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A characterization of weighted approximations by the Post–Widder and the Gamma operators, II

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Abstract

We present a characterization of the approximation errors of the Post–Widder and the Gamma operators in $L_p(0,\infty)$, $1 \le p \le \infty$, with a weight $x^{\gamma_0}(1+x)^{\gamma_\infty-\gamma_0}$ with arbitrary real γ_0, γ_∞. Characteristics of two types are used — weighted K-functionals of the approximated function itself and the classical fixed-step moduli of smoothness taken on simple modifications of it.
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1. Introduction

The Post–Widder operator is given by

$$(P_s f)(x) = \frac{1}{\Gamma(s)} \int_0^\infty f\left(\frac{xv}{s}\right) \mathrm{e}^{-v} v^s \frac{\mathrm{d}v}{v} \tag{1.1}$$

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and the Gamma operator is given by

$$(G_s f)(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty f\left(\frac{xs}{v}\right) e^{-v} v^{s+1} \frac{dv}{v}. \tag{1.2}$$

Here f is a measurable function defined on $(0,\infty)$ and satisfies mild growth conditions at 0 and at ∞, Γ denotes as usual the Gamma function and s is a positive real parameter.

For real α we denote the power function by $\chi^\alpha(x) = x^\alpha$ for $x > 0$. For real γ_0, γ_∞ we denote the weight that we are going to consider in this article by

$$w(x) = w(\gamma_0, \gamma_\infty; x) = \begin{cases} x^{\gamma_0} & \text{if } 0 < x \le 1; \\ x^{\gamma_\infty} & \text{if } 1 \le x < \infty. \end{cases} \tag{1.3}$$

For $r \in \mathbb{N}$, $1 \le p \le \infty$ and $D = \frac{d}{dx}$ we consider the weighted K-functionals:

$$\begin{aligned} K_w^r(f, t^r)_p &= K(f, t^r; L_p(w)(0,\infty), AC_{loc}^{r-1}, \chi^r D^r) \\ &= \inf\left\{ \|w(f-g)\|_p + t^r \|w\chi^r D^r g\|_p : g \in AC_{loc}^{r-1}(0,\infty) \right\}, \end{aligned} \tag{1.4}$$

defined for every $f \in \pi_{r-1} + L_p(w)(0,\infty)$ and $t > 0$. We have denoted by $L_p(w)(0,\infty)$ the set of all measurable functions f, defined on $(0,\infty)$, such that $wf \in L_p(0,\infty)$. The L_p-norm over the interval (a,b) is denoted by $\|\cdot\|_{p(a,b)}$, i.e.

$$\|F\|_{p(a,b)} = \left\{ \int_a^b |F(x)|^p \, dx \right\}^{1/p}, \quad F \in L_p(a,b),\ 1 \le p < \infty,$$

$$\|F\|_{\infty(a,b)} = \operatorname*{ess\,sup}_{x \in (a,b)} |F(x)|, \quad F \in L_\infty(a,b).$$

We assume that the norm is taken on $(0,\infty)$ when no interval is indicated in its notation. $AC_{loc}^k(a,b)$ denotes the set $\{g : g, g', \dots, g^{(k)} \in AC[\bar{a}, \bar{b}]\ \forall a < \bar{a} < \bar{b} < b\}$ and $AC[\bar{a}, \bar{b}]$ is the set of the absolutely continuous functions on $[\bar{a}, \bar{b}]$. Above and in what follows, $L_\infty(w)(0,\infty)$ can be replaced by the spaces $C(w)(0,\infty) = \{f : wf \in C(0,\infty)\}$, where $C(a,b)$ is the space of all continuous functions **bounded** on (a,b). When the function $g \in AC_{loc}^{r-1}(0,\infty)$ in (1.4) is such that either $f - g \notin L_p(w)(0,\infty)$ or $\chi^r D^r g \notin L_p(w)(0,\infty)$ we assume that $\|w(f-g)\|_p + t^r \|w\chi^r D^r g\|_p = +\infty$.

The following spaces of algebraic polynomials will be considered. Let i, j be integers. We set $\pi_{i,j} = \{c_i x^i + \cdots + c_j x^j : c_k \in \mathbb{R}\}$ if $0 \le i \le j$ and $\pi_{i,j} = \{0\}$ if $j < i$. For the space of all algebraic polynomials of degree $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, denoted as usual by π_k, we have $\pi_k = \pi_{0,k}$. Accordingly, we set $\pi_k = \{0\}$ for negative integers k.

In [8] we have established for $f \in L_p(w)(0,\infty)$, $1 \le p \le \infty$ and a weight of the type $w = \chi^\gamma$ (i.e. $\gamma_0 = \gamma_\infty = \gamma$) the equivalence

$$\|w(f - P_s f)\|_p \sim \|w(f - G_s f)\|_p \sim K_w^2(f, s^{-1})_p, \tag{1.5}$$

which contains a strong converse theorem of type A (in the terminology of [2]). Also in [8] the K-functional on the right-hand side of (1.5) was characterized in the terms of the classical fixed-step moduli of smoothness.

By $\Psi(f,t) \sim \Theta(f,t)$ we mean that there exists a positive constant c such that $c^{-1}\Theta(f,t) \le \Psi(f,t) \le c\Theta(f,t)$ for all f and t under consideration. In the paper we denote by c positive numbers independent of the functions f, the parameter t of the K-functional and the parameter

s of the operators. The numbers c may differ at each occurrence. Whenever necessary to indicate constants, which preserve their values throughout the article, we use the notation M, M_1, M_2, N, N_2. They will not depend on any of the parameters and in this sense they will be absolute constants.

Earlier contributions related to the inequalities in (1.5) (in the case $\gamma_0 = \gamma_\infty = \gamma$) are summarized in [6]. There are only few results in the case $\gamma_0 \neq \gamma_\infty$. The book of Ditzian and Totik [3] contains the direct estimate for weights (1.3) with arbitrary real exponents γ_0, γ_∞. The converse results for the same weights are given as a statement for the equivalent rates of convergence in terms of weighted Ditzian–Totik moduli (and hence weighted K-functionals).

One of the main results in the paper is a strong converse theorem of type A for the Post–Widder and the Gamma operators for a weight (1.3) with arbitrary real exponents γ_0, γ_∞. Let us note that the strong converse estimates of type A are optimal. Here we extend the research of [8], where, as we mentioned, the case $\gamma_0 = \gamma_\infty$ is considered. The extension is not trivial and requires a new idea because the strong converse inequalities of type A heavily rely on precise determination of the constants in some inequalities connected with the operators (see Section 2).

Theorem 1.1. *There are positive numbers N, M such that for every $\gamma_0, \gamma_\infty \in \mathbb{R}$, $s \geq N(\gamma_0^2 + \gamma_\infty^2 + 1)$, $1 \leq p \leq \infty$ and $f \in \pi_1 + L_p(w)(0, \infty)$ we have*

$$\|w(f - P_s f)\|_p \leq \left(2 + M\frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}} + M\frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}\right) K_w^2\left(f, \frac{1}{4s}\right)_p \tag{1.6}$$

and

$$K_w^2\left(f, \frac{1}{4s}\right)_p \leq \left(\kappa + M\frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}} + M\frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}\right) \|w(f - P_s f)\|_p \tag{1.7}$$

with

$$\kappa = \frac{21 - 4\sqrt{2}}{8 - 2\sqrt{2}} = 2.966824\ldots.$$

The same inequalities are true if P_s is replaced by G_s.

The direct inequality (1.6) is also proved in [3], but with an essentially bigger constant. The inverse inequality (1.7) is new for $\gamma_0 \neq \gamma_\infty$. It is established with a very small constant κ. Thus, the ratio $\|w(f - P_s f)\|_p / K_w^2(f, (4s)^{-1})_p$ is bounded between two numbers with ratio less than 6 when s is big enough! Note that Theorem 1.1 in the case $\gamma_0 = \gamma_\infty$ reduces to Theorem 1.1 from [8].

The relation $w(x) \leq c\bar{w}(x)$ for every $x \in (0, \infty)$ implies the inequalities $\|w(f - P_s f)\|_p \leq c\|\bar{w}(f - P_s f)\|_p$ and $K_w^r(f, t)_p \leq cK_{\bar{w}}^r(f, t)_p$ (with the same constant c). Hence Theorem 1.1 remains true (up to the value of the constants) if the weight (1.3) is replaced by any weight, which is equivalent to it on $(0, \infty)$, for example by

$$\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x) = x^{\gamma_0}(1 + x)^{\gamma_\infty - \gamma_0}. \tag{1.8}$$

The latter is more convenient for characterizing the weighted K-functionals with the classical moduli of smoothness (see Theorem 1.4).

Let us observe that in the case $\gamma_0 < \gamma_\infty$ we have $w = \max\{\chi^{\gamma_0}, \chi^{\gamma_\infty}\}$ and hence (1.6) and (1.7) easily follow from (1.5) (with twice bigger constants) because of $w \leq \chi^{\gamma_0} + \chi^{\gamma_\infty} \leq 2w$.

It does not seem that such a simple technique will work in the case $\gamma_0 > \gamma_\infty$ when $w = \min\{\chi^{\gamma_0}, \chi^{\gamma_\infty}\}$. The approach developed in Section 2 barely distinguishes between these two cases and provides constants which differ only in the remainder term from those obtained for $w = \chi^\gamma$ (i.e. $\gamma_0 = \gamma_\infty = \gamma$) in [8].

The K-functional (1.4) is characterized in [3, Chapter 6] by the weighted Ditzian–Totik moduli of smoothness. But it turns out that $K^r_{\mathrm{w}}(f, t^r)_p$ can also be characterized in terms of the classical moduli of smoothness, which are generally easier to compute. The second goal of our paper is to establish such characterizations. As usual, we denote by $\omega_r(F, t)_{p(J)}$ the classical unweighted fixed-step modulus of smoothness of order r of the function $F \in L_p(J)$, $J \subseteq \mathbb{R}$ is an interval, namely

$$\omega_r(F, t)_{p(J)} = \sup_{0<h\le t} \|\Delta^r_h F\|_{p(J)}.$$

We assume that $\Delta^r_h F(x) = 0$ if the argument of any of the summands of the finite differences $\Delta^r_h F(x)$ is outside J. Set $\omega_0(F, t)_{p(J)} = \|F\|_{p(J)}$. We use one and the same notation for a function F defined on $\mathbb{R}$ and for its restriction on some subinterval J.

In order to describe various conditions on the exponents γ_0 and γ_∞ in the definition of the weight w defined in (1.8) (or in (1.3)), we shall use the notation

$$\begin{aligned}
&\mathscr{T}_0(p) = (1/p, \infty),\\
&\mathscr{T}_i(p) = (-i - 1/p, 1 - i - 1/p), \quad i = 1, \dots, r-1,\\
&\mathscr{T}_r(p) = (-\infty, 1 - r - 1/p),\\
&\mathscr{T}_{exc}(p) = \{1 - r - 1/p, 2 - r - 1/p, \dots, -1/p\}.
\end{aligned}$$

For $r \in \mathbb{N}$, $i, j \in \mathbb{N}_0$, $j \le r$ and a weight $\bar{w}$ we define the linear operator $\mathscr{A}_{i,j-1}(\bar{w}) : L_{1,loc}(0,\infty) \to L_{1,loc}(\mathbb{R})$ by

$$\mathscr{A}_{i,j-1}(\bar{w}) f = (\bar{w}(f - \mathscr{L}_{i,j-1} f)) \circ \mathscr{E}, \tag{1.9}$$

where $\mathscr{E}(x) = \mathrm{e}^x$ and

$$(\mathscr{L}_{i,j-1} f)(x) = \sum_{n=i}^{j-1} a_n(f)\, x^n, \tag{1.10}$$

as $a_n : L_1(\alpha, \beta) \to \mathbb{R}$, $n = i, \dots, j-1$, $0 < \alpha < \beta$, are linear functionals. As usual, in (1.10) we assume that the sum is 0 if the upper bound is smaller than the lower.

We require $\mathscr{L}_{i,j-1}$ to satisfy the conditions:

(i) $|a_n(f)| \le c\,\|f\|_{1(\alpha,\beta)}$ for any $f \in L_1(\alpha, \beta)$, $n = i, \dots, j-1$;
(ii) $\mathscr{L}_{i,j-1} f = f$ for any $f \in \pi_{i,j-1}$;

and in some cases also one or both of the following conditions:

(iii) $\mathscr{L}_{i,j-1}(\chi^{i-1}) = 0$ if $i > 0$;
(iv) $\mathscr{L}_{i,j-1}(\chi^{j}) = 0$ if $j < r$.

Remark 1.2. For the proofs of the following theorems it is enough to replace (i) with

(i′) $|a_n(f)| \le c\,\|f\|_{p(\alpha,\beta)}$ for any $f \in L_p(\alpha, \beta)$, $n = i, \dots, j-1$.

We prefer to utilize (i) (which implies (i′)) in order for $a_n(f)$ to be easily computable for a given f. Simple examples of such operators $\mathscr{L}_{i,j-1}$ either satisfy (i) or satisfy (i′) for $p = \infty$.

Remark 1.3. The restrictions $\alpha > 0$ and $\beta < \infty$ used above can be relaxed to $\alpha = 0$ and/or $\beta = \infty$ at the cost of introducing additional weighted norm conditions.

We give explicit definitions of operators of the form (1.10) that satisfy conditions (i)–(ii) or (i)–(iv) in Section 6.

Following ideas of [5,8] in the two theorems below we characterize the K-functional $K_{\mathrm{w}}^r(f,t^r)_p$ by the unweighted fixed-step moduli of smoothness.

Theorem 1.4. *Let $r \in \mathbb{N}$, $i, j \in \mathbb{N}_0$, $i, j \le r$, $1 \le p \le \infty$ and $t_0 > 0$. Let also $\mathrm{w}(x) = \mathrm{w}(x;\gamma_0,\gamma_\infty)$ be defined in (1.8) with $\gamma_0 \in \mathscr{T}_i(p)$, $\gamma_\infty \in \mathscr{T}_j(p)$. Finally, let $\mathscr{A}_{i,j-1}$ be given by (1.9) as $\mathscr{L}_{i,j-1}$ satisfies conditions* (i) *and* (ii). *Then for every $f \in L_p(\mathrm{w})(0,\infty)$ and $0 < t \le t_0$ there holds*

$$K_{\mathrm{w}}^r(f,t^r)_p \sim \omega_r(\mathscr{A}_{i,j-1}(\chi^{1/p}\mathrm{w})f,t)_{p(\mathbb{R})} + t^r\,\|\mathscr{A}_{i,j-1}(\chi^{1/p}\mathrm{w})f\|_{p(\mathbb{R})}.$$

Let us explicitly note that for $j \le i$ we have $\mathscr{A}_{i,j-1}(\chi^{1/p}\mathrm{w})f = (\chi^{1/p}\mathrm{w}f)\circ\mathscr{E}$.

Theorem 1.5. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$ and $a, t_0 > 0$. Let also $\mathrm{w}(x) = \mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with $\gamma_0,\gamma_\infty \in \mathbb{R}$, and the integers i, j be determined by $\mathscr{T}_i(p) \cup \{1-i-1/p\} \ni \gamma_0$, $\mathscr{T}_j(p) \cup \{-j-1/p\} \ni \gamma_\infty$. We set $\ell_0 = 1$ if $\gamma_0 \in \mathscr{T}_{exc}(p)$, and $\ell_0 = 0$ otherwise. We set $\ell_\infty = 1$ if $\gamma_\infty \in \mathscr{T}_{exc}(p)$, and $\ell_\infty = 0$ otherwise. Let the integers i', j' be such that $0 \le i' \le i - \ell_0$ and $j + \ell_\infty \le j' \le r$. Let $\mathscr{A}_{i,j'-1}$ be given by (1.9) with an arbitrary $\mathscr{L}_{i,j'-1}$ satisfying conditions* (i) *and* (ii), *and also* (iii) *if $\gamma_0 \in \mathscr{T}_{exc}(p)$. Let $\mathscr{A}_{i',j-1}$ be given by (1.9) with an arbitrary $\mathscr{L}_{i',j-1}$ satisfying conditions* (i) *and* (ii), *and also* (iv) *if $\gamma_\infty \in \mathscr{T}_{exc}(p)$. Then for every $f \in L_p(\mathrm{w})(0,\infty)$ and $0 < t \le t_0$ there holds*

$$\begin{aligned} K_{\mathrm{w}}^r(f,t^r)_p \sim\ & \omega_r(\mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f,t)_{p(-\infty,a)} \\ & + t^{r-\ell_0}\omega_{\ell_0}(\mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f,t)_{p(-\infty,a)} \\ & + \omega_r(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f,t)_{p(-a,\infty)} \\ & + t^{r-\ell_\infty}\omega_{\ell_\infty}(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f,t)_{p(-a,\infty)}. \end{aligned} \tag{1.11}$$

As is well-known, the Post–Widder operator for integer s is actually the Post–Widder real inversion formula for the Laplace transform. Thus, Theorem 1.1 in combination with Theorem 1.4 or Theorem 1.5 gives us the rate of convergence of the Post–Widder real inversion formula measured by the structural properties of the original function [11, Ch. VII].

Remark 1.6. The hypotheses of Theorem 1.5 cover (with few exceptions depending on the specific values of p, γ_0 and γ_∞) the variety of indices i, j, i', j' for which (1.11) is true. The exact ranges of these indices are given in Remarks 5.11 and 5.12. We take advantage of the possibility to vary them in the proof of Theorem 5.15. Characterization (1.11) is most concise for $i' = i - \ell_0$ and $j' = j + \ell_\infty$. In each of these cases the polynomial $\mathscr{L}$ is a linear combination of the least number of monomials. The explicit form of the characterization is as follows.

