculus $f \mapsto f(T)$ satisfies the following properties*:

1. *It is consistent with the Riesz–Dunford functional calculus when $n = 1$.*
2. *It is a right module homomorphism.*
3. *$fg(T) = f(T)g(T)$ when f, g are represented by power series with real coefficients.*
4. *It is continuous on the space $\mathcal{B}_n^1(V_n) \times \mathcal{M}_{\sigma_S(T)} \to \mathcal{B}_n(V_n)$.*

Proof. 1. This fact has been shown above.

2. It is an immediate and follows by computations similar to those in Theorem 3.13.

3. The product of two power series with real coefficients is an s-monogenic function (see Proposition 2.13 in [2]). The statement follows from the definition of $f(T)$ (see (14)) and the S-resolvent equation (7).

4. It follows from the definition of $f(T)$ (see (14)) reasoning as for the Riesz–Dunford case. □

Remark 3.18. We can also consider operators $T \in \mathcal{B}_n^{0,1}(V_n)$: those are right-module homomorphisms for which we can prove parts 2–4 of the preceding theorem.

4. Slice-monogenic functional calculus for unbounded operators

We now show the development of a functional calculus for unbounded operators based on the calculus already obtained for bounded operators. Note that if T is a closed operator, the series $\sum_{n\geqslant 0} T^n s^{-1-n}$ does not converge. To overcome this difficulty, we observe that the right-hand side of formula (5) contains the inverse of the operator $T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}$. From a heuristical point of view, and for suitable $s \in \mathbb{R}^{n+1}$, the composition $(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})$ gives a bounded operator on suitable function spaces.

4.1. The S-resolvent operator for unbounded operators

Definition 4.1. Let V be a Banach space and V_n be the two-sided Banach module over $\mathbb{R}_n$ corresponding to $V \otimes \mathbb{R}_n$. Let $T_\mu : \mathcal{D}(T_\mu) \subset V \to V$ be linear closed densely defined operators for $\mu = 0, 1, \ldots, n$. Let

$$\mathcal{D}(T) = \left\{ v \in V_n\colon\ v = \sum_B v_B e_B,\ v_B \in \bigcap_{\mu=0}^{n} \mathcal{D}(T_\mu) \right\} \tag{15}$$

be the domain of the operator

$$T = T_0 + \sum_{j=1}^{n} e_j T_j, \quad T : \mathcal{D}(T) \subset V_n \to V_n.$$

Let us assume that:

(1) $\bigcap_{\mu=0}^{n} \mathcal{D}(T_\mu)$ is dense in V_n,
(2) $T - \bar{s}\mathcal{I}$ is densely defined in V_n,
(3) $\mathcal{D}(T^2) \subset \mathcal{D}(T)$ is dense in V_n,
(4) $T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}$ is one-to-one with range V_n.

The S-resolvent operator is defined by

$$S^{-1}(s, T) = -\big(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\big)^{-1}(T - \bar{s}\mathcal{I}). \tag{16}$$

Remark 4.2. We observe that, in principle, it is necessary also the following assumption:

(5) the operator $(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})$ is the restriction to the dense subspace $\mathcal{D}(T)$ of V_n of a bounded linear operator.

However this assumption is automatically fulfilled since it follows from the identity

$$\bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1}(T - \bar{s}\mathcal{I}) = T\bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1} - \bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1}\bar{s}\mathcal{I},$$

which is a consequence of

$$\bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1}T = T\bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1},$$

that can be easily verified applying on the left to both sides the operator $T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}$.

