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Stability Theory for Ordinary Differential Equations*

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1. Introduction

The stability theory presented here was developed in a series of papers ([*6*]–[*9*]). The purpose of this paper is to refine the fundamental theorems and to provide proofs for results which previously had only been stated. The applications of this theory have been discussed and illustrated in the above-mentioned references and in [*5*]. For a discussion of and references to the exploitation of the "invariance principle" used here see [*8*] and [*9*]. For difference equations and applications to numerical analysis see [*4*].

In order to see how much is gained when it is known that the limit sets of solutions have an invariance property and for completeness we give first in Section 2 the best result we know for locating limit sets of nonautonomous systems. This result is an improvement of a theorem given by Yoshizawa in [*12*]. Section 3 is for autonomous ordinary differential equations, and from Theorems 2 and 3 follow all of the classical Liapunov results on the stability and instability of these systems.

2. Nonautonomous Systems

We want first to define a "Liapunov function" relative to a nonautonomous system

$$\dot{x} = f(t, x). \tag{1}$$

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Let G be a set in R^n and let G^* be an open set of R^n containing $\bar{G}$, the closure of G. We assume that f is a continuous function on $[0, \infty) \times G^*$ into R^n.

Let V be a continuous function on $[0, \infty) \times G^*$ to R that is locally Lipschitzian; that is, corresponding to each (t, x) in $[0, \infty) \times G^*$ and some neighborhood N of (t_0, x) there is a constant L such that

$$| V(t, x) - V(t, y)| \leqslant L \mid x - y \mid \qquad \text{for all} \quad (t, y) \quad \text{in} \quad N.$$

Define

$$\dot{V}(t, x) = \liminf_{h \to 0^+} h^{-1}[V(t + h, x + hf(t, x)) - V(t, x)].$$

Now let $x(t)$ be a solution of (1) that remains in G for $t \geqslant 0$ and let $[0, \omega)$ be its maximal positive interval of definition (ω can be ∞). Then under our assumption that V is locally Lipschitzian, $\dot{V}$ is related to the rate of change of V along solutions by (see, e.g., [*13*] p. 3 with V replaced by $-V$)

$$\dot{V}(t, x(t)) = D_+V(t, x(t)), \tag{2}$$

where D_+ is the lower right-hand derivative (with respect to t). We then make the following observation:

LEMMA 1. *Let V be continuous and locally Lipschitzian on* $[0, \infty) \times G^*$ *to R, and let $x(t)$ be a solution of* (1) *that remains in G for all $t \in$* $[0, \omega)$, *the maximum positive interval of definition of $x(t)$. If $\dot{V}(t, x) \leqslant 0$ for all $t \in$* $[0, \omega)$ *and all $x \in G$, then $V(t, x(t))$ is differentiable almost everywhere on* $[0, \omega)$ *and on* $[0, \omega)$

$$V(t, x(t)) - V(0, x(0)) \leqslant \int_0^t \dot{V}(\tau, x(\tau))\, d\tau. \tag{3}$$

Proof. It follows from (2), since $V(t, x(t))$ is continuous on $[0, \omega)$, that $V(t, x(t))$ is nonincreasing (see [*10*], Sec. 34.1; note in this reference that monotonic decreasing means nonincreasing). Therefore $V(t, x(t))$ is differentiable almost everywhere and (3) follows (see Sec. 34.2 of [*10*]).

In applications it is usually true that V is C^1 (has continuous first partials). Then

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x)$$

is easily computed and

$$V(t, x(t)) - V(0, x(0)) = \int_0^t \dot{V}(\tau, x(\tau))\, d\tau.$$

Note now that as far as this lemma is concerned it really does not matter whether the right or left-hand limit is used to define $\dot{V}$. For functional differential equations $\dot{V}$ must be defined relative to solutions and there a right-hand limit must be used.

We shall say that V is a *Liapunov function of* (1) *on* G if it is continuous and locally Lipschitzian on $[0, \infty) \times G^*$ and if

(i) given x in $\bar{G}$ there is a neighborhood N of x such that $V(t, x)$ is bounded from below for all $t \geqslant 0$ and all x in $N \cap G$.

(ii) $\dot{V}(t, x) \leqslant -W(x) \leqslant 0$ for all $t \geqslant 0$ and all x in G, where W is continuous on $\bar{G}$.