For $\gamma_0 \in \mathscr{T}_i(p)$, $\gamma_\infty \in \mathscr{T}_j(p)$, $i' = i$ and $j' = j$ relation (1.11) takes the form

$$\begin{aligned} K_{\mathrm{w}}^r(f,t^r)_p \sim\ & \omega_r(\mathscr{A}_{i,j-1}(\chi^{\gamma_0+1/p})f,t)_{p(-\infty,a)} + t^r\,\|\mathscr{A}_{i,j-1}(\chi^{\gamma_0+1/p})f\|_{p(-\infty,a)} \\ & + \omega_r(\mathscr{A}_{i,j-1}(\chi^{\gamma_\infty+1/p})f,t)_{p(-a,\infty)} + t^r\,\|\mathscr{A}_{i,j-1}(\chi^{\gamma_\infty+1/p})f\|_{p(-a,\infty)}, \end{aligned}$$

and for $\gamma_0 = 1 - i - 1/p$, $0 < i \le r$, $\gamma_\infty \in \mathcal{T}_j(p)$, $i' = i - 1$ and $j' = j$ it takes the form

$$K^r_{\mathrm{w}}(f, t^r)_p \sim \omega_r(\mathcal{A}_{i,j-1}(\chi^{\gamma_0+1/p}) f, t)_{p(-\infty,a)} + t^{r-1}\omega_1(\mathcal{A}_{i,j-1}(\chi^{\gamma_0+1/p}) f, t)_{p(-\infty,a)}$$
$$+ \omega_r(\mathcal{A}_{i-1,j-1}(\chi^{\gamma_\infty+1/p}) f, t)_{p(-a,\infty)}$$
$$+ t^r \|\mathcal{A}_{i-1,j-1}(\chi^{\gamma_\infty+1/p}) f\|_{p(-a,\infty)}.$$

And similarly for $\gamma_\infty \in \mathcal{T}_{exc}(p)$. Note that the pass from $\gamma_0 \notin \mathcal{T}_{exc}(p)$ to $\gamma_0 \in \mathcal{T}_{exc}(p)$ not only changes $t^r\omega_0$ to $t^{r-1}\omega_1$ at the left end of the domain but also simultaneously affects the range for the index i' of the operator $\mathcal{A}_{i',j-1}$ acting at the other end.

The two quantities $\omega_r(F, t)_{p(J)} + t^r\|F\|_{p(J)}$ and $\omega_r(F, t)_{p(J)} + t^{r-1}\omega_1(F, t)_{p(J)}$ are not equivalent with constants independent of F and $t \in (0, 1]$. This is shown in [8, Remark 1.3] for any unbounded interval $J \subset \mathbb{R}$ but, of course, the same is true for finite intervals J.

Remark 1.7. If $f \in \pi_1 + L_p(w)(0, \infty)$ as in Theorem 1.1 and $\pi_1 \not\subset L_p(w)(0, \infty)$, then f is to be replaced by f_0 such that $f_0 \in L_p(w)(0, \infty)$ and $f - f_0 \in \pi_1$ when Theorems 1.4 and 1.5 are applied to the K-functional in Theorem 1.1.

Remark 1.8. Theorems 1.4 and 1.5 show the important role of the polynomials from π_{r-1} belonging to the space $L_p(\mathrm{w})$, that is the trivial class $\pi_{r-1} \cap L_p(\mathrm{w})(0, \infty)$ of the K-functional $K^r_{\mathrm{w}}(f, t^r)_p$. For future reference we recall

$$\pi_{i,r-1} \subset L_p(\chi^{\gamma_0})(0, 1) \iff \gamma_0 > -i - 1/p \quad \text{for } p < \infty \text{ or } \gamma_0 \ge -i \text{ for } p = \infty;$$
$$\pi_{0,j-1} \subset L_p(\chi^{\gamma_\infty})(1, \infty) \iff \gamma_\infty < 1 - j - 1/p$$
$$\text{for } p < \infty \text{ or } \gamma_\infty \le 1 - j \text{ for } p = \infty;$$
$$\pi_{i,j-1} \subset L_p(\mathrm{w})(0, \infty) \iff \gamma_0 > -i - 1/p, \gamma_\infty < 1 - j - 1/p$$
$$\text{for } p < \infty \text{ or } \gamma_0 \ge -i, \gamma_\infty \le 1 - j \text{ for } p = \infty.$$

Thus, if $p < \infty$, then $\pi_{r-1} \cap L_p(\mathrm{w})(0, \infty) \ne \{0\}$ iff $i < j$, where the integers i, j are determined by $\mathcal{T}_i(p) \cup \{1 - i - 1/p\} \ni \gamma_0$, $\mathcal{T}_j(p) \cup \{-j - 1/p\} \ni \gamma_\infty$. Also, if $p = \infty$, then $\pi_{r-1} \cap L_\infty(\mathrm{w})(0, \infty) \ne \{0\}$ iff $i < j$, where the integers i, j are determined by $\mathcal{T}_i(\infty) \cup \{-i\} \ni \gamma_0$, $\mathcal{T}_j(\infty) \cup \{1 - j\} \ni \gamma_\infty$.

In comparison with [8] difficulties of two new types have to be overcome in Theorems 1.4 and 1.5. First, this is the more complex structure of the space $L_p(\mathrm{w})(0, \infty)$ for some γ_0, γ_∞ compared to $L_p(\chi^\gamma)(0, \infty)$ as the structure of the subspaces of algebraic polynomials in each of them shows (cf. Remark 1.8). In order to cope with this problem we introduce the operators $\mathcal{L}_{i,j-1}$. Despite their effectiveness they, unfortunately, substantially complicate some proofs. Secondly, the belonging of at least one of γ_0, γ_∞ to $\mathcal{T}_{exc}(p)$ as $\gamma_0 \ne \gamma_\infty$ involves splitting of the singularities (see (5.6)), which, in turn, lessen the possibility for using Hardy's inequalities (see [9, p. 245, (9.9.8) and (9.9.9)]). Hence, we use appropriate integral representations of the derivatives (see Theorem 3.4) and modify Hardy's inequalities. The latter can be seen as a precise determination of the conditions on the weight w under which the inequality

$$\|\mathrm{w}\chi^k g^{(k)}\|_p \le c\|\mathrm{w}\chi^r g^{(r)}\|_p \tag{1.12}$$

follows for $\chi^r g^{(r)} \in L_p(\mathrm{w})(0, \infty)$ and $k < r$. But in many of the cases considered in this article the conditions of Hardy's inequalities are not met. So, under the additional assumption $g \in L_p(\mathrm{w})(0, \infty)$, we extend in Theorem 4.3 the range of (1.12) beyond the limits provided by Hardy's inequalities. As Remark 4.5 shows, the hypotheses of Theorem 4.3 are sharp for the validity of (1.12).

The paper is organized as follows. Section 2 contains the proof of Theorem 1.1 based on several inequalities related to the Post–Widder and the Gamma operators. In Section 3 we establish a representation of derivatives. In Section 4 we give a number of inequalities for the intermediate derivatives on which the proofs of the upper and lower estimates of the K-functional $K^r_{\mathrm{w}}(f,t^r)_p$ by the unweighted one are based. Theorems 1.4 and 1.5 are proved in Section 5, which also contains characterizations of the analogues of $K^r_{\mathrm{w}}(f,t^r)_p$ on the intervals $(0,a)$ and (a,∞) with $a>0$, as well as for spaces of continuous functions. In this section we show how several basic properties of $K^r_{\mathrm{w}}(f,t^r)_p$ can be derived from its characterization in Theorem 1.5. Finally, in Section 6 we explicitly construct operators $\mathscr{L}_{i,j-1}$ which satisfy conditions (i)–(ii) or (i)–(iv).

2. A characterization of the Post–Widder and the Gamma operator errors

The next theorem is basic for obtaining good upper bounds for the constants in Propositions 2.6–2.11. The functions from $L_{\infty,loc}(0,\infty)$ do not need to be bounded at 0 or at ∞.

Theorem 2.1. *Let $\xi,\eta\in\mathbb{R}$, $1\le p\le\infty$, $\psi\in L_{\infty,loc}(0,\infty)$. Set $\tilde{w}(x)=x^{\xi}$ for $0<x\le 1$ and $\tilde{w}(x)=x^{\eta}$ for $1\le x<\infty$. For every complex-valued $F\in L_p(\chi^{-1/p}\tilde{w})(0,\infty)$ denote*

$$G(x)=\int_0^\infty F(ux)\psi(u)\,\frac{\mathrm{d}u}{u},\qquad x\in(0,\infty). \tag{2.1}$$

Then

$$\|\chi^{-1/p}\tilde{w}G\|_{p(0,\infty)}\le(\theta_1+\theta_2)\|\chi^{-1/p}\tilde{w}F\|_{p(0,\infty)}, \tag{2.2}$$

where

$$\theta_1=\max\left\{\int_0^\infty|\psi(u)|u^{-\xi}\,\frac{\mathrm{d}u}{u},\int_0^\infty|\psi(u)|u^{-\eta}\,\frac{\mathrm{d}u}{u}\right\}, \tag{2.3}$$

$$\theta_2=\max\left\{\int_0^1\left|u^{-\xi}-u^{-\eta}\right||\psi(u)|\,\frac{\mathrm{d}u}{u},\int_1^\infty\left|u^{-\xi}-u^{-\eta}\right||\psi(u)|\,\frac{\mathrm{d}u}{u}\right\}. \tag{2.4}$$

Proof. Set $w_0=\chi^{\xi}$, $w_\infty=\chi^{\eta}$. Then $\tilde{w}=\max\{w_0,w_\infty\}$ iff $\xi\le\eta$ and $\tilde{w}=\min\{w_0,w_\infty\}$ iff $\xi\ge\eta$. Note that w_0 and w_∞ are multiplicative functions, i.e. $w_0(xy)=w_0(x)w_0(y)$ and $w_\infty(xy)=w_\infty(x)w_\infty(y)$ for every $x,y\in(0,\infty)$, but $\tilde{w}$ is not multiplicative when $\xi\ne\eta$.

The operator defined in (2.1) is linear. In view of the Riesz–Thorin theorem the statement will be established if we prove (2.2) for $p=1$ and for $p=\infty$.

First we deal with the case $p=1$. We have

$$\begin{aligned}\int_0^\infty\tilde{w}(x)|G(x)|\frac{\mathrm{d}x}{x}&\le\int_0^\infty\int_0^\infty\tilde{w}(x)|F(ux)|\frac{\mathrm{d}x}{x}|\psi(u)|\,\frac{\mathrm{d}u}{u}\\&=\int_0^\infty\int_0^\infty\tilde{w}\left(\frac{y}{u}\right)|F(y)|\frac{\mathrm{d}y}{y}|\psi(u)|\,\frac{\mathrm{d}u}{u}.\end{aligned} \tag{2.5}$$

Let us consider the weight $\tilde{w}(y/u)$ on the right-hand side of (2.5). We have $\tilde{w}(y/u)=w_0(y/u)$ if $0<y\le u<\infty$ and $\tilde{w}(y/u)=w_\infty(y/u)$ if $0<u\le y<\infty$. We aim to get a good upper bound for the difference $\tilde{w}(y/u) - w_0(y/u)$ in $0 < y \le 1, 0 < u < \infty$ and for the difference $\tilde{w}(y/u) - w_\infty(y/u)$ in $1 \le y < \infty, 0 < u < \infty$. We have $\tilde{w}(y/u) = w_0(y/u)$ if $0 < y \le 1, y \le u < \infty$ and $\tilde{w}(y/u) = w_\infty(y/u)$ if $1 \le y < \infty, 0 < u \le y$. So, it remains to consider the domains $\Omega_0 = \{(y,u) \in \mathbb{R}^2 : 0 < u \le y \le 1\}$ and $\Omega_\infty = \{(y,u) \in \mathbb{R}^2 : 1 \le y \le u < \infty\}$.

First, let $\xi \ge \eta$. Then we have

$$\tilde{w}(y/u) = w_\infty(y/u) \le w_0(y/u) = \frac{w_0(y)}{w_0(u)}, \quad (y,u) \in \Omega_0,$$

$$\tilde{w}(y/u) = w_0(y/u) \le w_\infty(y/u) = \frac{w_\infty(y)}{w_\infty(u)}, \quad (y,u) \in \Omega_\infty.$$

Using these inequalities in (2.5) we get

$$\begin{aligned}\int_0^\infty \tilde{w}(x)|G(x)|\frac{\mathrm{d}x}{x} &\le \int_0^\infty \frac{|\psi(u)|}{w_0(u)}\frac{\mathrm{d}u}{u} \cdot \int_0^1 \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y} + \int_0^\infty \frac{|\psi(u)|}{w_\infty(u)}\frac{\mathrm{d}u}{u} \\ &\quad \times \int_1^\infty \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y} \\ &\le \theta_1 \int_0^\infty \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y},\end{aligned}$$

which proves (2.2) for $p = 1$ and $\xi \ge \eta$.

Secondly, let $\xi \le \eta$. Then we have

$$\tilde{w}(y/u) = \frac{w_\infty(y)}{w_\infty(u)} \le \frac{w_0(y)}{w_\infty(u)} = \frac{w_0(y)}{w_0(u)} + \left[\frac{w_0(y)}{w_\infty(u)} - \frac{w_0(y)}{w_0(u)}\right], \quad (y,u) \in \Omega_0,$$

$$\tilde{w}(y/u) = \frac{w_0(y)}{w_0(u)} \le \frac{w_\infty(y)}{w_0(u)} = \frac{w_\infty(y)}{w_\infty(u)} + \left[\frac{w_\infty(y)}{w_0(u)} - \frac{w_\infty(y)}{w_\infty(u)}\right], \quad (y,u) \in \Omega_\infty.$$

Note that the terms in the square brackets are positive. Using these inequalities in (2.5) we get

$$\begin{aligned}&\int_0^\infty \tilde{w}(x)|G(x)|\frac{\mathrm{d}x}{x} \\ &\quad \le \int_0^\infty \frac{|\psi(u)|}{w_0(u)}\frac{\mathrm{d}u}{u} \cdot \int_0^1 \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y} + \int_0^\infty \frac{|\psi(u)|}{w_\infty(u)}\frac{\mathrm{d}u}{u} \cdot \int_1^\infty \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y} \\ &\qquad + \int_0^1 \int_u^1 w_0(y)|F(y)|\frac{\mathrm{d}y}{y}\left[\frac{1}{w_\infty(u)} - \frac{1}{w_0(u)}\right]|\psi(u)|\frac{\mathrm{d}u}{u} \\ &\qquad + \int_1^\infty \int_1^u w_\infty(y)|F(y)|\frac{\mathrm{d}y}{y}\left[\frac{1}{w_0(u)} - \frac{1}{w_\infty(u)}\right]|\psi(u)|\frac{\mathrm{d}u}{u} \\ &\quad \le \theta_1 \int_0^\infty \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y} + \theta_2 \int_0^1 w_0(y)|F(y)|\frac{\mathrm{d}y}{y} + \theta_2 \int_1^\infty w_\infty(y)|F(y)|\frac{\mathrm{d}y}{y} \\ &\quad = (\theta_1 + \theta_2)\int_0^\infty \tilde{w}(y)|F(y)|\frac{\mathrm{d}y}{y}.\end{aligned}$$

This completes the proof of (2.2) for $p = 1$.

Now, let us consider the case $p = \infty$. Let $\xi \le \eta$. For $0 < x \le 1$, using that $w_0(y) \le \tilde{w}(y)$ for every $y \in (0, \infty)$, we get

$$\begin{aligned}\tilde{w}(x)|G(x)| &= w_0(x)|G(x)| \\ &\le \int_0^\infty w_0(x)|F(ux)||\psi(u)| \frac{\mathrm{d}u}{u} = \int_0^\infty w_0(ux)|F(ux)| \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} \\ &\le \int_0^\infty \tilde{w}(ux)|F(ux)| \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} \le \int_0^\infty \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} \cdot \|\tilde{w}F\|_\infty \le \theta_1 \|\tilde{w}F\|_\infty.\end{aligned}$$

Similarly, for $1 \le x < \infty$ we get $w_\infty(x)|G(x)| \le \theta_1 \|\tilde{w}F\|_\infty$, which proves (2.2) for $p = \infty$ and $\xi \le \eta$.

Let $\xi \ge \eta$. For $0 < x \le 1$, using that $w_0(x) \le w_\infty(x)$, we get

$$\begin{aligned}\tilde{w}(x)|G(x)| &= w_0(x)|G(x)| \le \int_0^\infty w_0(x)|F(ux)||\psi(u)| \frac{\mathrm{d}u}{u} \\ &\le \int_0^{1/x} w_0(x)|F(ux)||\psi(u)| \frac{\mathrm{d}u}{u} + \int_{1/x}^\infty w_\infty(x)|F(ux)||\psi(u)| \frac{\mathrm{d}u}{u} \\ &= \int_0^{1/x} w_0(ux)|F(ux)| \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} + \int_{1/x}^\infty w_\infty(ux)|F(ux)| \frac{|\psi(u)|}{w_\infty(u)} \frac{\mathrm{d}u}{u} \\ &\le \left(\int_0^{1/x} \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} + \int_{1/x}^\infty \frac{|\psi(u)|}{w_\infty(u)} \frac{\mathrm{d}u}{u} \right) \|\tilde{w}F\|_\infty \\ &\le \left(\int_0^\infty \frac{|\psi(u)|}{w_0(u)} \frac{\mathrm{d}u}{u} + \int_{1/x}^\infty \left[\frac{|\psi(u)|}{w_\infty(u)} - \frac{|\psi(u)|}{w_0(u)} \right] \frac{\mathrm{d}u}{u} \right) \|\tilde{w}F\|_\infty \\ &\le (\theta_1 + \theta_2) \|\tilde{w}F\|_\infty.\end{aligned}$$

Similarly, for $1 \le x < \infty$, using that $w_\infty(x) \le w_0(x)$, we get

$$\begin{aligned}\tilde{w}(x)|G(x)| &= w_\infty(x)|G(x)| \\ &\le \left(\int_0^\infty \frac{|\psi(u)|}{w_\infty(u)} \frac{\mathrm{d}u}{u} + \int_0^{1/x} \left[\frac{|\psi(u)|}{w_0(u)} - \frac{|\psi(u)|}{w_\infty(u)} \right] \frac{\mathrm{d}u}{u} \right) \|\tilde{w}F\|_\infty \\ &\le (\theta_1 + \theta_2) \|\tilde{w}F\|_\infty,\end{aligned}$$

which completes the proof. □

Remark 2.2. In the proof of Theorem 2.1 for $p = \infty$ the case $\xi < \eta$, i.e. $\tilde{w} = \max\{\chi^\xi, \chi^\eta\}$, is simpler than the case $\xi > \eta$, i.e. $\tilde{w} = \min\{\chi^\xi, \chi^\eta\}$. But for $p = 1$ we have the opposite situation — the case $\xi > \eta$ is simpler than the case $\xi < \eta$!