Definition 4.3. Let $T : \mathcal{D}(T) \to V_n$ be a linear closed densely defined operator as in Definition 4.1. We define the S-resolvent set of T to be the set

$$\rho_S(T) = \bigl\{s \in \mathbb{R}^{n+1} \text{ such that } S^{-1}(s, T) \text{ exists and it is in } \mathcal{B}_n(V_n)\bigr\}. \tag{17}$$

We define the S-spectrum of T as the set

$$\sigma_S(T) = \mathbb{R}^{n+1} \setminus \rho_S(T). \tag{18}$$

Theorem 4.4 *(S-resolvent operator equation). Let $T : \mathcal{D}(T) \to V_n$ be a linear closed densely defined operator. Let $s \in \rho_S(T)$. Then $S^{-1}(s, T)$ satisfies the (S-resolvent) equation*

$$S^{-1}(s, T)s - TS^{-1}(s, T) = \mathcal{I}.$$

Proof. It follows by a direct computation replacing the S-resolvent operator into the S-resolvent equation. □

Theorem 4.5. *Let $T : \mathcal{D}(T) \to V_n$ be a linear closed densely defined operator. Let $s \in \rho_S(T)$. Then the S-resolvent operator can be represented by*

$$S^{-1}(s, T) = \sum_{n\geqslant 0}\bigl(\mathrm{Re}[s]\mathcal{I} - T\bigr)^{-n-1}\bigl(\mathrm{Re}[s] - s\bigr)^n \tag{19}$$

if and only if

$$|\underline{s}|\,\bigl\|\bigl(\mathrm{Re}[s]\mathcal{I} - T\bigr)^{-1}\bigr\| < 1. \tag{20}$$

Proof. Observe that

$$\begin{aligned}\bigl(T^2 - 2T\,\mathrm{Re}[s] + |s|^2\mathcal{I}\bigr)^{-1} &= \bigl(\mathcal{I} + |\underline{s}|^2\bigl(T - \mathrm{Re}[s]\mathcal{I}\bigr)^{-2}\bigr)^{-1}\bigl(T - \mathrm{Re}[s]\mathcal{I}\bigr)^{-2} \\ &= \sum_{n\geqslant 0}(-1)^n|\underline{s}|^{2n}\bigl(T - \mathrm{Re}[s]\mathcal{I}\bigr)^{-2n}\bigl(T - \mathrm{Re}[s]\mathcal{I}\bigr)^{-2},\end{aligned}$$

so we get

$$\begin{aligned} S^{-1}(s,T) &= \sum_{n\geqslant 0}(-1)^{n+1}|\underline{s}|^{2n}\big(T-\mathrm{Re}[s]\mathcal{I}\big)^{-2n-1}\big(\mathcal{I}-\big(T-\mathrm{Re}[s]\mathcal{I}\big)^{-1}\underline{s}\big) \\ &= \sum_{n\geqslant 0}\big(\mathrm{Re}[s]\mathcal{I}-T\big)^{-2n-1}\big(\mathrm{Re}[s]-s\big)^{2n}+\sum_{n\geqslant 0}\big(\mathrm{Re}[s]\mathcal{I}-T\big)^{-2n-2}\big(\mathrm{Re}[s]-s\big)^{2n+1} \\ &= \big(\mathrm{Re}[s]\mathcal{I}-T\big)^{-1}\sum_{n\geqslant 0}\big(\mathrm{Re}[s]\mathcal{I}-T\big)^{-n}\big(\mathrm{Re}[s]-s\big)^{n} \end{aligned}$$

which converges in the bounded linear operator space if and only if (20) holds. □

Remark 4.6. The previous result implies that given any element $s \in \rho_S(T)$, then any other $s' \in \mathbb{R}^{n+1}$ with $|\underline{s}'| = |\underline{s}|$ belongs to the resolvent set, independently on its real part s_0, in fact s' satisfies the inequality (20).

4.2. Functional calculus for unbounded operators

Let V be a Banach space and $T = T_0 + \sum_{j=1}^{m} e_j T_j$ where $T_\mu : \mathcal{D}(T_\mu) \to V$ are linear operators for $\mu = 0, 1, \dots, n$. If at least one of the T_j's is an unbounded operator then its resolvent is not defined at infinity. It is therefore natural to consider closed operators T for which the resolvent $S^{-1}(s,T)$ is not defined at infinity and to define the extended spectrum as

$$\overline{\sigma}_S(T) := \sigma_S(T) \cup \{\infty\}.$$

Let us consider $\overline{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ endowed with the natural topology: a set is open if and only if it is union of open discs $D(x,r)$ with center at points in $x \in \mathbb{R}^{n+1}$ and radius r, for some r, and/or union of sets the form $\{x \in \mathbb{R}^{n+1} \mid |x| > r\} \cup \{\infty\} = D'(\infty, r) \cup \{\infty\}$, for some r.