We want to see now what information can be obtained from a Liapunov function. Assume that V is a Liapunov function of (1) on G and it can be that $W = 0$. Let $G_1 \subset G$ be such that $V(t, x) < \alpha$ for all $t \geqslant 0$ and all $x \in G_1$, and let $G_2 \subset G$ be such that $V(t, x) \geqslant \alpha$ for all $t \geqslant 0$ and all $x \in G_2$. Then, if $x(t)$ is any solution of (1) that remains in G for all $t \geqslant 0$, $V(t, x(t))$ is nonincreasing and $x(t)$ cannot go from G_1 to G_2 in increasing time. It is this type of argument that yields sufficient conditions for boundedness and simple stability. For instance, one obtains immediately,

If (a) *V is a Liapunov function of* (1) *on G,* (b) *$V(t_0, x) < a$ for all $x \in G$ and some $t_0 \geqslant 0$, and* (c) *$\alpha(x) \leqslant V(t, x)$ on $[0, \infty) \times G$, where relative to G $\liminf_{|x| \to \infty} \alpha(x) = a$, then solutions which start in G at time t_0 and remain in the future in G are bounded in the future* ($a = \infty$ is the usual case).

We are mainly interested in seeing what additional information a Liapunov function can give on the asymptotic behaviors of solutions. Let $x(t)$ be any solution of (1) that remains in G for all $t \geqslant 0$ with $[0, \omega)$ its maximal positive interval of definition. If ω is finite, then $x(t)$ is unbounded in the future. We want our fundamental theorem on nonautonomous systems to include theorems on finite escape times, unboundedness and other instabilities, and therefore do not want to restrict ourselves to bounded solutions. For this reason we compactify the space R^n and denote the one-point compactification of R^n by R^n_∞. Let $d(x, y) = |x - y|$ denote the Euclidean distance between x and y, and define $d(x, \infty) = 1/|x|$. For Q a set in R^n_∞, define $d(x, Q) = \inf\{d(x, y); y \in Q\}$. Let $x(t)$ be a continuous function on $[0, \omega)$; then $x(t) \to Q$ as $t \to \omega^-$ means $d(x(t), Q) \to 0$ as $t \to \omega^-$. If we know that $x(t) \to Q$ as $t \to \omega^-$, where Q is not all of R^n_∞, then we have obtained information about the asymptotic behavior of $x(t)$ as $t \to \omega^-$. We would like to be able to find the smallest closed set Ω that $x(t)$ approaches as $t \to \omega^-$. This set Ω is well defined in R^n_∞ and is G. D. Birkhoff's positive limit set. The set Ω is nonempty, closed, and connected in R^n_∞.

If V is a Liapunov function for (1) on G, we define

$$E = \{x;\ W(x) = 0, x \in \bar{G}\} \qquad \text{and} \qquad E_\infty = E \cup \{\infty\}.$$

If $x(t)$ is a solution of (1) that remains in G for $t \geqslant 0$, then our first theorem gives sufficient conditions that $x(t) \to E_\infty$ as $t \to \infty$. Since the positive limit set Ω of $x(t)$ is contained in E_∞, which is a closed set in R^n_∞, this gives information about the asymptotic behavior of $x(t)$ as $t \to \omega^-$.

THEOREM 1. *Let V be a Liapunov function for (1) on G, and let $x(t)$ be a solution of (1) that remains in G for $t \geqslant t_0 \geqslant 0$ with $[t_0, \omega)$ the maximal future interval of definition of $x(t)$.*

(a) *If for each $p \in \bar{G}$ there is a neighborhood N of p such that $|f(t, x)|$ is bounded for all $t \geqslant 0$ and all x in $N \cap G$, then either $x(t) \to \infty$ as $t \to \omega^-$, or $\omega = \infty$ and $x(t) \to E_\infty$ as $t \to \infty$.*

(b) *If $W(x(t))$ is absolutely continuous and its derivative is bounded from above (or from below) almost everywhere on $[t_0, \omega)$ and if $\omega = \infty$, then $x(t) \to E_\infty$ as $t \to \infty$.*