Remark 2.3. Note that the differences between the two quantities under the max sign in (2.3) and (2.4) coincide, i.e.

$$\begin{aligned}&\left| \int_0^\infty |\psi(u)| u^{-\xi} \frac{\mathrm{d}u}{u} - \int_0^\infty |\psi(u)| u^{-\eta} \frac{\mathrm{d}u}{u} \right| \\ &\quad = \left| \int_0^1 \left| u^{-\xi} - u^{-\eta} \right| |\psi(u)| \frac{\mathrm{d}u}{u} - \int_1^\infty \left| u^{-\xi} - u^{-\eta} \right| |\psi(u)| \frac{\mathrm{d}u}{u} \right|.\end{aligned}$$

In the applications below the above quantity will have *smaller order* than θ_2, which in turn will have *smaller order* than θ_1. Let us also mention the obvious inequality $\theta_2 < \theta_1$ for every $\psi \not\equiv 0$.

For the applications of Theorem 2.1 in the proofs of Propositions 2.6–2.11 we need some notation and results established in [8]. For $\zeta \in \mathbb{R}$ and $s > \max\{0, \zeta\}$ we set

$$\kappa_1(\zeta, s) = \frac{s^\zeta \Gamma(s-\zeta)}{\Gamma(s)} = \frac{s^s}{\Gamma(s)} \int_0^\infty e^{-su} u^{s-\zeta} \frac{du}{u};$$

$$\kappa_j(\zeta, s) = \frac{s^{j-1}}{(2j-3)!\Gamma(s)} \int_0^\infty \int_1^{v/s} \left(\frac{v}{sy} - 1\right)^{2j-3} y^{-\zeta} \frac{dy}{y} e^{-v} v^s \frac{dv}{v}, \quad j = 2, 3;$$

$$\lambda_j(\zeta, s) = \frac{s^{\zeta-1}}{\Gamma(s)} \int_0^\infty |(v-s-2j+1)^2 - s - 2j + 1| e^{-v} v^{s-\zeta} \frac{dv}{v}, \quad j = 1, 2;$$

$$\lambda_3(\zeta, s) = \frac{s^{\zeta-\frac{1}{2}}}{\Gamma(s)} \int_0^\infty |v-s-2| e^{-v} v^{s-\zeta} \frac{dv}{v}.$$

Note that the signs of $(\frac{v}{sy} - 1)^{2j-3}$ and $(\frac{v}{s} - 1)$ in the definition of κ_2 and κ_3 coincide for every y from the integration range. Hence, the inner integral always has a non-negative value. This fact will be used in the proofs of Propositions 2.7 and 2.8.

The inequalities collected in the following lemma are established in Lemma 2.2, Propositions 2.7, 2.8, 2.9 and Remark 2.12 in [8].

Lemma 2.4. *There exists an absolute constant M_1 such that for every $s \geq \zeta^2 + 8$ and $\zeta \in \mathbb{R}$ we have*

$$|\kappa_1(\zeta, s) - 1| \leq M_1 \frac{\zeta^2+1}{s}; \tag{2.6}$$

$$\left|\kappa_2(\zeta, s) - \frac{1}{2}\right| \leq M_1 \frac{\zeta^2+1}{s}; \tag{2.7}$$

$$\left|\kappa_3(\zeta, s) - \frac{1}{8}\right| \leq M_1 \frac{\zeta^2+1}{s}; \tag{2.8}$$

$$\frac{s^{\zeta-1}}{\Gamma(s)} \int_0^\infty |(v-s-k)^2 - s - k| e^{-v} v^{s-\zeta} \frac{dv}{v} \leq \sqrt{2} + M_1 \frac{\zeta^2+1}{s}, \quad k = -1, 1, 3; \tag{2.9}$$

$$\frac{s^{\zeta-\frac{1}{2}}}{\Gamma(s)} \int_0^\infty |v-s-k| e^{-v} v^{s-\zeta} \frac{dv}{v} \leq 1 + M_1 \frac{\zeta^2+1}{s}, \quad k = -1, 2. \tag{2.10}$$

Lemma 2.5. *For every $s > 0$ and $k \in \mathbb{R}$ we have*

$$\frac{s^s}{\Gamma(s)} \int_0^\infty [su-s]^2 [su-s-k]^2 e^{-su} u^s \frac{du}{u} = 3s^2 + (k^2 - 4k + 6)s, \tag{2.11}$$

$$\frac{s^s}{\Gamma(s)} \int_0^\infty [su-s]^2 [(su-s-k)^2 - s - k]^2 e^{-su} u^s \frac{du}{u}$$
$$= 10s^3 + (16k^2 - 76k + 118)s^2 + (k^4 - 10k^3 + 45k^2 - 108k + 120)s. \tag{2.12}$$

Proof. Using the definition of $\Gamma(s)$ and its properties we get

$$\begin{aligned}
&\frac{s^s}{\Gamma(s)}\int_0^\infty [su-s]^2[su-s-k]^2 \mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}\\
&=\frac{1}{\Gamma(s)}\int_0^\infty [v-s]^2[v-s-k]^2\mathrm{e}^{-v}v^s\,\frac{\mathrm{d}v}{v}\\
&=[\Gamma(s+4)-2(2s+k)\Gamma(s+3)+(6s^2+6ks+k^2)\Gamma(s+2)\\
&\quad -2(s+k)s(2s+k)\Gamma(s+1)+(s+k)^2s^2\Gamma(s)]/\Gamma(s)\\
&=3s^2+(k^2-4k+6)s
\end{aligned}$$

and

$$\begin{aligned}
&\frac{s^s}{\Gamma(s)}\int_0^\infty [su-s]^2[(su-s-k)^2-s-k]^2\mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}\\
&=\frac{1}{\Gamma(s)}\int_0^\infty [v-s]^2[(v-s-k)^2-s-k]^2\mathrm{e}^{-v}v^s\,\frac{\mathrm{d}v}{v}\\
&=[\Gamma(s+6)-2(3s+2k)\Gamma(s+5)+(15s^2+20ks-2s+6k^2-2k)\Gamma(s+4)\\
&\quad -4(s+k)(5s^2+5ks-2s+k^2-k)\Gamma(s+3)\\
&\quad +(s+k)(15s^3+(25k-12)s^2+(11k-1)(k-1)s+k(k-1)^2)\Gamma(s+2)\\
&\quad -2(s+k)^2(s+k-1)(3s+k-1)s\Gamma(s+1)\\
&\quad +(s+k)^2(s+k-1)^2s^2\Gamma(s)]/\Gamma(s)\\
&=10s^3+(16k^2-76k+118)s^2+(k^4-10k^3+45k^2-108k+120)s. \quad \Box
\end{aligned}$$

In the proofs of Propositions 2.6–2.11 we shall apply the following estimates valid for every $\xi,\eta\in\mathbb{R}$:

$$\left|u^{-\xi}-u^{-\eta}\right|\le\begin{cases}|\xi-\eta|u^{-\mu}|u-1|, & 0<u\le 1;\\ |\xi-\eta|u^{-\nu}|u-1|, & 1\le u<\infty,\end{cases} \tag{2.13}$$

where $\mu=\max\{\xi,\eta\}+1$ and $\nu=\min\{\xi,\eta\}$. Now, we are ready to establish the main ingredients for the proof of Theorem 1.1.

Proposition 2.6. *There are positive numbers N_2, M_2 such that for every $\gamma_0,\gamma_\infty\in\mathbb{R}$, $s>N_2(\gamma_0^2+\gamma_\infty^2+1)$, $1\le p\le\infty$ and $f\in L_p(w)(0,\infty)$ we have*

$$\|wP_sf\|_p\le\kappa_1^*(\gamma_0,\gamma_\infty,s)\|wf\|_p, \tag{2.14}$$

where

$$\kappa_1^*(\gamma_0,\gamma_\infty,s)\le 1+M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}+M_2\frac{\gamma_0^2+\gamma_\infty^2+1}{s}.$$

Proof. From (1.1) we get the integral representation

$$(P_sf)(x)=\frac{s^s}{\Gamma(s)}\int_0^\infty f(xu)\mathrm{e}^{-su}u^s\frac{\mathrm{d}u}{u}.$$

Therefore we apply Theorem 2.1 with $\xi=\gamma_0+1/p$, $\eta=\gamma_\infty+1/p$, $\psi(u)=\mathrm{e}^{-su}u^ss^s/\Gamma(s)$ and get (2.14) with $\kappa_1^*(\gamma_0,\gamma_\infty,s)=\theta_1+\theta_2$, where θ_1,θ_2 are given in (2.3), (2.4). From (2.3) and (2.6) we get

$$\theta_1 = \frac{s^s}{\Gamma(s)} \max\left\{\int_0^\infty \mathrm{e}^{-su} u^{s-\xi}\,\frac{\mathrm{d}u}{u}, \int_0^\infty \mathrm{e}^{-su} u^{s-\eta}\,\frac{\mathrm{d}u}{u}\right\}$$
$$= \max\{\kappa_1(\xi, s), \kappa_1(\eta, s)\} \le 1 + M_1 \frac{\xi^2 + \eta^2 + 1}{s} \le 1 + M_2 \frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}.$$

In order to estimate θ_2 we apply the first inequality in (2.13), the Cauchy–Schwarz inequality, (2.6), the identity $\kappa_1(-2, s) - 2\kappa_1(-1, s) + \kappa_1(0, s) = s^{-1}$ and get

$$\frac{s^s}{\Gamma(s)} \int_0^1 \left|u^{-\xi} - u^{-\eta}\right| \mathrm{e}^{-su} u^s\,\frac{\mathrm{d}u}{u} \le |\xi - \eta| \frac{s^s}{\Gamma(s)} \int_0^\infty u^{-\mu}|u - 1|\mathrm{e}^{-su} u^s\,\frac{\mathrm{d}u}{u}$$
$$\le |\xi - \eta| \left\{\frac{s^s}{\Gamma(s)} \int_0^\infty u^{-2\mu} \mathrm{e}^{-su} u^s\,\frac{\mathrm{d}u}{u}\right\}^{1/2} \left\{\frac{s^s}{\Gamma(s)} \int_0^\infty (u - 1)^2 \mathrm{e}^{-su} u^s\,\frac{\mathrm{d}u}{u}\right\}^{1/2}$$
$$\le |\xi - \eta| \{\kappa_1(2\mu, s)\}^{1/2} \{\kappa_1(-2, s) - 2\kappa_1(-1, s) + \kappa_1(0, s)\}^{1/2}$$
$$\le \left\{1 + M_1 \frac{4\mu^2 + 1}{s}\right\}^{1/2} \frac{|\xi - \eta|}{\sqrt{s}} \le M_2 \frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}}.$$

Similarly, using the second estimate in (2.13), we get the same upper bound for the integral on $[1, \infty)$ as the one for $(0, 1]$ and complete the proof. □

Proposition 2.7. *There are positive numbers N_2, M_2 such that for every $\gamma_0, \gamma_\infty \in \mathbb{R}$, $s > N_2(\gamma_0^2 + \gamma_\infty^2 + 1)$, $1 \le p \le \infty$ and every g such that $\chi^2 D^2 g \in L_p(w)(0, \infty)$ we have*

$$\|w(P_s g - g)\|_p \le s^{-1} \kappa_2^*(\gamma_0, \gamma_\infty, s) \|w\chi^2 D^2 g\|_p, \tag{2.15}$$

where

$$\kappa_2^*(\gamma_0, \gamma_\infty, s) \le \frac{1}{2} + M_2 \frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}} + M_2 \frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}.$$

Proof. The following integral representation is obtained in the proof of Proposition 2.5 in [8]:

$$(P_s g)(x) - g(x) = \frac{1}{\Gamma(s)} \int_0^\infty \int_1^{v/s} \left(\frac{v}{su} - 1\right) (xu)^2 (D^2 g)(xu)\,\frac{\mathrm{d}u}{u}\,\mathrm{e}^{-v} v^s\,\frac{\mathrm{d}v}{v}.$$

Therefore we apply Theorem 2.1 with $\xi = \gamma_0 + 1/p$, $\eta = \gamma_\infty + 1/p$, $F = \chi^2 D^2 g$,

$$\psi(u) = \frac{1}{\Gamma(s)} \int_0^{su} \left(1 - \frac{v}{su}\right) \mathrm{e}^{-v} v^s\,\frac{\mathrm{d}v}{v} \quad \text{for } 0 < u < 1,$$
$$\psi(u) = \frac{1}{\Gamma(s)} \int_{su}^\infty \left(\frac{v}{su} - 1\right) \mathrm{e}^{-v} v^s\,\frac{\mathrm{d}v}{v} \quad \text{for } 1 < u < \infty$$

(and hence $G = P_s g - g$) and get (2.15) with $\kappa_2^*(\gamma_0, \gamma_\infty, s) = s\theta_1 + s\theta_2$, where θ_1, θ_2 are given in (2.3), (2.4). From (2.3) and (2.7) we get

$$s\theta_1 = \max\{\kappa_2(\xi, s), \kappa_2(\eta, s)\} \le \frac{1}{2} + M_1 \frac{\xi^2 + \eta^2 + 1}{s} \le \frac{1}{2} + M_2 \frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}.$$

In order to estimate $s\theta_2$ we apply the first inequality in (2.13), the Cauchy–Schwarz inequality, (2.7) and get

$$\begin{aligned} s\int_0^1 \left|u^{-\xi}-u^{-\eta}\right| |\psi(u)| \frac{\mathrm{d}u}{u} &= \frac{s}{\Gamma(s)}\int_0^s\int_1^{v/s} \left|u^{-\xi}-u^{-\eta}\right| \left(\frac{v}{su}-1\right)\frac{\mathrm{d}u}{u}\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v} \\ &\le |\xi-\eta|\frac{s}{\Gamma(s)}\int_0^s\int_1^{v/s} u^{-\mu}|u-1|\left(\frac{v}{su}-1\right)\frac{\mathrm{d}u}{u}\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v} \\ &\le |\xi-\eta|\left\{\frac{s}{\Gamma(s)}\int_0^\infty\int_1^{v/s} u^{-2\mu}\left(\frac{v}{su}-1\right)\frac{\mathrm{d}u}{u}\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v}\right\}^{1/2} \\ &\quad\times\left\{\frac{s}{\Gamma(s)}\int_0^\infty\int_1^{v/s}(u-1)^2\left(\frac{v}{su}-1\right)\frac{\mathrm{d}u}{u}\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v}\right\}^{1/2} \\ &= |\xi-\eta|\{\kappa_2(2\mu,s)\}^{1/2}\{\kappa_2(-2,s)-2\kappa_2(-1,s)+\kappa_2(0,s)\}^{1/2} \\ &\le |\xi-\eta|\left\{\frac{1}{2}+M_1\frac{4\mu^2+1}{s}\right\}^{1/2}\left\{\frac{10M_1}{s}\right\}^{1/2} \le M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}. \end{aligned}$$

Similarly, using the second estimate in (2.13), we get the same upper bound for the integral on $[1,\infty)$ as the one for $(0,1]$ and complete the proof. □

Proposition 2.8. *There are positive numbers N_2, M_2 such that for every $\gamma_0,\gamma_\infty\in\mathbb{R}$, $s > N_2(\gamma_0^2+\gamma_\infty^2+1)$, $1\le p\le\infty$ and every g such that $\chi^4D^4g\in L_p(w)(0,\infty)$ we have*

$$\left\|w\left(P_sg-g-\frac{\chi^2D^2g}{2s}-\frac{\chi^3D^3g}{3s^2}\right)\right\|_p \le \frac{\kappa_3^*(\gamma_0,\gamma_\infty,s)}{s^2}\|w\chi^4D^4g\|_p, \tag{2.16}$$

where

$$\kappa_3^*(\gamma_0,\gamma_\infty,s)\le\frac{1}{8}+M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}+M_2\frac{\gamma_0^2+\gamma_\infty^2+1}{s}.$$

Proof. The following integral representation is obtained in the proof of Proposition 2.6 in [8]:

$$\begin{aligned} &(P_sg)(x)-g(x)-\frac{1}{2}s^{-1}\chi^2(x)(D^2g)(x)-\frac{1}{3}s^{-2}\chi^3(x)(D^3g)(x) \\ &\quad=\frac{1}{6\Gamma(s)}\int_0^\infty\int_1^{v/s}\left(\frac{v}{su}-1\right)^3(xu)^4(D^4g)(xu)\frac{\mathrm{d}u}{u}\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v}. \end{aligned}$$

Therefore we apply Theorem 2.1 with $\xi=\gamma_0+1/p$, $\eta=\gamma_\infty+1/p$, $F=\chi^4D^4g$,

$$\psi(u)=\frac{1}{6\Gamma(s)}\int_0^{su}\left(1-\frac{v}{su}\right)^3\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v}\quad\text{for } 0<u<1,$$

$$\psi(u)=\frac{1}{6\Gamma(s)}\int_{su}^\infty\left(\frac{v}{su}-1\right)^3\mathrm{e}^{-v}v^s\frac{\mathrm{d}v}{v}\quad\text{for } 1<u<\infty$$

(and hence $G=P_sg-g-\frac{1}{2}s^{-1}\chi^2D^2g-\frac{1}{3}s^{-2}\chi^3D^3g$) and get (2.16) with $\kappa_3^*(\gamma_0,\gamma_\infty,s)=s^2\theta_1+s^2\theta_2$, where θ_1,θ_2 are given in (2.3), (2.4). From (2.3) and (2.8) we get

$$s^2\theta_1=\max\{\kappa_3(\xi,s),\kappa_3(\eta,s)\}\le\frac{1}{8}+M_1\frac{\xi^2+\eta^2+1}{s}\le\frac{1}{8}+M_2\frac{\gamma_0^2+\gamma_\infty^2+1}{s}.$$