Definition 4.7. We say that f is s-monogenic function at ∞ if $f(x)$ is an s-monogenic function in a set $D'(\infty, r)$ and $\lim_{x\to\infty} f(x)$ exists and it is finite. We define $f(\infty)$ to be the value of this limit.

Definition 4.8. Let $T : \mathcal{D}(T) \to V_n$ be a linear closed operator as in Definition 4.1. Let $U \subset \mathbb{R}^{n+1}$ be an open set such that:

(i) $\partial(U \cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$,
(ii) U contains the circularization of the S-spectrum $\sigma_S(T)$.

A function f is said to be locally s-monogenic on $\overline{\sigma}_S(T)$ if it is s-monogenic an open set $U \subset \mathbb{R}^{n+1}$ as above and at infinity.

We will denote by $\mathcal{M}_{\overline{\sigma}_S(T)}$ the set of locally s-monogenic functions on $\overline{\sigma}_S(T)$.

Consider $\alpha \in \mathbb{R}^{n+1}$ and the homeomorphism

$$\Phi : \overline{\mathbb{R}}^{n+1} \to \overline{\mathbb{R}}^{n+1} \quad \text{for } \alpha \in \mathbb{R}^{n+1}$$

defined by

$$p = \Phi(s) = (s-\alpha)^{-1}, \qquad \Phi(\infty) = 0, \qquad \Phi(\alpha) = \infty.$$

Definition 4.9. Let $T : \mathcal{D}(T) \to V_n$ be a linear closed operator as in Definition 4.1 with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{M}_{\overline{\sigma}_S(T)}$. Let us consider

$$\phi(p) := f\big(\Phi^{-1}(p)\big)$$

and the operator

$$A := (T - k\mathcal{I})^{-1}, \quad \text{for some } k \in \rho_S(T) \cap \mathbb{R}.$$

We define

$$f(T) = \phi(A). \tag{21}$$

Remark 4.10. Observe that, if $\alpha = k \in \mathbb{R}$, we have that:

(i) the function ϕ is s-monogenic because it is the composition of the function f which is s-monogenic and $\Phi^{-1}(p) = p^{-1} + k$ which is s-monogenic with real coefficients;
(ii) in the case $k \in \rho_S(T) \cap \mathbb{R}$ we have that $(T - k\mathcal{I})^{-1} = -S^{-1}(k, T)$.

Theorem 4.11. *If $k \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and Φ, ϕ are as above, then $\Phi(\overline{\sigma}_S(T)) = \sigma_S(A)$ and the relation $\phi(p) := f(\Phi^{-1}(p))$ determines a one-to-one correspondence between $f \in \mathcal{M}_{\overline{\sigma}_S(T)}$ and $\phi \in \mathcal{M}_{\overline{\sigma}_S(A)}$.*

Proof. First we consider the case $p \in \sigma_S(A)$ and $p \neq 0$. Recall that

$$S^{-1}(p, A) = -\big(A^2 - 2A\,\mathrm{Re}[p] + |p|^2\mathcal{I}\big)^{-1}(A - \overline{p}\mathcal{I}),$$

from which we obtain

$$\big(A^2 - 2A\,\mathrm{Re}[p] + |p|^2\mathcal{I}\big)S^{-1}(p, A) = -(A - \overline{p}\mathcal{I}).$$

Let us apply the operator A^{-2} on the left to get

$$\big(\mathcal{I} - 2\,\mathrm{Re}[p]A^{-1} + A^{-2}|p|^2\big)S^{-1}(p, A) = -\big(A^{-1} - A^{-2}\overline{p}\big).$$