Proof. Let $p \in R^n$ be a finite positive limit point of $x(t)$. Then there is an increasing sequence t_n such that $t_n \to \omega^-$ and $x(t_n) \to p$ as $n \to \infty$. Then (i) and (ii) in the definition of a Liapunov function imply that $V(t_n, x(t_n))$ is nonincreasing and bounded from below. Therefore $V(t_n, x(t_n)) \to c$ as $n \to \infty$, and again since $V(t, x(t))$ is nonincreasing, $V(t, x(t)) \to c$ as $t \to \omega^-$. We also know by Lemma 1 that on $[0, \omega)$

$$V(t, x(t)) - V(t_0, x(t_0)) \leqslant -\int_{t_0}^{t} W(x(\tau))\, d\tau,$$

and hence

$$\int_{t_0}^{\omega} W(x(\tau))\, d\tau < \infty.$$

Part (a). Assume that p is not in E. Then $W(p) > 2\delta > 0$, and within some neighborhood $N(2\epsilon, p)$ of radius 2ϵ about p, we would have $W(x) > \delta$ for x in $N(2\epsilon, p) \cap \bar{G}$. Now it could be that $x(t)$ remains in $N(2\epsilon, p)$ for all t in $[t_1, \omega)$ for some $t_1 \geqslant t_0$. If this were true, then ω would be ∞, and this would contradict $\int_{t_0}^{\infty} W(x(\tau))\, d\tau < \infty$. The other possibility is that $x(t)$ goes in and out of $N(2\epsilon, p)$ an infinite number of times. This would mean that $x(t)$ travels an infinite distance within $N(2\epsilon, p)$, since p is a positive limit point and $x(t)$ must enter $N(\epsilon, p)$ an infinite number of times. This means $x(t)$ travels an infinite distance in $N(2\epsilon, p)$. Since its speed $|\dot{x}(t)|$ is bounded in $N(2\epsilon, p)$ for ϵ sufficiently small, $x(t)$ must remain in $N(2\epsilon, p)$ an infinite length of time. Again this implies $\omega = \infty$, and this contradicts $\int_{t_0}^{\infty} W(x(\tau))\, d\tau < \infty$. Therefore $W(p) = 0$, and E contains all finite positive limit points of $x(t)$. The above also shows that if $x(t)$ has finite limit points then $\omega = \infty$, and this completes the proof of (a).

Part (b). Here we assume $\omega = \infty$, and because $\int_{t_0}^{\infty} W(x(\tau))\, d\tau < \infty$, the boundedness of the derivative of $W(x(t))$ almost everywhere from above (or from below) implies $W(x(t)) \to 0$ as $t \to \infty$. Since W is continuous, $W(p) = 0$, and this completes the proof of (b).

Theorem 1(a) stems from and is a modification of a result due to Yoshizawa (see Theorem 5 [*12*] or Theorem 14.1 [*13*]). If in Theorem 1(a) the condition that $W(x)$ is continuous is replaced by the condition that $W(x)$ is "positive definite with respect to a closed set E", (see [*12*] or [*13*]) then the result is more general than Yoshizawa's. In some instances Theorem 1(b) can be applied when Theorem 1(a) cannot. For an example that illustrates this and also shows that the conclusion of the theorem is the "best possible" see [*8*] or [*9*].

The smaller the set E the "better" is the Liapunov function, and the problem in applications is to find "good" Liapunov functions. It is also necessary to have information about which solutions remain in G in the future. In practice this is often done using more than one Liapunov function. Note that *if V^1 and V^2 are Liapunov functions for* (1) *on G, then $V = V^1 + V^2$ is also a Liapunov function for* (1) *on G and $E_\infty = E_\infty^1 \cap E_\infty^2$* . Another remark that is useful in applications is that *if E_∞ is made up of a number of components* (maximal connected sets) *and $x(t)$ remains in G for $t \geqslant 0$, then $x(t)$ approaches just one of these components*, since Ω is connected. For example, *if E is bounded, then either $x(t) \to \infty$ as $t \to \omega^-$ or $x(t) \to E$ as $t \to \infty$.*

The following result is also worth noting, and its proof is contained in the proof of Theorem 1. *If the Liapunov function in Theorem* 1 *does not depend upon t, then "$x(t) \to E_\infty$ as $t \to \infty$" can be replaced by "$x(t) \to (E \cap Q_c) \cup \{\infty\}$" for some c where $Q_c = \{x;\ V(x) = c\}$.* Thus, in the example $\dot{x} = y$, $\dot{y} = -p(t)\, y - x, p(t) \geqslant \delta > 0$ of [*8*] or [*9*] one can conclude using $V = x^2 + y^2$ for each solution $(x(t), y(t))$ not only that $y(t) \to 0$ as $t \to \infty$ but also $x(t) \to$ constant as $t \to \infty$.