In order to estimate $s^2\theta_2$ we apply the first inequality in (2.13), the Cauchy–Schwarz inequality, (2.8) and get

$$\begin{aligned}
s^2 \int_0^1 \left|u^{-\xi} - u^{-\eta}\right| |\psi(u)| \frac{\mathrm{d}u}{u} &= \frac{s^2}{6\Gamma(s)} \int_0^s \int_1^{v/s} \left|u^{-\xi} - u^{-\eta}\right| \left(\frac{v}{su} - 1\right)^3 \frac{\mathrm{d}u}{u} \mathrm{e}^{-v} v^s \frac{\mathrm{d}v}{v} \\
&\le |\xi - \eta| \frac{s^2}{6\Gamma(s)} \int_0^s \int_1^{v/s} u^{-\mu} |u-1| \left(\frac{v}{su} - 1\right)^3 \frac{\mathrm{d}u}{u} \mathrm{e}^{-v} v^s \frac{\mathrm{d}v}{v} \\
&\le |\xi - \eta| \left\{\frac{s^2}{6\Gamma(s)} \int_0^\infty \int_1^{v/s} u^{-2\mu} \left(\frac{v}{su} - 1\right)^3 \frac{\mathrm{d}u}{u} \mathrm{e}^{-v} v^s \frac{\mathrm{d}v}{v}\right\}^{1/2} \\
&\quad \times \left\{\frac{s^2}{6\Gamma(s)} \int_0^\infty \int_1^{v/s} (u-1)^2 \left(\frac{v}{su} - 1\right)^3 \frac{\mathrm{d}u}{u} \mathrm{e}^{-v} v^s \frac{\mathrm{d}v}{v}\right\}^{1/2} \\
&= |\xi - \eta| \left\{\kappa_3(2\mu, s)\right\}^{1/2} \left\{\kappa_3(-2, s) - 2\kappa_3(-1, s) + \kappa_3(0, s)\right\}^{1/2} \\
&\le |\xi - \eta| \left\{\frac{1}{8} + M_1 \frac{4\mu^2 + 1}{s}\right\}^{1/2} \left\{\frac{10 M_1}{s}\right\}^{1/2} \le M_2 \frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}}.
\end{aligned}$$

Similarly, using the second estimate in (2.13), we get the same upper bound for the integral on $[1, \infty)$ as the one for $(0, 1]$ and complete the proof. □

Proposition 2.9. *There are positive numbers N_2, M_2 such that for every $\gamma_0, \gamma_\infty \in \mathbb{R}$, $s > N_2(\gamma_0^2 + \gamma_\infty^2 + 1)$, $1 \le p \le \infty$ and every $f \in L_p(w)(0, \infty)$ we have*

$$\|w\chi^2 D^2 P_s f\|_p \le s\lambda_1^*(\gamma_0, \gamma_\infty, s)\|wf\|_p, \tag{2.17}$$

where

$$\lambda_1^*(\gamma_0, \gamma_\infty, s) \le \sqrt{2} + M_2 \frac{|\gamma_0 - \gamma_\infty|}{\sqrt{s}} + M_2 \frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}.$$

Proof. The following integral representation is obtained in the proof of Proposition 2.7 in [8]:

$$\chi^2(x)(D^2 P_s f)(x) = \frac{s^s}{\Gamma(s)} \int_0^\infty f(xu)[(su - s - 1)^2 - s - 1] \mathrm{e}^{-su} u^s \frac{\mathrm{d}u}{u}.$$

Therefore we apply Theorem 2.1 with $\xi = \gamma_0 + 1/p$, $\eta = \gamma_\infty + 1/p$, $\psi(u) = [(su - s - 1)^2 - s - 1]\mathrm{e}^{-su} u^s s^s / \Gamma(s)$ and get (2.17) with $\lambda_1^*(\gamma_0, \gamma_\infty, s) = s^{-1}\theta_1 + s^{-1}\theta_2$, where θ_1, θ_2 are given in (2.3), (2.4). From (2.3) and (2.9) with $k = 1$ we get

$$\begin{aligned}
s^{-1}\theta_1 &= \max\left\{\lambda_1(\xi, s), \lambda_1(\eta, s)\right\} \le \sqrt{2} + M_1 \frac{\xi^2 + \eta^2 + 1}{s} \\
&\le \sqrt{2} + M_2 \frac{\gamma_0^2 + \gamma_\infty^2 + 1}{s}.
\end{aligned}$$

In order to estimate $s^{-1}\theta_2$ we apply the first inequality in (2.13), the Cauchy–Schwarz inequality, (2.6), (2.12) with $k = 1$ and get

$$\begin{aligned}
&\frac{s^{s-1}}{\Gamma(s)} \int_0^1 \left|u^{-\xi} - u^{-\eta}\right| \left|(su - s - 1)^2 - s - 1\right| \mathrm{e}^{-su} u^s \frac{\mathrm{d}u}{u} \\
&\quad \le |\xi - \eta| \frac{s^{s-1}}{\Gamma(s)} \int_0^1 u^{-\mu} |u - 1| \left|(su - s - 1)^2 - s - 1\right| \mathrm{e}^{-su} u^s \frac{\mathrm{d}u}{u}
\end{aligned}$$

$$\le |\xi-\eta|\left\{\frac{s^s}{\Gamma(s)}\int_0^\infty u^{-2\mu}\mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}\right\}^{1/2}$$
$$\times\left\{\frac{s^{s-2}}{\Gamma(s)}\int_0^\infty \frac{(su-s)^2}{s^2}\left[(su-s-1)^2-s-1\right]^2\mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}\right\}^{1/2}$$
$$\le |\xi-\eta|\left\{1+M_1\frac{4\mu^2+1}{s^2}\right\}^{1/2}\left\{\frac{10s^3+58s^2+48s}{s^4}\right\}^{1/2}$$
$$\le M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}.$$

Similarly, using the second estimate in (2.13), we get the same upper bound for the integral on $[1,\infty)$ as the one for $(0,1]$ and complete the proof. □

Proposition 2.10. *There are positive numbers N_2, M_2 such that for every $\gamma_0,\gamma_\infty\in\mathbb{R}$, $s>N_2(\gamma_0^2+\gamma_\infty^2+1)$, $1\le p\le\infty$ and every g such that $\chi^2D^2g\in L_p(w)(0,\infty)$ we have*

$$\|w\chi^4D^4P_sg\|_p\le s\lambda_2^*(\gamma_0,\gamma_\infty,s)\|w\chi^2D^2g\|_p, \tag{2.18}$$

where

$$\lambda_2^*(\gamma_0,\gamma_\infty,s)\le\sqrt{2}+M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}+M_2\frac{\gamma_0^2+\gamma_\infty^2+1}{s}.$$

Proof. The following integral representation is obtained in the proof of Proposition 2.8 in [8]:

$$\chi^4(x)(D^4P_sg)(x)=\frac{s^s}{\Gamma(s)}\int_0^\infty(xu)^2(D^2g)(xu)[(su-s-3)^2-s-3]\mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}.$$

Now, we proceed as in the proof of Proposition 2.9 using (2.12) with $k=3$. □

Proposition 2.11. *There are positive numbers N_2, M_2 such that for every $\gamma_0,\gamma_\infty\in\mathbb{R}$, $s>N_2(\gamma_0^2+\gamma_\infty^2+1)$, $1\le p\le\infty$ and every g such that $\chi^2D^2g\in L_p(w)(0,\infty)$ we have*

$$\|w\chi^3D^3P_sg\|_p\le\sqrt{s}\lambda_3^*(\gamma_0,\gamma_\infty,s)\|w\chi^2D^2g\|_p, \tag{2.19}$$

where

$$\lambda_3^*(\gamma_0,\gamma_\infty,s)\le 1+M_2\frac{|\gamma_0-\gamma_\infty|}{\sqrt{s}}+M_2\frac{\gamma_0^2+\gamma_\infty^2+1}{s}.$$

Proof. The following integral representation is obtained in the proof of Proposition 2.9 in [8]

$$\chi^3(x)(D^3P_sg)(x)=\frac{s^s}{\Gamma(s)}\int_0^\infty(xu)^2(D^2g)(xu)[su-s-2]\mathrm{e}^{-su}u^s\,\frac{\mathrm{d}u}{u}.$$

Now, we proceed as in the proof of Proposition 2.9 using (2.10) instead of (2.9) and (2.11) instead of (2.12). □

Remark 2.12. If the Post–Widder operator P_s is replaced by the Gamma operator G_s, then Propositions 2.6–2.11 remain unchanged except Proposition 2.8, where (2.16) is to be replaced by

$$\left\|w\left(G_sg-g-\frac{\chi^2D^2g}{2(s-1)}-\frac{2\chi^3D^3g}{3(s-1)(s-2)}\right)\right\|_p\le\frac{\kappa_3^*(\gamma_0,\gamma_\infty,s)}{s^2}\|w\chi^4D^4g\|_p.$$

The reason is a different integral representation, namely

$$(G_s g)(x) - g(x) - \frac{\chi^2(x)(D^2g)(x)}{2(s-1)} - \frac{2\chi^3(x)(D^3g)(x)}{3(s-1)(s-2)}$$
$$= \frac{1}{6\Gamma(s+1)} \int_0^\infty \int_1^{s/v} \left(\frac{s}{vu} - 1\right)^3 (xu)^4 (D^4 g)(xu) \frac{\mathrm{d}u}{u} \mathrm{e}^{-v} v^{s+1} \frac{\mathrm{d}v}{v}.$$

The only modification in the proofs is the necessity to change the signs of ξ and η to the opposite, because the integral representations connected with G_s are naturally of the type

$$\int_0^\infty F(y^{-1}x)\tilde{\psi}(y)\frac{\mathrm{d}y}{y}.$$

So, a change of the variable $u = y^{-1}$ in the above integral has to be made before applying Theorem 2.1 and the inverse one $y = u^{-1}$ afterwards.

Proof of Theorem 1.1. We apply the proof of [8, Theorem 1.1] given in [8, Section 3] by simply replacing κ_j, λ_j there with κ_j^*, λ_j^* from Propositions 2.6–2.11 (and Remark 2.12) proved in this article as the parameters of the κ_j^*'s and λ_j^*'s are γ_0, γ_∞ and s. For the convenience of the reader we sketch the proof below.

The direct estimate is derived via a standard argument by means of Propositions 2.6 and 2.7 from the estimate

$$\|w(P_s f - f)\|_p \le \|w P_s(f-g)\|_p + \|w(P_s g - g)\|_p + \|w(f-g)\|_p$$

and taking the infimum over $g \in AC_{loc}^1(0,\infty)$ such that $g, \chi^2 D^2 g \in L_p(w)(0,\infty)$.

To establish the converse inequality we first note that

$$K_w^2\left(f, \frac{1}{4s}\right)_p \le \|w(f - P_s^2 f)\|_p + \frac{1}{4s}\|w\chi^2 D^2 P_s^2 f\|_p$$
$$\le (1+\kappa_1^*)\|w(f - P_s f)\|_p + \frac{1}{4s}\|w\chi^2 D^2 P_s^2 f\|_p. \tag{2.20}$$

In order to estimate the second summand above we use that

$$\frac{1}{2s}\|w\chi^2 D^2 P_s^2 f\|_p \le \left\| w\left(P_s^3 f - P_s^2 f - \frac{1}{2s}\chi^2 D^2 P_s^2 f - \frac{1}{3s^2}\chi^3 D^3 P_s^2 f\right)\right\|_p$$
$$+ \|w P_s^2(P_s f - f)\|_p + \frac{1}{3s^2}\|w\chi^3 D^3 P_s^2 f\|_p. \tag{2.21}$$

In view of Proposition 2.6, the second summand on the right-hand side of (2.21) is estimated by the L_p-norm of $w(f - P_s f)$. To achieve this for the first summand, we apply consecutively Proposition 2.8 (with $g = P_s^2 f$) and 2.10 (with $g = P_s f$), the relation

$$\|w\chi^2 D^2 P_s f\|_p \le \|w\chi^2 D^2 P_s^2 f\|_p + \|w\chi^2 D^2 P_s(f - P_s f)\|_p \tag{2.22}$$

and Proposition 2.9 (with $f - P_s f$ in the place of f). Similarly, to the third summand of (2.21) we apply Proposition 2.11 (with $g = P_s f$) and again (2.22) and Proposition 2.9 (with $f - P_s f$ in the place of f). Thus we arrive at the estimate

$$\frac{1}{4s}\|w\chi^2 D^2 P_s^2 f\|_p \le \frac{\kappa_1^{*2} + \kappa_3^*\lambda_1^*\lambda_2^* + 1/3\lambda_1^*\lambda_3^* s^{-1/2}}{2 - 4\kappa_3^*\lambda_2^* - 4/3\lambda_3^* s^{-1/2}}\|w(f - P_s f)\|_p. \tag{2.23}$$

The number $N \geq N_2$ from the hypotheses of the theorem is chosen in such way that the inequality

$$4\kappa_3^*\lambda_2^* + 4/3\lambda_3^* s^{-1/2} < 2$$

is satisfied. The converse inequality of the theorem follows from (2.20) and (2.23). □

3. An auxiliary derivative representation

A basic tool in the proof of the lower estimates of the K-functional $K_{\mathrm{w}}^r(f, t^r)_p$ by the unweighted fixed-step moduli of smoothness is a representation of the derivatives of a function $g \in AC_{loc}^{r-1}(0, \infty)$ such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0, \infty)$. To establish it we use the following assertions about the limit behaviour of the function at 0 and infinity.

Lemma 3.1 (*cf.* [8, *Corollary 4.3*]). *Let* $1 \leq p \leq \infty$.

(a) *Let* $G \in AC_{loc}(0, 1)$ *and* $G, \chi G' \in L_p(\chi^\gamma)(0, 1)$ *with* $\gamma \leq -1/p$ *if* $p < \infty$ *or* $\gamma < 0$ *if* $p = \infty$. *Then* $\lim_{x\to 0+0} G(x) = 0$.
(b) *Let* $G \in AC_{loc}(1, \infty)$ *and* $G, \chi G' \in L_p(\chi^\gamma)(1, \infty)$ *with* $\gamma \geq -1/p$ *if* $p < \infty$ *or* $\gamma > 0$ *if* $p = \infty$. *Then* $\lim_{x\to\infty} G(x) = 0$.

Proof. Let $p = 1, \gamma \leq -1$ or $1 < p \leq \infty, \gamma < -1/p$ in assertion (a). The condition on G' (and Hölder's inequality if $p > 1$) imply $G' \in L_1(0, 1)$; hence $G \in AC[0, 1]$. The assumption $|G(x)| \geq c > 0$ in a neighborhood of the origin would imply $\chi^\gamma \in L_p(0, 1)$, which contradicts $\gamma < -1/p$ (or $\gamma \leq -1$ for $p = 1$). Hence, there exists a sequence $\{\xi_n\}$ such that $\xi_n \to 0 + 0$ and $G(\xi_n) \to 0$ as $n \to \infty$, which in view of the continuity of G implies $\lim_{x\to 0+0} G(x) = 0$.

In the remaining case $1 < p < \infty, \gamma = -1/p$ we set $\tilde{G} = |G|^p \in L_1(\chi^{-1})(0, 1)$. From

$$\tilde{G}'(x) = p\, |G(x)|^{p-1} G'(x) \operatorname{sign} G(x),$$

$\chi^{-1+1/p}|G|^{p-1} \in L_{p'}(0, 1)$ with $p' = p/(p-1)$, $\chi^{1-1/p} G' \in L_p(0, 1)$ and Hölder's inequality we get $\tilde{G}' \in L_1(0, 1)$. Hence, $\tilde{G}$ satisfies the hypotheses of assertion (a) for $p = 1, \gamma = -1$ and then $\lim_{x\to 0+0} \tilde{G}(x) = 0$. The proof of assertion (a) is completed.

Assertion (b) is verified similarly. □

Remark 3.2. Lemma 3.1 is not true for the remaining values of γ and p. For instance, for $\gamma = 0$, $p = \infty$ counterexamples are given by $G(x) = 1$ or $G(x) = \sin \log x$.

From Lemma 3.1 we derive:

Lemma 3.3. *Let* $1 \leq p \leq \infty$, $r \in \mathbb{N}$, $\rho \in \mathbb{N}_0$, $\rho < r$.

(a) *Let* $g \in AC_{loc}^{r-1}(0, 1)$ *and* $g, \chi^r g^{(r)} \in L_p(\chi^{\gamma_0})(0, 1)$ *with* $\gamma_0 \leq -\rho - 1/p$ *if* $p < \infty$ *or* $\gamma_0 < -\rho$ *if* $p = \infty$. *Then*

$$\lim_{x\to 0+0} x^{\ell-\rho} g^{(\ell)}(x) = 0, \quad \ell = 0, 1, \ldots, r-1. \tag{3.1}$$

(b) *Let* $g \in AC_{loc}^{r-1}(1, \infty)$ *and* $g, \chi^r g^{(r)} \in L_p(\chi^{\gamma_\infty})(1, \infty)$ *with* $\gamma_\infty \geq -\rho - 1/p$ *if* $p < \infty$ *or* $\gamma_\infty > -\rho$ *if* $p = \infty$. *Then*

$$\lim_{x\to\infty} x^{\ell-\rho} g^{(\ell)}(x) = 0, \quad \ell = 0, 1, \ldots, r-1.$$

Proof. The hypotheses of assertion (a) and Proposition 4.1 imply

$$\chi^k g^{(k)} \in L_p(\chi^{\gamma_0})(0, 1), \quad k = 0, 1, \ldots, r. \tag{3.2}$$

Now, Lemma 3.1(a) with $G = g^{(m)}$ and $\gamma = \gamma_0 + m$ for $m = 0, \ldots, \rho$ implies $\lim_{x\to 0+0} g^{(m)}(x) = 0$ for $m = 0, \ldots, \rho$. Thus we get (3.1) for $\ell = \rho$.

Next, for $\ell = 0, \ldots, \rho - 1$, $\rho > 0$ by Taylor's formula at 0 as $g^{(\rho)} \in C[0, 1]$ we get

$$x^{\ell-\rho} g^{(\ell)}(x) = \frac{1}{(\rho - \ell - 1)!}\frac{1}{x}\int_0^x \left(1 - \frac{y}{x}\right)^{\rho-\ell-1} g^{(\rho)}(y)\,\mathrm{d}y.$$

Now, in view of $\lim_{x\to 0+0} g^{(\rho)}(x) = 0$ we get (3.1) for $\ell = 0, \ldots, \rho - 1$.