Now we use the relations

$$A^{-1} = T - k\mathcal{I}, \qquad A^{-2} = T^2 - 2kT + k^2\mathcal{I}$$

to get

$$\big(\mathcal{I} - 2\operatorname{Re}[p](T - k\mathcal{I}) + \big(T^2 - 2kT + k^2\mathcal{I}\big)|p|^2\big)S^{-1}(p, A)$$
$$= -\big(T - k\mathcal{I} - \big(T^2 - 2kT + k^2\mathcal{I}\big)\overline{p}\big).$$

Using the identities

$$s_0|p|^2 = k|p|^2 + p_0, \qquad |p|^2|s|^2 = k^2|p|^2 + 2p_0k + 1, \tag{22}$$

we have

$$\big(T^2|p|^2 - 2s_0|p|^2T + |s|^2|p|^2\mathcal{I}\big)S^{-1}(p, A) = -\big(T - k\mathcal{I} - \big(T^2 - 2kT + k^2\mathcal{I}\big)\overline{p}\big).$$

So we get the equalities

$$\begin{aligned}
S^{-1}(p, A) &= -\frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\big(T - k\mathcal{I} - \big(T^2 - 2kT + k^2\mathcal{I}\big)\overline{p}\big)\\
&= -\frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\big(T\overline{p}^{-1} - k\overline{p}^{-1}\mathcal{I} - T^2 + 2kT - k^2\mathcal{I}\big)\overline{p}\\
&= -\frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\big(-\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)\overline{p}\\
&\quad + \big(T\big(2k - 2s_0 + \overline{p}^{-1}\big) + \big(|s|^2 - k^2 - k\overline{p}^{-1}\big)\mathcal{I}\big)\overline{p}\big)\\
&= \frac{\overline{p}}{|p|^2}\mathcal{I} - \frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\big(T + \big(|s|^2 - k^2 - k\overline{p}^{-1}\big)\\
&\quad \times \big(2k - 2s_0 + \overline{p}^{-1}\big)^{-1}\mathcal{I}\big)\big(2k - 2s_0 + \overline{p}^{-1}\big)\overline{p}.
\end{aligned}$$

With some calculation we get

$$\begin{aligned}
S^{-1}(p, A) = p^{-1}\mathcal{I}&\\
&- \frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\big(T - \overline{s}\mathcal{I} + \big[\overline{s} + \big(|s|^2 - k^2 - k\overline{p}^{-1}\big)\\
&\times \big(2k - 2s_0 + \overline{p}^{-1}\big)^{-1}\big]\mathcal{I}\big)\big(2k - 2s_0 + \overline{p}^{-1}\big)\overline{p}
\end{aligned}$$

and also

$$S^{-1}(p, A) = p^{-1}\mathcal{I} + S^{-1}(s, T)\lambda - \frac{1}{|p|^2}\big(T^2 - 2s_0T + |s|^2\mathcal{I}\big)^{-1}\Lambda\mathcal{I},$$

where

$$\lambda := \big(2k - 2s_0 + \overline{p}^{-1}\big)\frac{\overline{p}}{|p|^2}$$

and

$$\Lambda := \big[\overline{s} + \big(|s|^2 - k^2 - k\overline{p}^{-1}\big)\big(2k - 2s_0 + \overline{p}^{-1}\big)^{-1}\big]\big(2k - 2s_0 + \overline{p}^{-1}\big)\overline{p}.$$

Using the identities (22) we have that

$$\lambda = \left(2k - 2s_0 + \overline{p}^{-1}\right)\frac{\overline{p}}{|p|^2} = \left(2k - 2s_0 + \overline{p}^{-1}\right)p^{-1}$$
$$= (2k - 2s_0 + \bar{s} - k)(s - k) = -(s - k)^2 = -p^{-2}$$

and with analogous calculation we get

$$\Lambda = \bar{s}\left(2k - 2s_0 + \overline{p}^{-1}\right)\overline{p} + \left(|s|^2 - k^2 - k\overline{p}^{-1}\right)\overline{p} = 0.$$

So

$$S^{-1}(p, A) = p^{-1}\mathcal{I} - S^{-1}(s, T)p^{-2},$$

but also

$$S^{-1}(s, T) = p\mathcal{I} - S^{-1}(p, A)p^2. \tag{23}$$

So $p \in \rho_S(A)$, $p \neq 0$ then $s \in \rho_S(T)$.

