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An Existence Theorem for Strong Liftings

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1. INTRODUCTION

Let (Z, μ) be a couple consisting of a locally compact Hausdorff space Z and a positive Radon measure μ on Z. By $\mathscr{L}^\infty(Z, \mu)$ we denote the algebra of bounded μ-measurable functions on Z in the sense of Bourbaki [1], by $L^\infty(Z, \mu)$ their classes modulo locally negligible functions, and by N^∞ the essential supremums norm on $\mathscr{L}^\infty(Z, \mu)$ and $L^\infty(Z, \mu)$; $L^\infty(Z, \mu)$ is a Banach algebra and a complete vector lattice [2, Chap. I].

A *strong lifting* for (Z, μ) is a map T from $L^\infty(Z, \mu)$ into $\mathscr{L}^\infty(Z, \mu)$ with the following properties:

(1) $T\dot{f} \in \dot{f}$ for $\dot{f}$ in $L^\infty(Z, \mu)$,

(2) $T1 = 1$,

(3) $\dot{f} \geqslant 0$ implies $T\dot{f} \geqslant 0$ everywhere,

(4) $T(\alpha\dot{f} + \beta\dot{g}) = \alpha T\dot{f} + \beta T\dot{g}$ for α, β in $\mathbf{R}$,

(5) $T(\dot{f}\dot{g}) = T\dot{f} \cdot T\dot{g}$,

(6) $T\dot{f} = f$ if f is continuous.

In order that there exist a strong lifting for (Z, μ) it is necessary that the support of μ be Z: the class $\dot{f}$ of a negligible function f has image 0 under T according to (4), and (6) implies that the only negligible continuous function is 0. We assume henceforth that all Radon measures appearing have support the whole space of their definition. While there is always a lifting

$$T : L^\infty(Z, \mu) \to \mathscr{L}^\infty(Z, \mu)$$

with properties (1) through (5), the existence of a strong lifting has only been proved in the case that Z is metrizable and for various special couples (Z, μ) [2, 3]. For these results and various applications we refer to the monograph by Ionescu Tulcea [2]. The question whether there is a strong lifting for arbitrary locally compact measure spaces (Z, μ) is still open. We prove in this context the following:

THEOREM. *Let Z be a locally compact Hausdorff space and μ, ν positive Radon measures on Z such that the support of μ is Z. If μ is absolutely continuous with respect to ν and if (Z, ν) admits a strong lifting then so does (Z, μ).*

This reduces the question whether there exists a strong lifting for arbitrary couples (Z, μ) to the case that Z is a product of unit intervals, as will be shown in Section 3.

2. Proof of the Theorem

We denote by $\dot f$ the class in $L^\infty(Z, \nu)$ of a function f in $\mathscr{L}^\infty(Z, \nu)$ and by $\bar f$ the class in $L^\infty(Z, \mu)$ of a function f in $\mathscr{L}^\infty(Z, \mu)$. A ν-measurable function f is also μ-measurable such that the identity map is an algebra homomorphism of $\mathscr{L}^\infty(Z, \nu)$ into $\mathscr{L}^\infty(Z, \mu)$. If f and g in $\mathscr{L}^\infty(Z, \nu)$ differ by a locally ν-negligible function then their images in $\mathscr{L}^\infty(Z, \mu)$ coincide μ-almost everywhere and we get a natural algebra homomorphism N of $L^\infty(Z, \nu)$ into $L^\infty(Z, \mu)$. N is surjective. For let $\bar f$ be a class in $L^\infty(Z, \mu)$ and f a Borel function in $\bar f$. Then f is ν-measurable and $\bar f$ is the image under N of $\dot f \in L^\infty(Z, \nu)$.[1] Let $\dot e$ be the supremum in $L^\infty(Z,\nu)$ of those elements in the kernel of N which are smaller than $\dot 1$. Then $\dot e$ is an idempotent in $L^\infty(Z, \nu)$ and $\dot e L^\infty(Z, \nu)$ is the kernel of N. Let $\bar f$ be in $L^\infty(Z, \mu)$ and f a Borel function in $\bar f$. The class $E\bar f := (\dot 1 - \dot e)\dot f$ does not depend on the choice of f in $\bar f$ such that E is an algebra homomorphism of $L^\infty(Z, \mu)$ into $L^\infty(Z, \nu)$. E is a right inverse for N.

