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HYPOHAMILTONIAN AND HYPOTRACEABLE GRAPHS

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Abstract. In this note hypohamiltonian and hypotraceable graphs are constructed.

1. Introduction

We adopt the notation and terminology of Harary [4]. However, the terms *vertices* and *edges* are used here instead of the terms *points* and *lines*, respectively, in [4], and the edge joining the vertices x and y is denoted by (x, y). A graph is *Hamiltonian*, resp. *traceable*, if it has a Hamiltonian cycle, resp. path. A graph G is *hypohamiltonian*, resp. *hypotraceable*, if it is not Hamiltonian, resp. traceable, but every vertex-deleted subgraph $G-v$ is Hamiltonian, resp. traceable.

By results of Gaudin, Herz and Rossi [2], Herz, Duby and Vigué [5], Lindgren [7] and Chvátal [1], the Petersen graph is the only hypohamiltonian graph with ≤ 10 vertices, there is no hypohamiltonian graph with 11 or 12 vertices, and for all $p \geq 13$ except possibly for $p = 14, 17, 19, 20, 25$, there is a hypohamiltonian graph with p vertices. Using the existence of hypohamiltonian graphs with 10, 13, 16 and 22 vertices, we shall obtain in this note, by simple constructions, hypohamiltonian graphs with p vertices for all $p \geq 13$ except for $p = 14, 17, 19$. This improves on the above-mentioned results for $p = 20$ and $p = 25$.

Kapoor, Kronk and Lick [6] mentioned as unsolved the problem whether hypotraceable graphs exist or not. However, Grünbaum [3] re-

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ports that a hypotraceable graph with 40 vertices has been found by J. Horton. We shall show that for $p = 34, 37, 39, 40$ and for all $p \geq 42$, there exists a hypotraceable graph with p vertices. Walter [8] gave an example of a connected graph in which the longest paths do not have a vertex in common. Clearly, every hypotraceable graph also has this property.

2. Hypohamiltonian graphs

Let G_1, G_2 be disjoint hypohamiltonian graphs. Assume that G_1, resp. G_2, contains a vertex x_0, resp. y_0, of degree 3 and let x_1, x_2, x_3, resp. y_1, y_2, y_3, be the vertices adjacent to x_0, resp. y_0. As pointed out by Bondy [1, p. 39], G_1, resp. G_2, contains none of the edges (x_1, x_2), (x_2, x_3), (x_1, x_3), resp. (y_1, y_2), (y_2, y_3), (y_1, y_3). Put $H_1 = G_1 - x_0$ and $H_2 = G_2 - y_0$. Let G be the graph obtained by identifying the vertices x_1, y_1 into a vertex z_1, the vertices x_2, y_2 into z_2 and the vertices x_3, y_3 into z_3 (see Fig. 1). We consider H_1 and H_2 as subgraphs of G.

Lemma 2.1. *G is hypohamiltonian.*

Proof. Suppose (reductio ad absurdum) that G has a Hamiltonian cycle. This cycle includes z_1, z_2 and z_3 and is therefore the union of a $z_1 - z_2$ path P^1, a $z_2 - z_3$ path P^2 and a $z_3 - z_1$ path P^3. Two of the paths P^1, P^2, P^3 (P^1 and P^2, say) are contained in one of the graphs H_1, H_2 (H_1, say). Then $P^1 \cup P^2$ is a Hamiltonian path of H_1 joining $z_1 = x_1$ and $z_3 = x_3$. Adding to this path the vertex x_0 and the edges (x_0, x_1), (x_0, x_3), we obtain a Hamiltonian cycle of G_1 which is a contradiction. So G is not Hamiltonian.

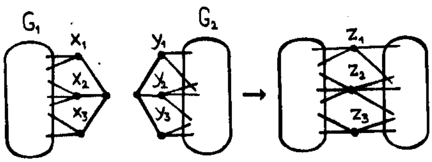

Fig. 1. Construction of hypohamiltonian graphs.

Let v be any vertex of G. Assume w.l.g. that v is a vertex of H_1. $G_1 - v$ is Hamiltonian so $G_1 - v - x_0 = H_1 - v$ has a Hamiltonian path P^1 joining two of the vertices x_1, x_2, x_3 (x_1 and x_2, say). $G_2 - y_3$ is Hamiltonian so $G_2 - y_3 - y_0 = H_2 - y_3$ has a Hamiltonian path P^2 joining y_1 and y_2. $P^1 \cup P^2$ is a Hamiltonian cycle of $G - v$, and the lemma is proved.

If we assume that each of G_1 and G_2 has a vertex distinct from x_0, resp. y_0, which has degree 3 and which is not adjacent to x_0, resp. y_0, then clearly also G has two non-adjacent vertices of degree 3. So if there exist hypohamiltonian graphs with p_1 and p_2 vertices, respectively, so that each of these has two non-adjacent vertices of degree 3, then there exists a hypohamiltonian graph which has two non-adjacent vertices of degree 3 and which has $p_1 + p_2 - 5$ vertices. The Petersen graph and each of the hypohamiltonian graphs with $6k + 10$ ($k \geq 1$) vertices constructed by Lindgren [7] and Sousselier [5] have two non-adjacent vertices of degree 3. Also the hypohamiltonian graph with 13 vertices constructed by Herz, Duby and Vigué [5, Fig. 8] has two non-adjacent vertices of degree 3. A set of integers M which contains 10, 13, 16, 22 and which has the property $p_1, p_2 \in M \Rightarrow p_1 + p_2 - 5 \in M$ contains all integers ≥ 13 except possibly 14, 17, 19. So we have the following:

Theorem 2.2. *For $p = 10$ and for $p \geq 13, p \neq 14, 17, 19$, there exists a hypohamiltonian graph which has p vertices and which has two non-adjacent vertices of degree* 3.