Further, for $\ell = \rho + 1, \ldots, r - 1$, $\rho < r - 1$, using

$$\left(x^{\ell-\rho} g^{(\ell-1)}(x)\right)' = (\ell - \rho)\, x^{\ell-\rho-1} g^{(\ell-1)}(x) + x^{\ell-\rho} g^{(\ell)}(x), \tag{3.3}$$

$$\left(x^{\ell-\rho} g^{(\ell-1)}(x)\right)'' = (\ell - \rho)(\ell - \rho - 1)\, x^{\ell-\rho-2} g^{(\ell-1)}(x) + 2(\ell - \rho)\, x^{\ell-\rho-1} g^{(\ell)}(x) + x^{\ell-\rho} g^{(\ell+1)}(x)$$

and (3.2) for $k = \ell-1, \ell, \ell+1$, we get that $\left(\chi^{\ell-\rho} g^{(\ell-1)}\right)'$, $\chi\left(\chi^{\ell-\rho} g^{(\ell-1)}\right)'' \in L_p(\chi^{\gamma_0+\rho})(0, 1)$. Consequently, by Lemma 3.1(a) with $G = \left(\chi^{\ell-\rho} g^{(\ell-1)}\right)'$ and $\gamma = \gamma_0 + \rho$ we get

$$\lim_{x\to 0+0} \left(x^{\ell-\rho} g^{(\ell-1)}(x)\right)' = 0, \quad \ell = \rho + 1, \ldots, r - 1. \tag{3.4}$$

Now, (3.1) with $\ell > \rho$ follows by induction from (3.3), (3.1) with $\ell = \rho$ and (3.4). This completes the proof of assertion (a).

Just similarly we verify assertion (b) as we use Lemma 3.1(b) and Taylor's expansion at $a > 1$. □

The next theorem contains the derivative representation that we shall extensively use. In its formulation we follow the convention that a sum is 0 if the upper boundary is smaller than the lower.

Theorem 3.4. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$, $\mu, \nu, k \in \mathbb{N}_0$ as $\mu \le \nu \le r$ and $k < r$, $a > 0$, and $x \in (0, \infty)$. Let also $g \in AC_{loc}^{r-1}(0, \infty)$ be such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0, \infty)$, where $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ is defined in (1.8). If $k < \mu$ we assume $\gamma_0 < 1 - \mu - 1/p$ for $p > 1$ or $\gamma_0 \le -\mu$ for $p = 1$, and if $\nu < r$ we assume $\gamma_\infty > -\nu - 1/p$ for $p > 1$ or $\gamma_\infty \ge -\nu - 1$ for $p = 1$. We set*

$$b_{r,n}(g, a) = \sum_{\ell=n}^{r-1} \frac{(-a)^{\ell-n}}{(\ell - n)!}\, g^{(\ell)}(a)$$

for $n = \mu, \ldots, \nu - 1$, $\mu < \nu$; $\tilde{\mu} = \max\{\mu, k\}$ and $\tilde{\nu} = \max\{\nu, k\}$. Then

$$g^{(k)}(x) = \sum_{n=\tilde{\mu}}^{\nu-1} \frac{x^{n-k}}{(n - k)!}\, b_{r,n}(g, a) + \sum_{n=r-\mu}^{r-k-1} \frac{(-1)^n x^{r-k-n-1}}{n!(r - k - n - 1)!}\int_0^x y^n g^{(r)}(y)\,\mathrm{d}y$$
$$+ \sum_{n=r-\nu}^{r-\tilde{\mu}-1} \frac{(-1)^n x^{r-k-n-1}}{n!(r - k - n - 1)!}\int_a^x y^n g^{(r)}(y)\,\mathrm{d}y$$
$$+ \sum_{n=0}^{r-\tilde{\nu}-1} \frac{(-1)^{n+1} x^{r-k-n-1}}{n!(r - k - n - 1)!}\int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y.$$

Proof. First, let us note that the integrals in the representation of $g^{(k)}(x)$ with 0 or ∞ as an integration boundary are finite in view of Hölder's inequality.

Let us denote respectively by $S_{m,k}(x)$, $m = 1, 2, 3, 4$, the four sums on the right-hand side of the formula of the theorem. We need to show that

$$S_{1,k}(x) + S_{2,k}(x) + S_{3,k}(x) + S_{4,k}(x) = g^{(k)}(x), \quad k = 0, \dots, r-1. \tag{3.5}$$

Let us observe that the convention for the sum notation implies

$$S_{1,k}(x) = 0, \quad k \geq \nu \text{ or } \mu = \nu, \tag{3.6}$$

$$S_{2,k}(x) = 0, \quad k \geq \mu, \tag{3.7}$$

$$S_{3,k}(x) = 0, \quad k \geq \nu \text{ or } \mu = \nu. \tag{3.8}$$

In the proof we extensively use the following formula obtained via integration by parts:

$$\int_\xi^\eta y^n g^{(r)}(y)\,\mathrm{d}y = n! \sum_{\ell=r-n-1}^{r-1} (-1)^{r-\ell-1} \left.\frac{y^{n+\ell-r+1} g^{(\ell)}(y)}{(n+\ell-r+1)!}\right|_\xi^\eta . \tag{3.9}$$

Using (3.9) with $\eta = x$ and $\xi \to 0$, Lemma 3.3(a) with $\rho = \mu - 1 \geq 0$, interchanging the order of summation, reordering the summands in the inner sum by setting $m = r - k - n - 1$ and considering separately the cases $\ell < \mu$ and $\ell \geq \mu$, we get

$$S_{2,k}(x) = g^{(k)}(x) + \sum_{\ell=\mu}^{r-1} \left[\sum_{m=0}^{\mu-k-1} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!}\, g^{(\ell)}(x), \quad k < \mu. \tag{3.10}$$

Similarly, by means of (3.9) with $\eta = x$ and $\xi = a$, interchanging the order of summation in the double sum containing $g^{(\ell)}(x)$, and reordering the summands in the inner sum by setting $m = r - k - n - 1$, we get for $k < \nu$ and $\mu < \nu$

$$S_{3,k}(x) = \sum_{\ell=\tilde{\mu}}^{r-1} \left[\sum_{m=\tilde{\mu}-k}^{\min\{\nu-k-1,\ell-k\}} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!}\, g^{(\ell)}(x) - S_{1,k}(x).$$

Hence

$$\begin{aligned} S_{3,k}(x) = {} & \sum_{\ell=\mu}^{\nu-1} \left[\sum_{m=\mu-k}^{\ell-k} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!}\, g^{(\ell)}(x) \\ & + \sum_{\ell=\nu}^{r-1} \left[\sum_{m=\mu-k}^{\nu-k-1} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!}\, g^{(\ell)}(x) - S_{1,k}(x), \\ & k < \mu < \nu, \end{aligned} \tag{3.11}$$

and

$$S_{3,k}(x) = g^{(k)}(x) + \sum_{\ell=\nu}^{r-1} \left[\sum_{m=0}^{\nu-k-1} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!}\, g^{(\ell)}(x) - S_{1,k}(x),$$
$$\mu \leq k < \nu. \tag{3.12}$$

As in the proof of (3.10), using now (3.9) with $\xi = x$ and $\eta \to \infty$ and Lemma 3.3(b) with $\rho = \nu$ we get for $\nu < r$

$$S_{4,k}(x) = \sum_{\ell=\tilde{\nu}}^{r-1} \left[\sum_{m=\tilde{\nu}-k}^{\ell-k} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!} g^{(\ell)}(x).$$

Hence

$$S_{4,k}(x) = \sum_{\ell=\nu}^{r-1} \left[\sum_{m=\nu-k}^{\ell-k} (-1)^m \binom{\ell-k}{m} \right] \frac{(-x)^{\ell-k}}{(\ell-k)!} g^{(\ell)}(x), \quad k < \nu, \tag{3.13}$$

and

$$S_{4,k}(x) = g^{(k)}(x), \quad k \ge \nu. \tag{3.14}$$

Now, (3.10), (3.8) if $\mu = \nu$ or (3.11) if $\mu < \nu$, and (3.13) imply (3.5) for $k = 0, \dots, \mu - 1$, $\mu > 0$; (3.7), (3.12) and (3.13) imply (3.5) for $k = \mu, \dots, \nu - 1$, $\mu < \nu$; and, finally, (3.6)–(3.8) and (3.14) imply (3.5) for $k = \nu, \dots, r-1$, $\nu < r$. $\square$

Remark 3.5. The case $\nu < \mu$ under the hypotheses of the theorem is covered by the case $\mu = \nu$. Let us observe that if $\nu \le \mu$, then the space $L_p(\mathrm{w})(0, \infty)$ is rather narrow; in particular, it does not contain any non-zero polynomial of degree less than r. For $\mu = \nu$ the formula of the theorem takes the form

$$\begin{aligned} g^{(k)}(x) = &\sum_{n=r-\mu}^{r-k-1} \frac{(-1)^n x^{r-k-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\, \mathrm{d}y \\ &+ \sum_{n=0}^{r-\mu-1} \frac{(-1)^{n+1} x^{r-k-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\, \mathrm{d}y, \quad k < \mu, \end{aligned}$$

and

$$g^{(k)}(x) = \sum_{n=0}^{r-k-1} \frac{(-1)^{n+1} x^{r-k-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\, \mathrm{d}y, \quad \mu \le k < r.$$

Also, let us note that if we do not impose any restriction on the weight w at 0 (i.e. we set $\mu = 0$), we get representations which do not contain integrals of the form $\int_0^x y^n g^{(r)}(y)\, \mathrm{d}y$. Similarly, if we do not impose any restriction on w at infinity (i.e. we set $\nu = r$), we get representations without integrals of the form $\int_x^\infty y^n g^{(r)}(y)\, \mathrm{d}y$.

4. Inequalities for intermediate derivatives

In the proof of the characterization of the K-functional $K_{\mathrm{w}}^r(f, t^r)_p$ we use several inequalities for the intermediate derivatives. The following inequalities are well-known (see e.g. [1, Ch. 2, Theorem 5.6]):

$$(b-a)^k \, \|g^{(k)}\|_{p[a,b]} \le c \left(\|g\|_{p[a,b]} + (b-a)^r \, \|g^{(r)}\|_{p[a,b]} \right), \tag{4.1}$$

for every $g \in W_p^r[a, b]$ and $k = 0, 1, \dots, r$, and

$$\|g^{(k)}\|_{p(J)} \le c \left(\|g\|_{p(J)} + \|g^{(r)}\|_{p(J)} \right), \tag{4.2}$$

for every $g \in W_p^r(J)$ and $k = 0, 1, \dots, r$, where $J = (-\infty, \infty)$ or $J = (-\infty, a)$ or $J = (a, \infty)$, $a \in \mathbb{R}$. The constant c in (4.1) and (4.2) depends only on r. Through the arguments used in the proof of [8, Proposition 4.1] (see also [4, Lemma 1]) on the basis of (4.1) we establish

Proposition 4.1. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8) with $\gamma_0, \gamma_\infty \in \mathbb{R}$ and J be any of the intervals $(0, a)$, (a, ∞) or $(0, \infty)$, where $a > 0$. Then for every $g \in AC_{loc}^{r-1}(J)$ such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(J)$ we have*

$$\|\mathrm{w}\chi^k g^{(k)}\|_{p(J)} \le c\left(\|\mathrm{w}g\|_{p(J)} + \|\mathrm{w}\chi^r g^{(r)}\|_{p(J)}\right), \quad k = 0, 1, \dots, r, \tag{4.3}$$

where the constant c depends only on γ_0, γ_∞ and r.

To establish the characterizations of $K_{\mathrm{w}}^r(f, t^r)_p$ given in the Introduction we shall need several improvements of the inequality of the last proposition with the first term on the right missing. These inequalities are either consequences or modifications of Hardy's inequalities.

For the proofs we set

$$\psi_{m,n}(\xi, \eta; x) = x^{m-n-1} \int_\xi^\eta y^n g^{(m)}(y)\, \mathrm{d}y \tag{4.4}$$

for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $0 \le \xi, \eta \le \infty$ and $g \in AC_{loc}^{m-1}(0, \infty)$ provided that the integral is well defined.

Proposition 4.2. *Let $1 \le p \le \infty$, $r \in \mathbb{N}$, $j \in \mathbb{N}_0$, $j \le r$, $a > 0$ and $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8). Let also $g \in AC_{loc}^{r-1}(0, \infty)$ be such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0, \infty)$. The following assertions hold true:*

(a) *If $\gamma_0 < 1 - r - 1/p$, then*

$$\|\chi^{\gamma_0+k} g^{(k)}\|_{p(0,a)} \le c\, \|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,a)}, \quad k = 0, \dots, r-1.$$

(b) *If $\gamma_0 < 1 - j - 1/p$, $\gamma_\infty \in \mathscr{T}_j(p)$, $j > 0$, then*

$$\|\chi^{\gamma_\infty+k} g^{(k)}\|_{p(a,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = 0, \dots, j-1.$$

(c) *If $\gamma_\infty > -j - 1/p$, $j < r$, then*

$$\|\chi^{\gamma_\infty+k} g^{(k)}\|_{p(a,\infty)} \le c\, \|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(a,\infty)}, \quad k = j, \dots, r-1.$$

(d) *If $\gamma_0, \gamma_\infty > -j - 1/p$, $j < r$, then*

$$\|\chi^{\gamma_0+k} g^{(k)}\|_{p(0,a)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = j, \dots, r-1.$$

The constant c is independent of g.

Proof. Let $\gamma_0 < 1 - r - 1/p$. By Theorem 3.4 with $\mu = \nu = r$ we have for $k = 0, \dots, r-1$

$$x^k g^{(k)}(x) = \sum_{n=0}^{r-k-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\, \mathrm{d}y.$$

Now, since $\gamma_0 + r - n - 1 < -1/p$ for $n = 0, \dots, r-1$, Hardy's inequality implies

$$\|\chi^{\gamma_0} \psi_{r,n}(0, \cdot; \cdot)\|_{p(0,a)} \le c\|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,a)}, \quad k, n = 0, \dots, r-1,$$

where $\psi_{r,n}(0, x; x)$ is given by (4.4). Hence (a) follows.

To prove (b) we get by Theorem 3.4 with $\mu = \nu = j$ the representation

$$x^k g^{(k)}(x) = \sum_{n=r-j}^{r-k-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y + \sum_{n=0}^{r-j-1} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y$$

for $k = 0, \dots, j-1$. Since $\chi^{\gamma_\infty + r - n - 1} \in L_p(a, \infty)$ for $n \ge r - j$ and also $\gamma_0 < 1 - j - 1/p$, we get by Hölder's inequality for $n \ge r - j$

$$\|\chi^{\gamma_\infty} \psi_{r,n}(0, a; \cdot)\|_{p(a,\infty)} \le c\, \|\chi^n g^{(r)}\|_{1(0,a)} \le c\, \|\chi^{\gamma_0 + r} g^{(r)}\|_{p(0,a)}. \tag{4.5}$$

Since $\gamma_\infty + r - n - 1 < -1/p$ for $n \ge r - j$, Hardy's inequality yields for $n \ge r - j$

$$\|\chi^{\gamma_\infty} \psi_{r,n}(a, \cdot; \cdot)\|_{p(a,\infty)} \le c\, \|\chi^{\gamma_\infty + r} g^{(r)}\|_{p(a,\infty)}. \tag{4.6}$$

Relations (4.5), (4.6) and Minkowski's inequality imply for $n \ge r - j$

$$\begin{aligned} \|\chi^{\gamma_\infty} \psi_{r,n}(0, \cdot; \cdot)\|_{p(a,\infty)} &\le \|\chi^{\gamma_\infty} \psi_{r,n}(0, a; \cdot)\|_{p(a,\infty)} + \|\chi^{\gamma_\infty} \psi_{r,n}(a, \cdot; \cdot)\|_{p(a,\infty)} \\ &\le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}. \end{aligned}$$

Thus (b) is established for $j = r$. To finish the proof for $j < r$ we also need to observe that, since $\gamma_\infty + r - n - 1 > -1/p$ for $n \le r - j - 1$, $j < r$, Hardy's inequality implies for $n \le r - j - 1$, $j < r$,

$$\|\chi^{\gamma_\infty} \psi_{r,n}(\cdot, \infty; \cdot)\|_{p(a,\infty)} \le c\, \|\chi^{\gamma_\infty + r} g^{(r)}\|_{p(a,\infty)}.$$

Assertions (c) and (d) are established like (a) and (b) respectively using the representation from Theorem 3.4 with $\nu = j$ and with $k \ge j$. $\square$

Combining inequalities given in the last proposition, we get:

Theorem 4.3. *Let $1 \le p \le \infty$, $i, j \in \mathbb{N}_0$, $r \in \mathbb{N}$, $i, j \le r$ and $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8). We set*

$$m = \begin{cases} 0 & \text{if } \gamma_0 \in \mathscr{T}_i(p), \gamma_\infty \in \mathscr{T}_j(p), j \le i; \\ i & \text{if } \gamma_0 = 1 - i - 1/p,\ \gamma_\infty \in \mathscr{T}_j(p) \cup \{1 - j - 1/p\}, i > 0, j \le i; \\ j & \text{if } \gamma_0 \in \mathscr{T}_i(p),\ \gamma_\infty = 1 - j - 1/p, 0 < j \le i; \\ j & \text{if } \gamma_0 \in \mathscr{T}_i(p) \cup \{1 - i - 1/p\}, \gamma_\infty \in \mathscr{T}_j(p) \cup \{1 - j - 1/p\}, i < j. \end{cases}$$

If $m < r$, then for $g \in AC_{loc}^{r-1}(0, \infty)$ such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0, \infty)$ we have

$$\|\mathrm{w}\chi^k g^{(k)}\|_{p(0,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = m, m+1, \dots, r-1. \tag{4.7}$$

The constant c is independent of g.

Proof. If $\gamma_0, \gamma_\infty > -\max\{i, j\} - 1/p$ and $i, j < r$, then Proposition 4.2(c) and (d) imply

$$\|\mathrm{w}\chi^k g^{(k)}\|_{p(0,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = \max\{i, j\}, \dots, r-1, \tag{4.8}$$

which verifies the assertion of the theorem in the following cases:

- $\gamma_0, \gamma_\infty \in \mathscr{T}_0(p)$;
- $\gamma_0 = 1 - i - 1/p, \gamma_\infty \in \mathscr{T}_j(p) \cup \{1 - j - 1/p\}, 0 < i < r, j \le i$;

- $\gamma_0 \in \mathcal{T}_j(p), \gamma_\infty = 1 - j - 1/p, 0 < j < r$;
- $\gamma_0 \in \mathcal{T}_i(p) \cup \{1 - i - 1/p\}, \gamma_\infty \in \mathcal{T}_j(p) \cup \{1 - j - 1/p\}, i < j < r$.

