JOURNAL OF COMBINATORIAL THEORY 5, 192–197 (1968)

A Partial Characterization of Clique Graphs

RONALD C. HAMELINK

Department of Mathematics, Michigan State University

East Lansing, Michigan 48823

Communicated by R. C. Bose

ABSTRACT

A partial characterization of clique graphs is given here, including a method for constructing a graph having a given graph as its clique graph, provided the given graph meets certain conditions. In addition, an example is presented to show the existence of graphs which are not clique graphs.

1. DEFINITIONS AND NOTATIONS

Graphs in this paper will be ordered pairs of sets, the first set, the vertex set, which is non-empty; and the second, the edge set, consisting of two element subsets of the vertex set. For a graph G, $V(G)$ and $E(G)$ will denote its vertex set and its edge set, respectively. A *subgraph* F of G is a graph where $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A subgraph F of graph G will be called a *section graph* (or full subgraph) of G if $E(F) = \{\{x, y\}/x, y \in V(F$ and $\{x, y\} \in E(G)\}$. A *clique* C of graph G is a section graph of G which is complete and is contained in no larger complete subgraph of G (i.e., C is a maximal complete subgraph of G). A *clique graph* of G is a graph F and a 1-1, onto mapping α from $V(F)$ to the set of cliques of G which preserves incidence; that is $x, y \in V(F)$ are adjacent iff $V(\alpha(x)) \cap V(\alpha(y)) \neq \emptyset$

The definition of clique graph implies at once that if F_1 and F_2 are clique graphs of the same graph G then F_1 is isomorphic to F_2. In this sense we will speak of "the" clique graph of G and denote it by $K(G)$. Also from the definition of a clique it is clear that each $x \in V(G)$ belongs to at least one clique.

Vertices which belong to exactly one clique will be called *unicliqual vertices*, otherwise *multicliqual vertices*.

The principal problem under consideration is this; how can we tell if a given graph H is the clique graph of some graph G, and if it is, how do we find one such graph G?

2. A Partition of $V(H)$

Let H be an arbitrary, but fixed graph, and $K_1, K_2, \ldots, K_n$ denote the cliques of H. Let $X = \{1, 2, \ldots, n\}$ and let Y be the set of non-empty subsets of X. Define

$$W_S = \bigcap \{V(K_i) \mid i \in S\} \qquad \text{for} \quad S \in Y,$$

and

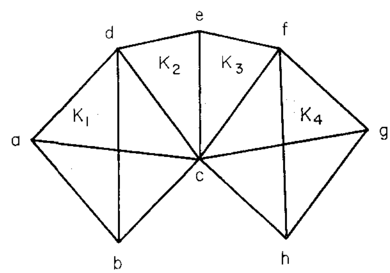
$$V_S = W_S - \bigcup \{W_T \mid T \in Y, S \subset T, \qquad \text{and} \quad S \neq T\},$$

and

$$N_S = \text{the number of elements in } V_S.$$

Example:

H:



$$X = \{1, 2, 3, 4\}$$

$V_X = \{c\}$	$V_{\{3,4\}} = \{f\}$
$V_{\{1,2\}} = \{d\}$	$V_{\{1\}} = \{a, b\}$
$V_{\{2,3\}} = \{e\}$	$V_{\{4\}} = \{g, h\}$

The remaining set V_S are empty. In this example the sets V_S which are non-empty partition $V(H)$. We will prove that this is a general result.

Lemma 1. *If* $x \in V(H)$ and $S = \{i \mid x \in V(K_i)\}$ *then* $x \in V_S$.

Proof: $x \in V(K_i)$ for $i \in S$ implies that $x \in W_S$. If $T \supset S$ and $T \neq S$, then for $j \in T - S$ $x \notin V(K_j)$ which implies $x \notin W_T$, so $x \in W_S - W_T$ for each such T. Hence $x \in W_S - \bigcup \{W_T \mid T \in Y, S \subset T, \text{and } S \neq T\} = V_S$.

Lemma 2. *If* $S \neq T$, *S and* $T \in Y$, *then* $V_S \cap V_T = \emptyset$.