3. Hypotraceable graphs

Let G_1, G_2, G_3, G_4 be disjoint hypohamiltonian graphs. Assume that for $i = 1, 2, 3, 4$, G_i has a vertex x_i of degree 3. Let y_i^1, y_i^2, y_i^3 be the vertices adjacent to x_i. Put $H_i = G_i - x_i$. We identify the vertices y_1^3, y_2^3 into a vertex z_1 and the vertices y_3^3, y_4^3 into a vertex z_2. Then we add the edges (y_1^1, y_3^1), (y_1^2, y_3^2), (y_2^1, y_4^1), (y_2^2, y_4^2). The resulting graph is denoted G (see Fig. 2). The graphs H_i are considered as subgraphs of G. F_1 denotes the union of H_1, H_3 together with the edges (y_1^1, y_3^1), (y_1^2, y_3^2), and F_2 denotes the union of H_2, H_4 together with the edges (y_2^1, y_4^1), (y_2^2, y_4^2).

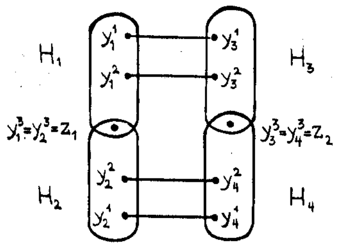

Fig. 2. Construction of hypotraceable graphs.

Lemma 3.1. *The graph G constructed above is hypotraceable.*

Proof. We shall first prove that G is not traceable. Suppose therefore (reductio ad absurdum) that G has a Hamiltonian path joining w_1, w_2, say. This path includes z_1 and z_2 and is therefore the union of a w_1 – z_1 path P^1, a $z_1 - z_2$ path P^2 and a $z_2 - w_2$ path P^3. One or both of the paths P^1, P^3 may have length zero. We may assume that P^2 is entirely contained in either F_1 or F_2 (F_1, say). Then P^2 includes precisely one of the edges (y_1^1, y_3^1), (y_1^2, y_3^2) ((y_1^2, y_3^2), say). At least one of the paths P^1, P^3 (P^3, say) is entirely contained in F_2. P^1 is entirely contained in either F_1 or F_2.

Case 1. *P^1 is entirely contained in F_2*. In this case P^2 is a Hamiltonian path of F_1. It is easy to see that P^2 contains a Hamiltonian path of H_1 joining $z_1 = y_1^3$ and y_1^2. But then G_1 is Hamiltonian which is a contradiction.

Case 2. *P^1 is entirely contained in F_1*. In this case $P^1 \cup P^2$ is a Hamiltonian path of F_1. If P^1 includes the edge (y_1^1, y_3^1), then $P^1 \cup P^2$ contains a Hamiltonian path of H_1 joining y_1^1 and y_1^2. But then G_1 is Hamiltonian which is a contradiction. If P^1 on the other hand does not include (y_1^1, y_3^1), then P^2 contains a Hamiltonian path of H_3 joining $z_2 = y_3^3$ and y_3^2. But then G_3 is Hamiltonian which is a contradiction.

So we have proved that G is not traceable. Let v be any vertex of G. We shall show that $G - v$ is traceable. By symmetry, we may assume that v is a vertex of H_1. $G_1 - v$ is Hamiltonian so $H_1 - v$ has a Hamiltonian path P^1 joining two of the vertices y_1^1, y_1^2, y_1^3.

Case (a). *P^1 joins y_1^1 and y_1^2*. $H_2 - y_2^3$ has a Hamiltonian path P^2 joining y_2^1, y_2^2 and for $i = 3, 4$, $H_i - y_i^1$ has a Hamiltonian path P^i joining y_i^2, y_i^3. Then

$$\{y_3^1\} \cup \{(y_3^1, y_1^1)\} \cup P^1 \cup \{(y_1^2, y_3^2)\} \cup P^3 \cup P^4$$
$$\cup \{(y_4^2, y_2^2)\} \cup P^2 \cup \{(y_2^1, y_4^1)\} \cup \{y_4^1\}$$

is a Hamiltonian path of $G - v$.

Case (b). *P^1 joins y_1^3 and one of the vertices y_1^1, y_1^2 (y_1^2, say).* Let P^2 be a Hamiltonian path of $H_2 - y_2^1$ joining $y_2^3 = y_1^3$ and y_2^2 and let P^4 be a Hamiltonian path of $H_4 - y_4^3$ joining y_4^1 and y_4^2. H_3 is Hamiltonian and has therefore a Hamiltonian path P^3 starting at y_3^2. Then

$$P^3 \cup \{(y_3^2, y_1^2)\} \cup P^1 \cup P^2 \cup \{(y_2^2, y_4^2)\} \cup P^4$$
$$\cup \{(y_2^1, y_4^1)\} \cup \{y_2^1\}$$

is a Hamiltonian path of $G - v$.

So $G - v$ is traceable and the lemma is proved.

If there exist hypohamiltonian graphs with p_1, p_2, p_3, p_4 vertices respectively, so that each of these has a vertex of degree 3, then by Lemma 3.1 there exists a hypotraceable graph with $p_1 + p_2 + p_3 + p_4 - 6$ vertices. Combining this with Theorem 2.2, we easily obtain the following:

Theorem 3.2. *For $p = 34, 37, 39, 40$ and for all $p \geq 42$, there exists a hypotraceable graph with p vertices.*

The smallest hypotraceable graph obtained in this way is shown in Fig. 3.

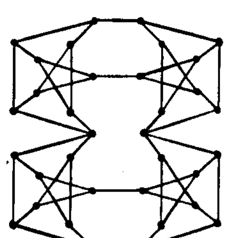

Fig. 3. A hypotraceable graph with 34 vertices.

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