Now take $s \in \rho_S(T)$ and observe that, from the definitions of $S^{-1}(s, T)$ and of A, with analogous calculation as above, we get

$$S^{-1}(s, T) = -AS^{-1}(p, A)p,$$

so if $s \in \rho_S(T)$ then $p \in \rho_S(A)$, $p \neq 0$.

The point $p = 0$ belongs to $\sigma_S(A)$ since $S^{-1}(0, A) = A^{-1} = T - k\mathcal{I}$ is unbounded. The last part of the statement is evident from the definition of Φ. □

Theorem 4.12. *Let $T : \mathcal{D}(T) \to V_n$ be a linear closed operator as in Definition* 4.1 *with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{M}_{\overline{\sigma}_S(T)}$. Then operator $f(T)$ defined in* (21) *is independent of $k \in \rho_S(T) \cap \mathbb{R}$.*

Let W, be an open set such that $\overline{\sigma}_S(T) \subset W$ and let f be an s-monogenic function on $W \cup \partial W$. Set $W_I = W \cap L_I$ for $I \in \mathbb{S}$ be such that its boundary ∂W_I is positively oriented and consists of a finite number of rectifiable Jordan curves. Then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi}\int_{\partial W_I} S^{-1}(s, T)\, ds_I\, f(s). \tag{24}$$

Proof. The first part of the statement follows from the validity of formula (24) since the integral is independent of k.

Given $k \in \rho_S(T) \cap \mathbb{R}$ and the set W we can assume that $k \notin W_I \cup \partial W_I$, $\forall I \in \mathbb{S}$, since otherwise, by the Cauchy theorem, we can replace W by W', on which f is s-monogenic, such that $k \notin W_I' \cup \partial W_I'$, without altering the value of the integral (24). Moreover, the integral (24) is independent of the choice of $I \in \mathbb{S}$, thanks to the structure of the spectrum (see Theorem 3.8) and an argument similar to the one used to prove Theorem 3.12.

We have that $\mathcal{V}_I := \Phi^{-1}(W_I)$ is an open set that contains $\sigma_S(T)$ and its boundary $\partial\mathcal{V}_I = \Phi^{-1}(\partial W_I)$ is positively oriented and consists of a finite number of rectifiable Jordan curves. Using the relation (23) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial W_I} S^{-1}(s,T)\,ds_I\,f(s)\\
&= -\frac{1}{2\pi}\int_{\partial\mathcal{V}_I} \big(p\mathcal{I} - S^{-1}(p,A)p^2\big)p^{-2}\,dp_I\,\phi(p)\\
&= -\frac{1}{2\pi}\int_{\partial\mathcal{V}_I} p^{-1}\,dp_I\,\phi(p) + \frac{1}{2\pi}\int_{\partial\mathcal{V}_I} S^{-1}(p,A)\,dp_I\,\phi(p)\\
&= -\mathcal{I}\phi(0) + \phi(A)
\end{aligned}$$

now by definition $\phi(A) = f(T)$ and $\phi(0) = f(\infty)$ we obtain

$$\frac{1}{2\pi}\int_{\partial W_I} S^{-1}(s,T)\,ds_I\,f(s) = -\mathcal{I}f(\infty) + f(T). \qquad \square$$

Acknowledgments

The first and the third authors are grateful to Chapman University for the hospitality during the period in which this paper was written. They are also indebted to G.N.S.A.G.A. of INdAM and the Politecnico di Milano for partially supporting their visit. The authors thank the anonymous referee for the valuable suggestions to improve the paper.

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