Next, if $\gamma_0 < 1 - i - 1/p$, $\gamma_\infty > -j - 1/p$ and $j < i$, then Proposition 4.2(a) and (c) with $r = i$ imply

$$\|\mathrm{w}\chi^k g^{(k)}\|_{p(0,\infty)} \le c\,\|\mathrm{w}\chi^i g^{(i)}\|_{p(0,\infty)}, \quad k = j, \dots, i-1, \tag{4.9}$$

which together with (4.8) yields the assertion of the theorem in the cases:

- $\gamma_0 \in \mathcal{T}_i(p), i > 0, \gamma_\infty \in \mathcal{T}_0(p)$;
- $\gamma_0 \in \mathcal{T}_i(p), \gamma_\infty = 1 - j - 1/p, 0 < j < i$.

Finally, if $\gamma_0, \gamma_\infty < 1 - j - 1/p$, $j > 0$, then Proposition 4.2(a) and (b) with $r = j = i$ imply

$$\|\mathrm{w}\chi^k g^{(k)}\|_{p(0,\infty)} \le c\|\mathrm{w}\chi^j g^{(j)}\|_{p(0,\infty)}, \quad k = 0, \dots, j-1, \tag{4.10}$$

which verifies the theorem in the case $\gamma_0, \gamma_\infty \in \mathcal{T}_r(p)$. Inequalities (4.10) and (4.8) with $0 < i = j < r$ imply the theorem for $\gamma_0, \gamma_\infty \in \mathcal{T}_i(p)$, $0 < i < r$; inequalities (4.10) and (4.9) with $i = r$ imply the theorem for $\gamma_0 \in \mathcal{T}_r(p)$, $\gamma_\infty \in \mathcal{T}_j(p)$, $0 < j < r$; and inequalities (4.10), (4.9) and (4.8) imply the theorem for $\gamma_0 \in \mathcal{T}_i(p)$, $\gamma_\infty \in \mathcal{T}_j(p)$, $0 < j < i < r$.

Thus the proof is completed. $\square$

Remark 4.4. Let us note that in terms of γ_0, γ_∞ the condition $m = r$ is equivalent to $\gamma_0 = 1 - r - 1/p$, or $\gamma_\infty = 1 - r - 1/p$, or $\gamma_\infty < 1 - r - 1/p < \gamma_0$.

Remark 4.5. Theorem 4.3 is exact in the following sense. The inequality (4.7) is not true for $k = m - 1$ provided that $m \ne 0$. Indeed, let $\phi \in C^\infty(\mathbb{R})$ be a fixed function with support in $[1, 2]$. For arbitrary $\delta \in (0, 1)$ we set $g_{1,\delta}(x) = x^{m-1}\phi(x^{-\delta})$ and $g_{2,\delta}(x) = x^{m-1}\phi(x^\delta)$. Let $\gamma \in \mathbb{R}$ be arbitrary. We observe that $g_{1,\delta}, \chi^r g_{1,\delta}^{(r)} \in L_p(\mathrm{w}(1 - m - 1/p, \gamma))(0, \infty)$ and $g_{2,\delta}, \chi^r g_{2,\delta}^{(r)} \in L_p(\mathrm{w}(\gamma, 1 - m - 1/p))(0, \infty)$ for $1 \le p \le \infty$. Moreover, we have

$$\begin{aligned}
&\|\mathrm{w}(1 - m - 1/p, \gamma)\chi^k g_{1,\delta}^{(k)}\|_{p(0,\infty)} \sim \delta^{-1/p}, \quad k = 0, \dots, m-1,\\
&\|\mathrm{w}(1 - m - 1/p, \gamma)\chi^k g_{1,\delta}^{(k)}\|_{p(0,\infty)} \sim \delta^{1-1/p}, \quad k = m, \dots, r,\\
&\|\mathrm{w}(\gamma, 1 - m - 1/p)\chi^k g_{2,\delta}^{(k)}\|_{p(0,\infty)} \sim \delta^{-1/p}, \quad k = 0, \dots, m-1,\\
&\|\mathrm{w}(\gamma, 1 - m - 1/p)\chi^k g_{2,\delta}^{(k)}\|_{p(0,\infty)} \sim \delta^{1-1/p}, \quad k = m, \dots, r.
\end{aligned}$$

If γ_0 or γ_∞ are in $\mathcal{T}_{exc}(p)$, then $g_{1,\delta}$ or $g_{2,\delta}$, respectively, with $\delta \to 0$ provides a counterexample to (4.7) with $k = m - 1$. A counterexample in the remaining cases with $m > 0$, which are described by $\gamma_0 > 1 - m - 1/p$, $\gamma_\infty \in \mathcal{T}_m(p)$, is provided by $g = \chi^{m-1}$.

Remark 4.6. In view of Theorem 4.3 and Remark 4.5 we can decrease the order of the derivative k (starting from $r - 1$) in (4.7) until three conditions: $\chi^k \notin L_p(\mathrm{w})(0, \infty)$, $\gamma_0 + k \ne -1/p$ and $\gamma_\infty + k \ne -1/p$ are satisfied. In all the cases considered in Theorem 4.3 we have $\pi_{r-1} \cap L_p(\mathrm{w})(0, \infty) \subseteq \pi_{m-1}$ and $\gamma_0 + k, \gamma_\infty + k \ne -1/p$, $k = m, \dots, r-1$.

Remark 4.7. Let us observe that if $\gamma_0 < \gamma_\infty$, then Theorem 4.3 follows from the assertion for $\gamma_0 = \gamma_\infty$, established in [8, Corollary 4.2], because in this case we have $\mathrm{w} \sim \max\{\chi^{\gamma_0}, \chi^{\gamma_\infty}\}$.

Now, we proceed to the analogue of Proposition 4.2 and Theorem 4.3 in the case when there exist monomials χ^k with $k \in \{m, \dots, r-1\}$ in $L_p(\mathrm{w})(0,\infty)$.

Proposition 4.8. *Let* $1 \le p \le \infty$, $i \in \mathbb{N}_0$, $j, r \in \mathbb{N}$ *as* $i < j \le r$, $a > 0$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ *be defined in* (1.8) *and the linear operator* $\mathscr{L}_{i,j-1}$ *given by* (1.10) *satisfy conditions* (i)–(ii). *We set* $\bar{\alpha} = \min\{a, \alpha\}$ *and* $\bar{\beta} = \max\{a, \beta\}$. *Let also* $g \in AC_{loc}^{r-1}(0,\infty)$ *be such that* $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0,\infty)$. *The following assertions hold true:*

(a) *If* $\gamma_0 \in \mathscr{T}_i(p)$ *and* $\gamma_\infty > -j - 1/p$, $j < r$, *then*

$$\|\chi^{\gamma_0+k}(g - \mathscr{L}_{i,j-1}g)^{(k)}\|_{p(0,a)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = 0, \dots, r-1.$$

(b) *If* $\gamma_0 \in \mathscr{T}_i(p)$, *then*

$$\|\chi^{\gamma_0+k}(g - \mathscr{L}_{i,r-1}g)^{(k)}\|_{p(0,a)} \le c\,\|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,\bar{\beta})}, \quad k = 0, \dots, r-1.$$

(c) *If* $\gamma_\infty \in \mathscr{T}_j(p)$, *then*

$$\|\chi^{\gamma_\infty+k}(g - \mathscr{L}_{0,j-1}g)^{(k)}\|_{p(a,\infty)} \le c\,\|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(\bar{\alpha},\infty)}, \quad k = 0, \dots, r-1.$$

(d) *If* $\gamma_0 < 1 - i - 1/p$, $i > 0$, *and* $\gamma_\infty \in \mathscr{T}_j(p)$, *then*

$$\|\chi^{\gamma_\infty+k}(g - \mathscr{L}_{i,j-1}g)^{(k)}\|_{p(a,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = 0, \dots, r-1.$$

(e) *If* $\gamma_0 \in \mathscr{T}_i(p)$ *and* $\gamma_\infty \in \mathscr{T}_j(p)$, *then*

$$\|\mathrm{w}\chi^k(g - \mathscr{L}_{i,j-1}g)^{(k)}\|_{p(0,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = 0, \dots, r-1.$$

The constant c *is independent of* g.

Remark 4.9. Note that in the hypotheses of items (a), (d) and (e) above we have $\pi_{r-1} \cap L_p(\mathrm{w})(0,\infty) \subseteq \pi_{i,j-1}$; in the hypothesis of (b) we have $\pi_{r-1} \cap L_p(\chi^{\gamma_0})(0,\bar{\beta}) = \pi_{i,r-1}$; and in the hypothesis of (c) we have $\pi_{r-1} \cap L_p(\chi^{\gamma_\infty})(\bar{\alpha},\infty) = \pi_{0,j-1}$ (cf. Remark 1.8). Consequently, by property (ii) of $\mathscr{L}_{i,j-1}$ the left-hand side of each of the inequalities above is 0 whenever g is a polynomial of degree less than r which belongs to the respective weighted L_p-space.

Proof of Proposition 4.8. Proposition 4.1 implies that it is sufficient to prove the assertions only for $k = 0$. Each of the hypotheses of (a)–(e) imply $\gamma_0 < 1-i-1/p$ for $i > 0$ and $\gamma_\infty > -j-1/p$ for $j < r$. Then by Theorem 3.4 with $\mu = i$, $\nu = j$, $k = 0$, and property (ii) of $\mathscr{L}_{i,j-1}$ we get

$$g - \mathscr{L}_{i,j-1}g = Rg - \mathscr{L}_{i,j-1}(Rg), \tag{4.11}$$

where

$$\begin{aligned}(Rg)(x) = {} & \sum_{n=r-i}^{r-1} \frac{(-1)^n x^{r-n-1}}{n!(r-n-1)!} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=r-j}^{r-i-1} \frac{(-1)^n x^{r-n-1}}{n!(r-n-1)!} \int_a^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=0}^{r-j-1} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y.\end{aligned}$$

First we shall prove (a) and (b). Since $\gamma_0 \in \mathscr{T}_i(p)$ we get by Hardy's inequalities that

$$\|\chi^{\gamma_0}\psi_{r,n}(0,\cdot;\cdot)\|_{p(0,\bar\beta)} \le c\,\|\chi^{\gamma_0+r}g^{(r)}\|_{p(0,\bar\beta)}, \quad n \ge r-i,\ i>0, \tag{4.12}$$

$$\|\chi^{\gamma_0}\psi_{r,n}(a,\cdot;\cdot)\|_{p(0,\bar\beta)} \le c\,\|\chi^{\gamma_0+r}g^{(r)}\|_{p(0,\bar\beta)}, \quad n \le r-i-1, \tag{4.13}$$

where the functions $\psi_{r,n}$ are defined in (4.4).

Next, since $\chi^{\gamma_0+r-n-1} \in L_p(0,\bar\beta)$ for $n \le r-j-1$ and also $\gamma_\infty > -j-1/p$ for $j<r$ we get by Hölder's inequality

$$\|\chi^{\gamma_0}\psi_{r,n}(a,\infty;\cdot)\|_{p(0,\bar\beta)} \le c\,\|\chi^{\gamma_\infty+r}g^{(r)}\|_{p(a,\infty)}, \quad n \le r-j-1,\ j<r. \tag{4.14}$$

Relations (4.13) and (4.14) imply

$$\|\chi^{\gamma_0}\psi_{r,n}(\cdot,\infty;\cdot)\|_{p(0,\bar\beta)} \le c\,\|\mathrm{w}\chi^{r}g^{(r)}\|_{p(0,\infty)}, \quad n \le r-j-1,\ j<r. \tag{4.15}$$

Now, inequalities (4.12), (4.13) and (4.15) imply

$$\|\chi^{\gamma_0}Rg\|_{p(0,\bar\beta)} \le c\,\|\mathrm{w}\chi^{r}g^{(r)}\|_{p(0,\infty)}, \quad j<r, \tag{4.16}$$

and (4.12) and (4.13) imply

$$\|\chi^{\gamma_0}Rg\|_{p(0,\bar\beta)} \le c\,\|\chi^{\gamma_0+r}g^{(r)}\|_{p(0,\bar\beta)}, \quad j=r. \tag{4.17}$$

Further, using property (i) of $\mathscr{L}_{i,j-1}$ and Hölder's inequality we get

$$\|\chi^{\gamma_0}\mathscr{L}_{i,j-1}(Rg)\|_{p(0,a)} \le c\,\|Rg\|_{1(\alpha,\beta)} \le c\,\|Rg\|_{p(\alpha,\beta)} \le c\,\|\chi^{\gamma_0}Rg\|_{p(0,\bar\beta)}. \tag{4.18}$$

Now, relations (4.11), (4.18) and (4.16) imply (a), and (4.11) and (4.18) and (4.17) imply (b).

Assertions (c) and (d) follow from (4.11) and the estimates

$$\|\chi^{\gamma_\infty}\psi_{r,n}(0,\cdot;\cdot)\|_{p(\bar\alpha,\infty)} \le c\|\mathrm{w}\chi^{r}g^{(r)}\|_{p(0,\infty)}, \quad n \ge r-i,\ i>0,$$
$$\|\chi^{\gamma_\infty}\psi_{r,n}(a,\cdot;\cdot)\|_{p(\bar\alpha,\infty)} \le c\|\chi^{\gamma_\infty+r}g^{(r)}\|_{p(\bar\alpha,\infty)}, \quad n \ge r-j,$$
$$\|\chi^{\gamma_\infty}\psi_{r,n}(\cdot,\infty;\cdot)\|_{p(\bar\alpha,\infty)} \le c\|\chi^{\gamma_\infty+r}g^{(r)}\|_{p(\bar\alpha,\infty)}, \quad n \le r-j-1,\ j<r,$$
$$\|\chi^{\gamma_\infty}\mathscr{L}_{i,j-1}(Rg)\|_{p(a,\infty)} \le c\,\|\chi^{\gamma_\infty}Rg\|_{p(\bar\alpha,\infty)},$$

which are verified as above.

Finally, assertion (e) follows directly from (a)–(d). □

5. A characterization of $K^r_{\mathrm{w}}(f,t^r)_p$ by the unweighted fixed-step moduli of smoothness

Let $J \subseteq \mathbb{R}$ be an open interval. For $r \in \mathbb{N}$, $F \in L_p(J)$ and $t>0$ we denote the unweighted K-functional by

$$K^r(F,t^r)_{p(J)} = \inf\left\{\|F-G\|_{p(J)} + t^r\|G^{(r)}\|_{p(J)} : G \in AC^{r-1}_{loc}(J)\right\}.$$

For $r=0$ we set $K^0(F,t^0)_{p(J)} = \|F\|_{p(J)}$. As is known (see e.g. [1, Ch. 6, Theorem 2.4])

$$K^r(F,t^r)_{p(J)} \sim \omega_r(F,t)_{p(J)}. \tag{5.1}$$

We shall also need the following characterization of another K-functional, which is a simple modification of the one above.

Lemma 5.1. *For $r \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le t_0$, an open interval $J \subseteq \mathbb{R}$ and $F \in L_p(J)$ there holds*

$$\inf\left\{\|F-G\|_{p(J)} + t^r\|G^{(r)}\|_{p(J)} + t^r\|G'\|_{p(J)} : G \in AC_{loc}^{r-1}(J)\right\}$$
$$\sim \omega_r(F,t)_{p(J)} + t^{r-1}\omega_1(F,t)_{p(J)}.$$

The assertion of this lemma can be established as in [8, Lemma 5.2].

We shall prove the upper and lower estimates of the K-functional $K_{\mathrm{w}}^r(f,t^r)_p$ separately as for each of them it is necessary to distinguish between two main cases: $j \le i$ and $i < j$, where i, j are determined by $\mathscr{T}_i(p) \ni \gamma_0$ and $\mathscr{T}_j(p) \ni \gamma_\infty$. According to Remark 1.8 the trivial class $\pi_{r-1} \cap L_p(\mathrm{w})(0,\infty)$ of the K-functional $K_{\mathrm{w}}^r(f,t^r)_p$ is $\{0\}$ for $j \le i$, whereas for $i < j$ it is $\pi_{i,j-1} \ne \{0\}$.

5.1. Upper estimates

The following theorem establishes the upper estimate of $K_{\mathrm{w}}^r(f,t^r)_p$ by the unweighted K-functionals. Although it is valid for all real γ_0, γ_∞, it will be used in the case $\gamma_0, \gamma_\infty \ne 1-r-1/p, \dots, -1/p$.

Theorem 5.2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $t_0 > 0$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8) with $\gamma_0, \gamma_\infty \in \mathbb{R}$. Then for $f \in L_p(\mathrm{w})(0,\infty)$, $q \in \pi_{r-1} \cap L_p(\mathrm{w})(0,\infty)$, $F = (\chi^{1/p}\mathrm{w}(f-q)) \circ \mathscr{E}$ and $0 < t \le t_0$ there holds*

$$K_{\mathrm{w}}^r(f,t^r)_p \le c\left(K^r(F,t^r)_{p(\mathbb{R})} + t^r\|F\|_{p(\mathbb{R})}\right).$$

Proof. First, let us observe that since $K_{\mathrm{w}}^r(f,t^r)_p = K_{\mathrm{w}}^r(f-q,t^r)_p$, it is enough to establish the theorem with $q = 0$.

In all the proofs in this section we follow a standard K-functional argument: in order to prove the assertion of the theorem, it is enough to show that for every function $G \in AC_{loc}^{r-1}(\mathbb{R})$ such that $G, G^{(r)} \in L_p(\mathbb{R})$ there exists a function $g \in AC_{loc}^{r-1}(0,\infty)$ such that

$$\|\mathrm{w}(f-g)\|_{p(0,\infty)} \le c\,\|F-G\|_{p(\mathbb{R})} \tag{5.2}$$

and

$$\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)} \le c\left(\|G\|_{p(\mathbb{R})} + \|G^{(r)}\|_{p(\mathbb{R})}\right). \tag{5.3}$$

Indeed, from (5.2) and (5.3) we get for every t such that $0 < t \le t_0$ and $G \in AC_{loc}^{r-1}(\mathbb{R})$ such that $G, G^{(r)} \in L_p(\mathbb{R})$ the estimate

$$\begin{aligned} K_{\mathrm{w}}^r(f,t^r)_p &\le \|\mathrm{w}(f-g)\|_{p(0,\infty)} + t^r\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)} \\ &\le c\left(\|F-G\|_{p(\mathbb{R})} + t^r\|G^{(r)}\|_{p(\mathbb{R})} + t^r\|F\|_{p(\mathbb{R})}\right). \end{aligned}$$

Taking the infimum on G in the above inequality we get the assertion of the theorem.