Proof: Assume $x \in V_S \cap V_T$. Then $x \in W_S \cap W_T$, which implies $x \in W_{S \cup T}$. But, since $S \subset S \cup T$, then $S \neq S \cup T$ would imply that

$x \notin V_S$, so that $S = S \cup T$. Likewise we see that $T = S \cup T$ and then $S = T$ follows.

Lemmas 1 and 2 constitute the proof of the following theorem.

THEOREM 1. *Let H be a graph, then the non-empty sets in $\{V_S \mid S \in Y\}$ form a partition of $V(H)$.*

An immediate result is the following.

COROLLARY. *The order of H is $\sum N_S$, where the sum is taken over all $S \in Y$.*

The partitioning of $V(H)$ we have defined will be referred to as the partition induced by the cliques, or as the clique partition of $V(H)$. In the next section the clique partition will play a central role in our construction of a graph whose clique graph is known.

3. CONSTRUCTION

For the graph H we now construct a graph G which under suitable restrictions on H will have H as its clique graph. (see Theorem 4).

Let

$$V(G) = \{v_1, v_2, ..., v_n\} \cup V(H), \qquad n = \text{number of cliques of } H$$

and

$$E(G) = \{\{v_i, v_j\} \mid \text{for some } S \in Y,\ i \text{ and } j \in S, \text{ and } V_S \neq \emptyset\}$$
$$\cup \{\{x, v_i\} \mid x \in V_S \text{ and } i \in S\}.$$

THEOREM 2. *The section graph F, of G, on the vertices $\{v_1, v_2, ..., v_n\}$ with the mapping $\alpha(v_i) = K_i$, is the clique graph of H.*

PROOF: Let $x \in V(K_i) \cap V(K_j)$, then let $S \in Y$ be such that $x \in V_S$. By Lemma 1, i and j are in S so $\{v_i, v_j\} \in E(G)$. If $V(K_i) \cap V(K_j) = \emptyset$, then for each $x \in V(H)$, if $x \in V_S$, both i and j cannot be in S, hence for no $S \in Y$, i and $j \in S$, if $V_S \neq \emptyset$. So $\{v_i, v_j\} \notin E(G)$. α as defined is clearly 1-1 and onto, and we see that it preserves incidence.

LEMMA 3. *Each section graph C_x of G for $x \in V(H)$ on the vertices $\{x\} \cup \{v_i \mid i \in S \text{ where } x \in V_S\}$ is a clique in G.*

PROOF: If $i \in S$ and $x \in V_S$, then $\{x, v\} \in E(G)$, and if i and $j \in S$ for $x \in V_S$ it follows that $\{v_i, v_j\} \in E(G)$; hence C_x is a complete subgraph of G. If $k \notin S$, then $\{v_k, x\} \notin E(G)$, so C_x is a maximal complete subgraph of G and is therefore a clique of G, with the unicliqual vertex x.

LEMMA 4. *The mapping $\beta : V(H)$ into the set of cliques of G given by $\beta(x) = C_x$ is a* 1-1 *mapping which preserves incidence.*

PROOF: If $\{x, y\} \in E(H)$, then some clique K_i of H contains both x and y, so $i \in S \cap T$ where $x \in V_S$ and $y \in V_T$. Then $v_i \in V(C_x) \cap V(C_y)$. If $\{x, y\} \notin E(H)$ then for each $i \in S$, for $x \in V_S$, $y \notin V(K_i)$. Hence if $v_i \in V(C_x)$ it follows that $v_i \notin V(C_y)$, and $V(C_x) \cap V(C_y) = \emptyset$.

We obtain from Lemma 4 the following theorem.