3. Autonomous Systems. An Invariance Principle

For the autonomous system

$$\dot{x} = f(x) \tag{4}$$

we assume that f is a continuous function on G^* to R^n where G^* is an open set of R^n. Let V be locally Lipschitzian from G^* to R where $\bar{G}$ is a subset of G^*. The definition of a Liapunov function then becomes simpler. We shall say that V is a *Liapunov function for* (4) *on* G if $\dot{V} \leqslant 0$ on G, where

$$\dot{V}(x) = \liminf_{h \to 0^+} h^{-1}[V(x + hf(x)) - V(x)].$$

Define

$$E = \{x;\ \dot V(x) = 0, x \in \bar G\},$$

and let M be the union of all solutions that remain in E on their maximal interval of definition. Exploiting the fact that for autonomous systems the limit sets have an invariance property, we then obtain the following result which says that a Liapunov function on G gives information about all positive and negative limit sets of solutions which remain in G.

THEOREM 2. *Let V be a Liapunov function for* (4) *on G and let $x(t)$ be a solution of* (4) *with maximal interval of definition* (α, ω), $\alpha < 0 < \omega$. *If $x(t)$ is in G for t in* $[0, \omega)((\alpha, 0])$, *then either* $x(t) \to \infty$ *as* $t \to \omega^-(t \to \alpha^+)$ *or* $\omega = \infty$ $(\alpha = -\infty)$ *and* $x(t) \to M_\infty = M \cup \{\infty\}$ *as* $t \to \infty$ $(t \to -\infty)$.

Proof. If we assume that $V(x)$ is C^1 on G^*, then this result is a corollary of Theorem 1. However without making this assumption we obtain an easy proof using the invariance of limit sets. We consider the case $t \geqslant 0$. Now just as before in the proof of Theorem 1 we have that $V(x(t)) \to c$ as $t \to \omega^-$. Let Γ^+ denote the set of finite positive limit points of $x(t)$. If Γ^+ is empty, then $x(t) \to \infty$ as $t \to \omega^-$. If Γ^+ is nonempty, then just as in the proof of Theorem 1 we conclude that $\omega = \infty$. This set Γ^+ has the property that if $q \in \Gamma^+$ then at least one solution starting at q remains in Γ^+ on its maximal interval of definition (see [*3*], p. 145, Theorem 1.2). As in Theorem 1, $V = c$ on Γ^+. Since Γ^+ has the above invariance property, $\dot V = 0$ on Γ^+, $\Gamma^+ \subset E$, and hence $\Gamma^+ \subset M$. Then $\Omega = \Gamma^+ \cup \{\infty\} \subset M_\infty$, and $x(t) \to M_\infty$ as $t \to \infty$. Replacing t by $-t$ and V by $-V$ takes care of the case $t \to -\infty$ and this completes the proof.

We shall now consider some consequences of this theorem that have turned out to be of particular importance in applications. The original versions of these results appeared in [*5*], and what is given here is a refinement of those results. Let Q be a set $\subset G^0$ and let $x(t, x^0)$ denote any solution of (4) satisfying $x(0, x^0) = x^0$ with (α, ω) denoting the maximal interval of definition of $x(t, x^0)$. If there is a neighborhood N of Q, such that $x^0 \in N$ implies $x(t, x^0) \to Q$ as $t \to \omega^-$ for each solution $x(t, x^0)$, we say that Q is an *attractor*. A set Q is said to be *stable* if given a neighborhood N of Q there is a neighborhood N_0 of Q such that $x^0 \in N_0$ implies that each solution $x(t, x^0)$ is in N for $t \in [0, \omega)$. A stable attractor is said to be *asymptotically stable.* A set Q is said to be *positively invariant* if for each $x^0 \in Q$, each $x(t, x^0)$ is in Q for $t \in [0, \omega)$. We then have as an immediate consequence of Theorem 2,

COROLLARY 1. *Let G be a bounded, open, positively invariant set. If V is a Liapunov function for* (4) *on G and $M \subset G$, then M is an attractor and G is in its region of attraction.*

Proof. By Theorem 2 each solution starting in G approaches M. Now M is the maximal invariant set in E and is therefore closed (since the closure of an invariant set is invariant). Hence G is a neighborhood of M, and M is an attractor.