Let $G \in AC_{loc}^{r-1}(\mathbb{R})$ be such that $G, G^{(r)} \in L_p(\mathbb{R})$. We set $g = \chi^{-1/p}\mathrm{w}^{-1}(G \circ \log) = (W^{-1}G) \circ \log$, where $W = (\chi^{1/p}\mathrm{w}) \circ \mathscr{E}$. Then by a change of the variable we see that (5.2) is valid as an equality with $c = 1$.

To prove (5.3) we write

$$\begin{aligned}\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)} &= \|\mathrm{w}\chi^r((W^{-1}G)\circ\log)^{(r)}\|_{p(0,\infty)}\\ &= \left\|\mathrm{w}\sum_{\ell=1}^{r} m_{r,\ell}(W^{-1}G)^{(\ell)}\circ\log\right\|_{p(0,\infty)}\\ &\le c\sum_{\ell=1}^{r}\|W(W^{-1}G)^{(\ell)}\|_{p(\mathbb{R})} \qquad (5.4)\end{aligned}$$

with appropriate integers $m_{r,\ell}$. To estimate $\|W(W^{-1}G)^{(\ell)}\|_{p(\mathbb{R})}$ for $\ell = 1,\dots,r$ we first apply the Leibniz rule and get

$$(W^{-1}(x)G(x))^{(\ell)} = W^{-1}(x)\sum_{k=0}^{\ell}\left[\sum_{n=0}^{\ell-k} b_{\ell,k,n}\left(\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)^n\right]G^{(k)}(x)$$

with some numbers $b_{\ell,k,n} = b_{\ell,k,n}(\gamma_0+1/p,\gamma_\infty+1/p)$. Next we only need to observe that

$$\left|\sum_{n=0}^{\ell-k} b_{\ell,k,n}\left(\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)^n\right| \le c,\quad x\in\mathbb{R},$$

and use (4.2) to get for $\ell = 1,\dots,r$

$$\|W(W^{-1}G)^{(\ell)}\|_{p(\mathbb{R})} \le c\sum_{k=0}^{\ell}\|G^{(k)}\|_{p(\mathbb{R})} \le c\left(\|G\|_{p(\mathbb{R})} + \|G^{(r)}\|_{p(\mathbb{R})}\right). \qquad (5.5)$$

Inequalities (5.4) and (5.5) imply (5.3) and complete the proof. □

To solve the cases when one or both of the γ's belong to $\mathscr{T}_{exc}(p)$, we treat the singularities separately by splitting the interval $(0,\infty)$. For J an interval of the type $(0,a)$ or (a,∞) with $0 < a < \infty$ and $\gamma \in \mathbb{R}$ we set

$$\begin{aligned}K^r_{\chi^\gamma}(f,t^r)_{p(J)} &= K(f,t^r; L_p(\chi^\gamma)(J), AC^{r-1}_{loc}, \chi^r D^r)\\ &= \inf\left\{\|\chi^\gamma(f-g)\|_{p(J)} + t^r\|\chi^{\gamma+r}D^r g\|_{p(J)} : g\in AC^{r-1}_{loc}(J)\right\}.\end{aligned}$$

According to [7, Lemma 7.1] (see also [1, Ch. 6, Lemma 2.3]) for $A > 1$, every $f \in L_p(\mathrm{w})(0,\infty)$ and $0 < t \le t_0$ there holds

$$K^r_{\mathrm{w}}(f,t^r)_p \sim K^r_{\chi^{\gamma_0}}(f,t^r)_{p(0,A)} + K^r_{\chi^{\gamma_\infty}}(f,t^r)_{p(1/A,\infty)}. \qquad (5.6)$$

Theorem 5.3. *Let $r\in\mathbb{N}$, $1\le p\le\infty$, $a, t_0 > 0$, $0 < t \le t_0$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with $\gamma_0,\gamma_\infty\in\mathbb{R}$. For $f\in L_p(\mathrm{w})(0,\infty)$ we set $F_0 = (\chi^{\gamma_0+1/p}(f-q_0))\circ\mathscr{E}$ and $F_\infty = (\chi^{\gamma_\infty+1/p}(f-q_\infty))\circ\mathscr{E}$, where $q_0\in\pi_{r-1}\cap L_p(\chi^{\gamma_0})(0,1)$ and $q_\infty\in\pi_{r-1}\cap L_p(\chi^{\gamma_\infty})(1,\infty)$ are arbitrary. Let $\ell_0 = 1$ if $\gamma_0\in\mathscr{T}_{exc}(p)$, and $\ell_0 = 0$ otherwise. Let $\ell_\infty = 1$ if $\gamma_\infty\in\mathscr{T}_{exc}(p)$, and $\ell_\infty = 0$ otherwise. Then we have*

$$\begin{aligned}K^r_{\mathrm{w}}(f,t^r)_p &\le c\big(K^r(F_0,t^r)_{p(-\infty,a)} + t^{r-\ell_0}K^{\ell_0}(F_0,t^{\ell_0})_{p(-\infty,a)}\\ &\quad + K^r(F_\infty,t^r)_{p(-a,\infty)} + t^{r-\ell_\infty}K^{\ell_\infty}(F_\infty,t^{\ell_\infty})_{p(-a,\infty)}\big).\end{aligned}$$

Proof. Let $A = e^a > 1$. In view of (5.6), $K^r_{\chi\gamma_0}(f, t^r)_{p(0,A)} = K^r_{\chi\gamma_0}(f - q_0, t^r)_{p(0,A)}$ and $K^r_{\chi\gamma_\infty}(f, t^r)_{p(1/A,\infty)} = K^r_{\chi\gamma_\infty}(f - q_\infty, t^r)_{p(1/A,\infty)}$, it is enough to prove the inequalities

$$K^r_{\chi\gamma_0}(f, t^r)_{p(0,A)} \le c\left(K^r(F_0, t^r)_{p(-\infty,a)} + t^{r-\ell_0} K^{\ell_0}(F_0, t^{\ell_0})_{p(-\infty,a)}\right) \tag{5.7}$$

with $F_0 = (\chi^{\gamma_0+1/p} f) \circ \mathscr{E}$ and

$$K^r_{\chi\gamma_\infty}(f, t^r)_{p(1/A,\infty)} \le c\left(K^r(F_\infty, t^r)_{p(-a,\infty)} + t^{r-\ell_\infty} K^{\ell_\infty}(F_\infty, t^{\ell_\infty})_{p(-a,\infty)}\right) \tag{5.8}$$

with $F_\infty = (\chi^{\gamma_\infty+1/p} f) \circ \mathscr{E}$. The proofs of (5.7) and (5.8) are quite similar and we shall give only that of the former.

For every $G \in AC^{r-1}_{loc}(-\infty, a)$ such that $G, G^{(r)} \in L_p(-\infty, a)$ we set $g = \chi^{-\gamma_0-1/p}(G \circ \log)$. Just as in the proof of Theorem 5.2, the inequality (5.7) with $\ell_0 = 0$ follows for an arbitrary real γ_0 from the relations

$$\|\chi^{\gamma_0}(f - g)\|_{p(0,A)} = \|F_0 - G\|_{p(-\infty,a)} \tag{5.9}$$

and

$$\|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,A)} \le c\left(\|G\|_{p(-\infty,a)} + \|G^{(r)}\|_{p(-\infty,a)}\right),$$

which are verified as in the proof of Theorem 5.2.

Let $\gamma_0 = -i - 1/p$, where $i \in \mathbb{N}_0$ and $i < r$. In view of Lemma 5.1 and the equivalence (5.1), relation (5.7) with $\ell_0 = 1$ follows from (5.9) and

$$\|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,A)} \le c\left(\|G'\|_{p(-\infty,a)} + \|G^{(r)}\|_{p(-\infty,a)}\right).$$

To verify the inequality above let us observe that $r - i \ge 1$ and we actually have with appropriate integers $m_{\ell,k}$

$$\begin{aligned}
\|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,A)} &= \|\chi^{-1/p}\chi^{r-i}(\chi^i(G \circ \log))^{(r)}\|_{p(0,A)} \\
&= \left\|\chi^{-1/p} \sum_{\ell=r-i}^{r} \binom{r}{\ell} \frac{i!}{(i+\ell-r)!}\, \chi^{\ell}(G \circ \log)^{(\ell)}\right\|_{p(0,A)} \\
&= \left\|\chi^{-1/p} \sum_{\ell=r-i}^{r} \binom{r}{\ell} \frac{i!}{(i+\ell-r)!} \sum_{k=1}^{\ell} m_{\ell,k}\, G^{(k)} \circ \log\right\|_{p(0,A)} \\
&\le c \sum_{k=1}^{r} \left\|\chi^{-1/p} G^{(k)} \circ \log\right\|_{p(0,A)} \\
&= c \sum_{k=1}^{r} \|G^{(k)}\|_{p(-\infty,a)} \le c\left(\|G'\|_{p(-\infty,a)} + \|G^{(r)}\|_{p(-\infty,a)}\right),
\end{aligned}$$

where at the last step we have applied (4.2). □

Remark 5.4. Let us note that actually (5.7) and (5.8) hold with $\ell_0 = 0$ for any $\gamma_0 \in \mathbb{R}$ and/or $\ell_\infty = 0$ for any $\gamma_\infty \in \mathbb{R}$. In particular, for any $\gamma_0, \gamma_\infty \in \mathbb{R}$ we have

$$\begin{aligned}
K^r_{w}(f, t^r)_p \le\ & c\left(K^r(F_0, t^r)_{p(-\infty,a)} + t^r \|F_0\|_{p(-\infty,a)}\right. \\
&\left. + K^r(F_\infty, t^r)_{p(-a,\infty)} + t^r \|F_\infty\|_{p(-a,\infty)}\right).
\end{aligned}$$

5.2. Lower estimates

In the proof of the lower estimates of $K^r_{\mathrm{w}}(f,t^r)_p$ by unweighted K-functionals, we shall use the following assertion, which is verified directly.

Proposition 5.5. *Let the linear operator $\mathscr{L}_{i,j-1}$ be defined by (1.10) and satisfy condition (i) and let $\pi_{i,j-1} \subset L_p(\mathrm{w})(0,\infty)$. Then $\mathscr{L}_{i,j-1} : L_p(\mathrm{w})(0,\infty) \to L_p(\mathrm{w})(0,\infty)$ is bounded.*

We also need a combinatorial identity, which follows from Vandermonde's convolution formula (see [10, Ch. 1, (5c)]). For the sake of completeness we give its short proof.

Lemma 5.6. *Let $n, m \in \mathbb{N}$. Then*

$$\sum_{k=0}^{\min\{n,m\}} (-1)^{m-k} \binom{m}{k} \frac{(n+m-k-1)!}{(n-k)!} = 0.$$

Proof. The identity follows from

$$0 \equiv \left(x^{-n} \cdot x^n\right)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} \left(x^{-n}\right)^{(m-k)} \left(x^n\right)^{(k)}$$

$$= \sum_{k=0}^{\min\{n,m\}} \binom{m}{k} \left[(-1)^{m-k} \frac{(n+m-k-1)!}{(n-1)!} x^{k-n-m}\right] \left[\frac{n!}{(n-k)!} x^{n-k}\right]$$

$$= n\, x^{-m} \sum_{k=0}^{\min\{n,m\}} (-1)^{m-k} \binom{m}{k} \frac{(n+m-k-1)!}{(n-k)!}. \quad \square$$

First, we shall prove the lower estimate of $K^r_{\mathrm{w}}(f,t^r)_p$ by means of unweighted K-functionals for $\gamma_0, \gamma_\infty \notin \mathscr{T}_{exc}(p)$.

Theorem 5.7. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $t_0 > 0$, $\mathrm{w}(x) = \mathrm{w}(x;\gamma_0,\gamma_\infty)$ be defined in (1.8) with $\gamma_0 \in \mathscr{T}_i(p)$, $\gamma_\infty \in \mathscr{T}_j(p)$. For $f \in L_p(\mathrm{w})(0,\infty)$ we set*

$$F = (\chi^{1/p} \mathrm{w}(f - \mathscr{L}_{i,j-1} f)) \circ \mathscr{E},$$

where $\mathscr{L}_{i,j-1}$ is given by (1.10) and satisfies conditions (i) and (ii). Then for $\ell = 0, 1, \ldots, r$ and $0 < t \le t_0$ there holds

$$t^{r-\ell} K^\ell(F, t^\ell)_{p(\mathbb{R})} \le c\, K^r_{\mathrm{w}}(f,t^r)_p.$$

Proof. We follow the standard K-functional argument used in the proof of Theorem 5.2. Let $g \in AC^{r-1}_{loc}(0,\infty)$ and $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0,\infty)$. We set $G = (\chi^{1/p}\mathrm{w}(g - \mathscr{L}_{i,j-1} g)) \circ \mathscr{E}$.

Let $j \le i$. Then $\mathscr{L}_{i,j-1} = 0$ by definition. First, just by a change of the variable we get

$$\|F - G\|_{p(\mathbb{R})} = \|\mathrm{w}(f-g)\|_{p(0,\infty)}. \tag{5.10}$$

Next, for $\ell = 1, 2, \ldots, r$ we have with some integers $n_{\ell,k}$

$$\begin{aligned}
\|G^{(\ell)}\|_{p(\mathbb{R})} &= \|((\chi^{1/p}\mathrm{w}g) \circ \mathscr{E})^{(\ell)}\|_{p(\mathbb{R})} = \left\| \sum_{k=1}^{\ell} n_{\ell,k} \mathscr{E}^k \left((\chi^{1/p}\mathrm{w}g)^{(k)} \circ \mathscr{E} \right) \right\|_{p(\mathbb{R})} \\
&\le c \sum_{k=1}^{\ell} \|\chi^{k-1/p}(\chi^{1/p}\mathrm{w}g)^{(k)}\|_{p(0,\infty)} \\
&\le c \left(\|\mathrm{w}g\|_{p(0,\infty)} + \|\chi^{r-1/p}(\chi^{1/p}\mathrm{w}g)^{(r)}\|_{p(0,\infty)} \right), \qquad (5.11)
\end{aligned}$$

where at the last step we have used Proposition 4.1 with $J = (0, \infty)$. Inequality (5.11) is also true for $\ell = 0$ in view of (5.10) with $f = F = 0$. To estimate the term $\|\chi^{r-1/p}(\chi^{1/p}\mathrm{w}g)^{(r)}\|_{p(0,\infty)}$ we apply the Leibniz rule to get

$$x^{r-1/p}(x^{1/p}\mathrm{w}(x)g(x))^{(r)} = \mathrm{w}(x) \sum_{k=0}^{r} \left[\sum_{n=0}^{r-k} d_{k,n} \left(\frac{x}{1+x} \right)^{r-k-n} \right] x^k g^{(k)}(x)$$

with some numbers $d_{k,n} = d_{k,n}(\gamma_0 + 1/p, \gamma_\infty + 1/p)$. Next, since

$$\left| \sum_{n=0}^{r-k} d_{k,n} \left(\frac{x}{1+x} \right)^{r-k-n} \right| \le c, \quad x \ge 0,$$

we get by means of Proposition 4.1 with $J = (0, \infty)$

$$\begin{aligned}
\|\chi^{r-1/p}(\chi^{1/p}\mathrm{w}g)^{(r)}\|_{p(0,\infty)} &\le c \sum_{k=0}^{r} \|\mathrm{w}\chi^k g^{(k)}\|_{p(0,\infty)} \\
&\le c\,(\|\mathrm{w}g\|_{p(0,\infty)} + \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}). \qquad (5.12)
\end{aligned}$$

Theorem 4.3 implies

$$\|\mathrm{w}g\|_{p(0,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)},$$

which together with (5.11) and (5.12) gives the inequalities

$$\|G^{(\ell)}\|_{p(\mathbb{R})} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0, 1, \ldots, r. \qquad (5.13)$$

Finally, (5.10) and (5.13) imply for $\ell = 0, 1, \ldots, r$ and $0 < t \le t_0$

$$\begin{aligned}
t^{r-\ell} K^r(F, t^\ell)_{p(\mathbb{R})} &\le t^{r-\ell}\|F - G\|_{p(\mathbb{R})} + t^r \|G^{(\ell)}\|_{p(\mathbb{R})} \\
&\le c \left(\|\mathrm{w}(f-g)\|_{p(0,\infty)} + t^r \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)} \right),
\end{aligned}$$

which proves the theorem in the case $j \le i$ by taking the infimum over g.

To establish the assertion for $i < j$, we, first, observe that Proposition 5.5 implies

$$\begin{aligned}
\|F - G\|_{p(\mathbb{R})} &\le \|\mathrm{w}(f-g)\|_{p(0,\infty)} + \|\mathrm{w}\mathscr{L}_{i,j-1}(f-g)\|_{p(0,\infty)} \\
&\le c\,\|\mathrm{w}(f-g)\|_{p(0,\infty)}.
\end{aligned}$$

Next, we establish the estimates

$$\|G^{(\ell)}\|_{p(\mathbb{R})} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0, 1, \ldots, r,$$

just in the same way as in the proof of the first part as we replace g with $g - \mathscr{L}_{i,j-1}g$ and use Proposition 4.8(e) instead of Theorem 4.3. □

To treat the cases when one or both of the γ's belong to the set $\mathscr{T}_{exc}(p)$, we shall prove several lower estimates, which correspond to the terms in the upper estimate of Theorem 5.3.