THEOREM 3. *H is isomorphic to a section graph of $K(G)$.*

THEOREM 4. *Let H be a graph with clique graph $K(H)$, and associated mapping $\gamma : V(K(H))$ to cliques of H. If for each clique C in $K(H)$ the intersection of the cliques in $\gamma(V(C))$ is not empty, then H is the clique graph of the graph G constructed above.*

PROOF: The mapping β of Lemma 4 already is 1-1 and incidence preserving. All that is needed now is to establish that β is onto. Let C be any clique in G. If C contains an $x \in V(H)$, then $C = C_x$ because x is a unicliqual vertex. Suppose $V(C) \subseteq \{v_1, v_2, ..., v_n\}$. Then the graph F of Theorem 2 is isomorphic to $K(H)$, hence $\gamma^{-1}(\alpha(V(C))$ is a clique in $K(H)$. By hypothesis the cliques in $\gamma(\gamma^{-1}(\alpha(V(C)))) = \alpha(V(C))$ have a non-empty intersection. Then let y be a vertex common to all the cliques of H in $\alpha(V(C))$. If $K_i \in \alpha(V(C))$, then $\{y, v_i\} \in E(G)$. Hence $y \in V(H)$ is adjacent to every vertex in C, which contradicts the assumption that C is a clique in G. So β is onto $K(G) = H$.

COROLLARY. *Any graph H is a section graph of a clique graph $\bar{H}$ where* $| V(\bar{H}) | = 1 + | V(H) |$.

PROOF: Adjoin to the graph H one new vertex a and make a adjacent to every vertex in H. Call this graph $\bar{H}$. Every clique in $\bar{H}$ contains the vertex a, hence by Theorem 4 $\bar{H}$ is a clique graph.

4. COUNTEREXAMPLE

In this section we will see that for some graphs the Corollary to Theorem 4 is the best result possible. The graph G below is now a general graph and not necessarily the one constructed in Section 3.

THEOREM 5. *Let $H = K(G)$ with α the associated mapping given by $\alpha(u) = C_u$. Let C be a clique in H with $V(C) = \{x, y, z\}$. Then either $V(C_x) \cap V(C_y) \subseteq V(C_z)$, or $V(C_y) \cap V(C_z) \subseteq V(C_x)$, or $V(C_z) \cap V(C_x) \subseteq V(C_y)$.*

PROOF: Assume that the conclusion does not hold. Let u, v, and w be, respectively, in $(V(C_x) \cap V(C_y)) - V(C_z)$, $(V(C_y) \cap V(C_z)) - V(C_x)$, and $(V(C_z) \cap V(C_x)) - V(C_y)$. Then the section graph on u, v, and w is complete in G, hence is contained in a clique C_t of G. C_t is distinct from C_x, C_y, and C_z, so t is adjacent to x, y, and z in H contradicting the assumption that the section graph on $\{x, y, z\}$ is a clique in H.

THEOREM 6. *Any graph H containing a clique T on 3 vertices $\{x, y, z\}$ and 3 other cliques A, B and C so related that*

$$V(T) \cap V(A) = \{x, y\},$$

$$V(T) \cap V(B) = \{y, z\},$$

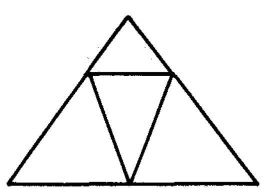
and

$$V(T) \cap V(C) = \{z, x\}$$

is not the clique graph of any graph.

PROOF: Assume that H was a clique graph of G, with associated mapping α, where we use the notation $\alpha(u) = C_u$. Using Theorem 5, we may without loss of generality assume that $V(C_x) \cap V(C_y) \subseteq V(C_z)$. Let $a \in V(A) - \{x, y\}$. Then a is adjacent to x and y but not z, so C_a intersects C_x and C_y but not C_z. Let $u \in V(C_a) \cap V(C_x)$, $v \in V(C_a) \cap V(C_y)$, and $w \in V(C_x) \cap V(C_y)$. These 3 vertices are distinct and mutually adjacent hence are contained in a clique C_t of G. Now C_t is distinct from C_x, C_y, C_z, but has a non-empty intersection with each of them. Hence $t \in V(H)$ is adjacent to x, y, and z, which contradicts the assumption that T is a clique in H.

EXAMPLE: The following six vertex graph is not the clique graph of any graph.



5. Conclusion

The answer to the question of whether a particular graph H is a clique graph depends, apparently, on the clique graph of H and the clique partition of $V(H)$. The clique partition may also be of use in other parts of graph theory.

Acknowledgment

Professor B. M. Stewart suggested Theorem 4 which is a strengthening of the author's original theorem where $V_X \neq \emptyset$ was required.