Now let S be a strong lifting for (Z, ν) and define $T'\bar f := SE\bar f$ for $\bar f$ in $L^\infty(Z, \mu)$. Then $T' : L^\infty(Z, \mu) \to \mathscr{L}^\infty(Z, \mu)$ has obviously properties (1), (3), (4), and (5). If f is continuous then

$$T'\bar f = S(1 - \dot e)\dot f = (1 - S\dot e)\, Sf = (1 - S\dot e) f$$

such that $T'\bar f = f$ on the set $\{z \text{ in } Z \mid S\dot e(z) = 0\}$. Its complement is the locally μ-negligible set $\{z \text{ in } Z \mid Se(z) = 1\}$. For every point z in the latter set we select a character χ_z of $L^\infty(Z, \mu)$ which extends the character $\varphi \to \varphi(z)$ of $C(Z)$, the set of continuous functions on Z. The map T defined by

$$T\bar f(z) := \begin{cases} T'\bar f(z) & \text{if} \quad S\dot e(z) = 0 \\ \chi_z(\bar f) & \text{if} \quad S\dot e(z) = 1 \end{cases}$$

is obviously a strong lifting for (Z, μ).

3. A Corollary

Let Z be a locally compact Hausdorff space, μ a positive Radon measure on Z. Let β be an infinite cardinal. We say that (Z, μ) is locally of order β if there

[1] *Added in proof.* A Borel function is here understood to be a function whose trace on every compact set is Borel in the usual sense.

is a collection $(K_i)_{i \in I}$ of disjoint compact subsets of Z whose union has a locally negligible complement and such that the topology induced on every K_i has a basis of cardinality not exceeding β. An example for such a space is the β-fold product

$$\prod^{\beta} := \prod_{\alpha<\beta} I_\alpha$$

of unit intervals I_α.

COROLLARY. *In order that there is a strong lifting for every couple* (Z, μ) *which is locally of order* β, *it is necessary and sufficient that* $(\prod^\beta, \nu)$ *admit a strong lifting for every positive Radon measure* ν. *In particular, if every couple* $(\prod, \nu)$ *consisting of a product* $\prod$ *of unit intervals and a positive Radon measure* ν *admits a strong lifting then so does any couple* (Z, μ).

The necessity is obvious. The proof of the sufficiency essentially repeats arguments due to Ionescu Tulcea [2, 4, 5]. For i in I, let K_i' be the support of the restriction $\mu_{|K_i}$ of μ to K_i. The family $(K_i')_{i \in I}$ has the properties of (K_i) and the additional one that $K_i' = \operatorname{supp} \mu_{|K_i'}$. We embed K_i' continuously into a β-fold product $\prod_i$ of unit intervals and denote by μ_i the image of $\mu_{|K_i'}$ under the embedding. Let λ denote the Lebesgue measure on $\prod_i$, i.e., the product of Lebesgue measure on the unit interval. Then $\nu_i = \lambda + \mu_i$ has support $\prod_i$ and according to assumption there is a strong lifting for $(\prod_i, \nu_i)$. According to [2, Chapt. VIII, No. 2, Proposition 2.1], there is a strong lifting for the restriction of ν_i to $K_i' \subset \prod_i$ and our theorem yields a strong lifting T_i for $(K_i', \mu_{|K_i'})$. As before (cf. also [2, Chap. VIII, No. 2, Propositions 1 and 2.2]) we select for every z in $Z - \bigcup_{i \in I} K_i'$ a character X_z extending the character $\phi \to \phi(z)$ of $C(Z)$ and put

$$Tf(z) = \begin{cases} T_i f(z) & \text{if} \quad z \in K_i' \\ X_z(f) & \text{elsewhere.} \end{cases}$$

T is a strong lifting for (Z, μ), and the corollary is proved.

REFERENCES

1. N. BOURBAKI, "Integration," Hermann, Paris, 1952.
2. A. AND C. IONESCU TULCEA, "Topics in the Theory of Lifting," Springer, New York, 1969.
3. A. AND C. IONESCU TULCEA, On the existence of a lifting commuting with the left translations of an arbitrary locally compact group, *Proc. Fifth Berkeley Symp. Math. Statist. Probab.*, Univ. of California Press, 1966.
4. A. AND C. IONESCU TULCEA, On the lifting property, III, *Bull. Amer. Math. Soc.* **70** (1964), 193–197.
5. A. AND C. IONESCU TULCEA, On the lifting property, IV—Disintegration of measures, *Ann. Inst. Fourier* **14** (1964), 445–472.