In using this result one looks for a Liapunov function such that $Q_c = \{x;\ V(x) < c\}$ or some component of Q_c is the set G in this corollary. If, in Corollary 1, M is a single point p and if it is known that V is positive-definite relative to p, then it follows that p is asymptotically stable. The next theorem shows that even without the assumption that V is positive-definite p is actually asymptotically stable. An example is given in [8] to show how information about asymptotic stability can be obtained using Liapunov functions which are not positive-definite. Even though in most applications "good" Liapunov functions are positive-definite, this result shows it is not necessary to check this, and this in itself is an advantage since, except for quadratic forms, there are no computable criteria for positive-definiteness.

THEOREM 3. *If, in addition, to the conditions of Corollary 1 we assume* (i) *solutions* $x(t, x^0)$ *of* (4) *are unique and* (ii) V *is constant on the boundary of* M, *then* M *is asymptotically stable and* G *is in its region of asymptotic stability.*

Proof. Assume that M is not stable. Then there is a neighborhood $N \subset G$ of M such that given any positive integer n there is a y_n such that $d(y_n, M) < 1/n$ and for some $\tau_n > 0$ the point $x(\tau_n, y_n)$ is on the boundary of N. Using compactness we can conclude that there is a sequence x_n and a sequence $t_n > 0$ such that $x_n \to x^0$ and $x(t_n; x_n) \to y$ as $n \to \infty$. We know then that x^0 is on the boundary of M and that y is on the boundary of N. Note also that all solutions are defined on $[0, \infty)$, since G is bounded and positively invariant. Define relative to the fixed sequence x_n

$$\gamma^+ = \{z;\ x(t_n, x_n) \to z \text{ as } n \to \infty \text{ for some sequence } t_n > 0, z \text{ not in } M\}.$$

Since $y \in \gamma^+$, γ^+ is nonempty and is in G but outside M. Note that it must be that $t_n \to \infty$ as $n \to \infty$. If t_n contained a convergent subsequence $s_j = t_{n_j}$ with a finite limit s, then $x(s_j, x_{n_j}) \to z = x(s, x^0)$. But M is an invariant set and since $x^0 \in M$ this contradicts z not in M. Given $t \in (-\infty, \infty)$, $x(t + t_n, x_n)$ is defined for n sufficiently large and $x(t + t_n, x_n) = x(t, x(t_n, x_n)) \to x(t, z)$ as $n \to \infty$. Therefore $x(t, z)$ is defined on $(-\infty, \infty)$ and $x(t, z) \in \gamma^+$ for all t. Hence γ^+ is an invariant set in G not contained in M. Then for $z \in \gamma^+$

$$V(z) = \lim_{n\to\infty} V(x(\tau_n; x_n)) \leqslant \lim_{n\to\infty} V(x_n) = V(x_0) = c.$$

where c is the value of V on M. But, since $x(t, z) \to M$ as $t \to \infty$, $V(z) \geqslant c$. Therefore $V = c$ on γ^+, and, since γ^+ is an invariant set, $\dot{V} = 0$ on γ^+. This then implies that $\gamma^+ \subset M$, and this contradiction completes the proof.

The results of this paper go beyond classical Liapunov theory and also provide a unification of that theory. Their greatest value will undoubtedly turn out to be their extension to other types of systems (see, for example [*1*]). One direction of extension has been to abstract dynamical systems. Zubov did the classical Liapunov theory in [*14*], and this has been further extended particularly by Auslander and Seibert (see [*11*] and the references given there). This has been revealing, has produced new results for stability theory of autonomous differential equations, and can include results of the type given here (see [*11*]). Unfortunately, however, this theory for abstract dynamical systems does not contain Hale's extension to functional differential equations in [*1*] and certainly will not be suitable for extensions to partial differential equations. Two of the reasons for this are that the state space is assumed to be locally compact and the motions are assumed to define a group. These are not the only difficulties, and Hale and Infante in [*2*] have recently taken a big step forward in this direction with the introduction of what they call an "extended dynamical system". It is hoped that this will be applicable to certain classes of problems arising from partial differential equations.

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