Theorem 5.8. *Let $r \in \mathbb{N}$, $i, j \in \mathbb{N}_0$, $i, j \le r$, $1 \le p \le \infty$, $a, t_0 > 0$, $0 < t \le t_0$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8) with $\gamma_0, \gamma_\infty \in \mathbb{R}$. For $f \in L_p(\mathrm{w})(0,\infty)$ we set $F_0 = (\chi^{\gamma_0+1/p} f) \circ \mathscr{E}$ and $F_\infty = (\chi^{\gamma_\infty+1/p} f) \circ \mathscr{E}$. We have:*

(a) *For $\gamma_0 \in \mathscr{T}_i(p)$ and either $\gamma_\infty > -i - 1/p, i < r$ or $\gamma_\infty \in \mathbb{R}, i = r$ there holds*

$$t^{r-\ell} K^\ell(F_0, t^\ell)_{p(-\infty,a)} \le c\, K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 0, 1, \dots, r;$$

(b) *For $\gamma_0 = 1 - i - 1/p, i > 0$ and either $\gamma_\infty > -i - 1/p, i < r$ or $\gamma_\infty \in \mathbb{R}, i = r$ there holds*

$$t^{r-\ell} K^\ell(F_0, t^\ell)_{p(-\infty,a)} \le c\, K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 1, \dots, r;$$

(c) *For $\gamma_\infty \in \mathscr{T}_j(p)$ and either $\gamma_0 \in \mathbb{R}, j = 0$ or $\gamma_0 < 1 - j - 1/p, j > 0$ there holds*

$$t^{r-\ell} K^\ell(F_\infty, t^\ell)_{p(-a,\infty)} \le c\, K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 0, 1, \dots, r;$$

(d) *For $\gamma_\infty = -j - 1/p, j < r$ and either $\gamma_0 \in \mathbb{R}, j = 0$ or $\gamma_0 < 1 - j - 1/p, j > 0$ there holds*

$$t^{r-\ell} K^\ell(F_\infty, t^\ell)_{p(-a,\infty)} \le c\, K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 1, \dots, r.$$

Proof. We follow the method used in the proof of the previous theorem. For the proof of (a) and (b) we set $G = (\chi^{\gamma_0+1/p} g) \circ \mathscr{E}$, where $g \in AC^{r-1}_{loc}(0,\infty)$ is such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0,\infty)$.

First, by a change of the variable we get

$$\|F_0 - G\|_{p(-\infty,a)} \le c\, \|\mathrm{w}(f-g)\|_{p(0,\infty)}. \tag{5.14}$$

Assertion (a) follows from (5.14) and

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0, 1, \dots, r. \tag{5.15}$$

To prove (5.15), we first get, as in the proof of (5.11)–(5.12),

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\, (\|\chi^{\gamma_0} g\|_{p(0,A)} + \|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,A)}), \quad \ell = 0, 1, \dots, r, \tag{5.16}$$

where $A = \mathrm{e}^a$. The inequality

$$\|\chi^{\gamma_0} g\|_{p(0,A)} \le c\, \|\chi^{\gamma_0+i} g^{(i)}\|_{p(0,A)} \tag{5.17}$$

is trivial for $i = 0$ and follows for $i = 1, \dots, r$ from Proposition 4.2(a) with $k = 0, r = i$ because in this case $\gamma_0 < 1 - i - 1/p$. Consequently, if $i = r$, (5.16) and (5.17) imply (5.15) for $\gamma_0 < 1 - r - 1/p$ and any real γ_∞.

If $i < r$, we use Proposition 4.2(d) with $k = j = i$ to get for $\gamma_0, \gamma_\infty > -i - 1/p$

$$\|\chi^{\gamma_0+i} g^{(i)}\|_{p(0,A)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)},$$

which together with (5.17) yields

$$\|\chi^{\gamma_0} g\|_{p(0,A)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}$$

and hence by (5.16) we get (5.15) for $i < r$ as well. Thus the proof of assertion (a) is completed.

Assertion (b) follows from (5.14) and

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 1, \dots, r. \tag{5.18}$$

To establish the above inequalities, we get for $\ell = 1, 2, \ldots, r$ as in the proof of (5.11)

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\left(\|\chi^{1-1/p}(\chi^{1-i}g)'\|_{p(0,A)} + \|\chi^{r-1/p}(\chi^{1-i}g)^{(r)}\|_{p(0,A)}\right). \tag{5.19}$$

Note that if $r = 1$, then $i = 1$ and the last inequality implies directly (5.18) for $\ell = r = 1$ and any real γ_∞. So let us assume that $r > 1$.

If $i < r$, then $\gamma_0, \gamma_\infty > -i - 1/p$ and Proposition 4.2(d) with $j = i$ implies

$$\|\chi^{\gamma_0+k}g^{(k)}\|_{p(0,A)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = i, \ldots, r-1. \tag{5.20}$$

Hence we get (5.18) for $i = 1$. For $i > 1$ the Leibniz rule gives for $m = 1, \ldots, r$

$$\begin{aligned} &x^{m-1/p}(x^{1-i}g(x))^{(m)} \\ &= \frac{x^{1-i-1/p}}{(i-2)!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(i+m-k-2)!\,x^k g^{(k)}(x). \end{aligned} \tag{5.21}$$

In view of (5.19)–(5.21) to establish (5.18) with $\gamma_0 = 1 - i - 1/p$, it is enough to prove the inequality

$$\begin{aligned} &\left\|\chi^{1-i-1/p}\sum_{k=0}^{\min\{i-1,m\}}(-1)^{m-k}\binom{m}{k}(i+m-k-2)!\,\chi^k g^{(k)}\right\|_{p(0,A)} \\ &\le c\,\|\chi^{1-1/p}g^{(i)}\|_{p(0,A)} \end{aligned} \tag{5.22}$$

for $m = 1, \ldots, r$. To accomplish this we apply Theorem 3.4 with $\mu = i - 1 > 0$ and $r = \nu = i$ to get for $k = 0, \ldots, i-1$ the representation

$$\begin{aligned} x^k g^{(k)}(x) &= g^{(i-1)}(a)\frac{x^{i-1}}{(i-k-1)!} + \frac{x^{i-1}}{(i-k-1)!}\int_a^x g^{(i)}(y)\,\mathrm{d}y \\ &\quad + \sum_{n=1}^{i-k-1}\frac{(-1)^n x^{i-n-1}}{n!(i-k-n-1)!}\int_0^x y^n g^{(i)}(y)\,\mathrm{d}y \\ &= g^{(i-1)}(x)\frac{x^{i-1}}{(i-k-1)!} + \sum_{n=1}^{i-k-1}\frac{(-1)^n x^{i-n-1}}{n!(i-k-n-1)!}\int_0^x y^n g^{(i)}(y)\,\mathrm{d}y. \end{aligned}$$

Now, taking into consideration Lemma 5.6 with $n = i - 1$, we get for $m = 1, \ldots, r$

$$\begin{aligned} &x^{1-i-1/p}\sum_{k=0}^{\min\{i-1,m\}}(-1)^{m-k}\binom{m}{k}(i+m-k-2)!\,x^k g^{(k)}(x) \\ &= \sum_{n=1}^{i-1}\rho_{i-1,m,n}\,x^{-n-1/p}\int_0^x y^n g^{(i)}(y)\,\mathrm{d}y, \end{aligned}$$

where

$$\begin{aligned} \rho_{i,m,n} &= \sum_{k=0}^{\min\{i-n,m\}}(-1)^{m+n-k}\binom{m}{k}\frac{(i+m-k-1)!}{n!(i-k-n)!} \\ &= (-1)^{m+n}\binom{i-1}{n-1}\frac{(m+n-1)!}{n!} \end{aligned} \tag{5.23}$$

as the last equality follows from [10, Ch. 1, (5a)]. Finally, Hardy's inequality implies (5.22).

For $i = r$, (5.18) follows from (5.19), (5.21) and (5.22), and, consequently, no restrictions are imposed on γ_∞. Thus the proof of (b) is completed.

For the proof of (c) and (d) we set $G = (\chi^{\gamma_\infty+1/p} g) \circ \mathscr{E}$, where $g \in AC_{loc}^{r-1}(0,\infty)$ is such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0,\infty)$.

Just as above we get

$$\|F_\infty - G\|_{p(-a,\infty)} \le c\,\|\mathrm{w}(f-g)\|_{p(0,\infty)}. \tag{5.24}$$

Assertion (c) follows from (5.24) and the inequalities

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0,1,\ldots,r.$$

They are verified as in the proof of (a) as the estimate

$$\|\chi^{\gamma_\infty} g\|_{p(1/A,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}$$

follows in the case $j = 0$ from Proposition 4.2(c) with $k = j = 0$ and hence no restrictions on γ_0 are imposed, and in the case $j > 0$ from Proposition 4.2(b) with $k = 0$.

Assertion (d) follows from (5.24) and

$$\|G^{(\ell)}\|_{p(-a,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 1,\ldots,r. \tag{5.25}$$

To prove the last inequalities we get as in the proof of (5.11)

$$\|G^{(\ell)}\|_{p(-a,\infty)} \le c\left(\|\chi^{1-1/p}(\chi^{-j}g)'\|_{p(1/A,\infty)} + \|\chi^{r-1/p}(\chi^{-j}g)^{(r)}\|_{p(1/A,\infty)}\right). \tag{5.26}$$

If $r = 1$, then $j = 0$ and (5.26) directly implies (5.25) for $\ell = r = 1$ and any $\gamma_0 \in \mathbb{R}$.

Let $r > 1$. The inequality

$$\|\chi^{\gamma_\infty+k} g^{(k)}\|_{p(1/A,\infty)} \le c\,\|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(1/A,\infty)}, \quad k = j+1,\ldots,r. \tag{5.27}$$

is trivial for $k = r$ and for $k = j+1,\ldots,r-1$ (and hence $j < r-1$) follows from Proposition 4.2(c) with $j+1$ instead of j since $\gamma_\infty > -j-1-1/p$. From (5.26) and (5.27) we get (5.25) for $j = 0$ and any real γ_0. For $j > 0$ by the Leibniz rule we have for $m = 1,\ldots,r$

$$x^{m-1/p}(x^{-j}g(x))^{(m)} = \frac{x^{-j-1/p}}{(j-1)!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(j+m-k-1)!\,x^k g^{(k)}(x). \tag{5.28}$$

Now, in view of (5.26)–(5.28) to establish (5.25) it is enough to prove

$$\left\|\chi^{-j-1/p}\sum_{k=0}^{\min\{j,m\}}(-1)^{m-k}\binom{m}{k}(j+m-k-1)!\,\chi^k g^{(k)}\right\|_{p(1/A,\infty)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad m = 1,\ldots,r, \tag{5.29}$$

with $1 \le j < r$. To do this we apply Theorem 3.4 with $a = 1/A$, $\mu = j$ and $\nu = j+1 \le r$ to get for $k = 0,\ldots,j$ the representation

$$x^k g^{(k)}(x) = (Qg)(x)\,\frac{x^j}{(j-k)!} + (\widehat{R}_k g)(x),$$

where

$$(Qg)(x) = \sum_{\ell=j}^{r-1}\frac{(-A)^{j-\ell}}{(\ell-j)!}\,g^{(\ell)}(1/A) + \frac{(-1)^{r-j-1}}{(r-j-1)!}\int_{1/A}^{x} y^{r-j-1}g^{(r)}(y)\,\mathrm{d}y \tag{5.30}$$

and

$$(\widehat{R}_k g)(x) = \sum_{n=r-j}^{r-k-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\,dy + \sum_{n=0}^{r-j-2} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,dy.$$

Using the representations above and taking into account Lemma 5.6 with $n = j$, we get

$$x^{-j-1/p} \sum_{k=0}^{\min\{j,m\}} (-1)^{m-k} \binom{m}{k} (j+m-k-1)!\, x^k g^{(k)}(x) = \sum_{n=r-j}^{r-1} \rho'_{j,m,n} x^{r-j-n-1-1/p} \int_0^x y^n g^{(r)}(y)\,dy + \sum_{n=0}^{r-j-2} \rho''_{j,m,n} x^{r-j-n-1-1/p} \int_x^\infty y^n g^{(r)}(y)\,dy, \tag{5.31}$$

where

$$\rho'_{j,m,n} = \sum_{k=0}^{\min\{r-n-1,m\}} (-1)^{m+n-k} \binom{m}{k} \frac{(j+m-k-1)!}{n!(r-k-n-1)!} = (-1)^{m+n} \binom{j+m+n-r}{n} \binom{j-1}{r-n-1}, \tag{5.32}$$

$$\rho''_{j,m,n} = \sum_{k=0}^{\min\{j,m\}} (-1)^{m+n-k+1} \binom{m}{k} \frac{(j+m-k-1)!}{n!(r-k-n-1)!}, \tag{5.33}$$

as to calculate $\rho'_{j,m,n}$ we again used [10, Ch. 1, (5a)].

Since $\chi^{r-j-n-1-1/p} \in L_p(1/A, \infty)$ for $n \geq r-j$ and also $\gamma_0 < 1 - j - 1/p$, we get by Hölder's inequality

$$\|\chi^{\gamma_\infty} \psi_{r,n}(0, 1/A; \cdot)\|_{p(1/A,\infty)} \leq c\, \|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,1/A)}, \quad n \geq r-j, \tag{5.34}$$

where $\psi_{r,n}$ is defined in (4.4).

By Hardy's inequalities we get

$$\|\chi^{\gamma_\infty} \psi_{r,n}(1/A, \cdot; \cdot)\|_{p(1/A,\infty)} \leq c\, \|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(1/A,\infty)}, \quad n \geq r-j, \tag{5.35}$$

and

$$\|\chi^{\gamma_\infty} \psi_{r,n}(\cdot, \infty; \cdot)\|_{p(1/A,\infty)} \leq c\, \|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(1/A,\infty)}, \quad n \leq r-j-2. \tag{5.36}$$

Inequalities (5.34) and (5.35) imply

$$\|\chi^{\gamma_\infty} \psi_{r,n}(0, \cdot; \cdot)\|_{p(1/A,\infty)} \leq c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad n \geq r-j. \tag{5.37}$$

Finally, (5.31), (5.37) and (5.36) imply (5.29). This completes the proof of assertion (d). □

Remark 5.9. If $\chi^{-j-1}g, \chi^{r-j-1}g^{(r)} \in L_1(1,\infty)$, then we have by Lemma 3.3(b) and (3.9)

$$(Qg)(x) = \frac{(-1)^{r-j}}{(r-j-1)!}\int_x^\infty y^{r-j-1}g^{(r)}(y)\,\mathrm{d}y$$

for Qg given in (5.30). The above condition does not follow from the hypotheses of Theorem 5.8(d) when $p > 1$.

Theorem 5.10. *Let $r, j \in \mathbb{N}$, $i \in \mathbb{N}_0$, $i < j \le r$, $1 \le p \le \infty$, $a, t_0 > 0$, $0 < t \le t_0$ and $\mathrm{w}(x) = \mathrm{w}(\gamma_0, \gamma_\infty; x)$ be defined in (1.8) with $\gamma_0, \gamma_\infty \in \mathbb{R}$. For $f \in L_p(\mathrm{w})(0,\infty)$ we set*

$$F_0 = (\chi^{\gamma_0+1/p}(f - \mathscr{L}_{i,j-1}f)) \circ \mathscr{E} \quad \text{and} \quad F_\infty = (\chi^{\gamma_\infty+1/p}(f - \mathscr{L}_{i,j-1}f)) \circ \mathscr{E},$$

where $\mathscr{L}_{i,j-1}$ is given by (1.10).

(a) *Let $\mathscr{L}_{i,j-1}$ satisfy conditions (i) and (ii). Then for $\gamma_0 \in \mathscr{T}_i(p)$ and either $\gamma_\infty > -j-1/p$, $j < r$ or $\gamma_\infty \in \mathbb{R}$, $j = r$ there holds*

$$t^{r-\ell}K^\ell(F_0, t^\ell)_{p(-\infty,a)} \le c\,K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 0, 1, \ldots, r.$$

(b) *Let $\mathscr{L}_{i,j-1}$ satisfy conditions (i)–(iii). Then for $\gamma_0 = 1 - i - 1/p$, $i > 0$, and either $\gamma_\infty > -j-1/p$, $j < r$ or $\gamma_\infty \in \mathbb{R}$, $j = r$ there holds*

$$t^{r-\ell}K^\ell(F_0, t^\ell)_{p(-\infty,a)} \le c\,K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 1, \ldots, r.$$

(c) *Let $\mathscr{L}_{i,j-1}$ satisfy conditions (i) and (ii). Then for $\gamma_\infty \in \mathscr{T}_j(p)$ and either $\gamma_0 \in \mathbb{R}$, $i = 0$ or $\gamma_0 < 1 - i - 1/p$, $i > 0$ there holds*

$$t^{r-\ell}K^\ell(F_\infty, t^\ell)_{p(-a,\infty)} \le c\,K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 0, 1, \ldots, r.$$

(d) *Let $\mathscr{L}_{i,j-1}$ satisfy conditions (i), (ii) and (iv). Then for $\gamma_\infty = -j - 1/p$, $j < r$, and either $\gamma_0 \in \mathbb{R}$, $i = 0$ or $\gamma_0 < 1 - i - 1/p$, $i > 0$ there holds*

$$t^{r-\ell}K^\ell(F_\infty, t^\ell)_{p(-a,\infty)} \le c\,K^r_{\mathrm{w}}(f, t^r)_p, \quad \ell = 1, \ldots, r.$$

Proof. Let $g \in AC^{r-1}_{loc}(0,\infty)$ be such that $g, \chi^r g^{(r)} \in L_p(\mathrm{w})(0,\infty)$. We set $\tilde{g} = g - \mathscr{L}_{i,j-1}g$. Let us note that $\tilde{g}^{(k)} = g^{(k)}$ for $k \ge j$ and $\tilde{g}, \chi^r\tilde{g}^{(r)} \in L_p(\mathrm{w})(0,\infty)$.

For the proof of assertions (a) and (b) we set $G = (\chi^{\gamma_0+1/p}\tilde{g}) \circ \mathscr{E}$. First, by a change of the variable and Proposition 5.5 we get

$$\|F_0 - G\|_{p(-\infty,a)} \le c\,\|\mathrm{w}(f-g)\|_{p(0,\infty)}. \tag{5.38}$$

Assertion (a) follows from (5.38) and

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0, 1, \ldots, r. \tag{5.39}$$

By (5.16) we have

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\,(\|\chi^{\gamma_0}\tilde{g}\|_{p(0,A)} + \|\chi^{\gamma_0+r}g^{(r)}\|_{p(0,A)}), \quad \ell = 0, 1, \ldots, r, \tag{5.40}$$

where $A = \mathrm{e}^a$. Next, respectively by Proposition 4.8(a) and Proposition 4.8(b) with $k = 0$ we have

$$\|\chi^{\gamma_0}\tilde{g}\|_{p(0,A)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad j < r, \tag{5.41}$$

$$\|\chi^{\gamma_0}\tilde{g}\|_{p(0,A)} \le c\,\|\chi^{\gamma_0+r}g^{(r)}\|_{p(0,\bar{\beta})}, \quad j = r, \tag{5.42}$$

where $\bar{\beta} = \max\{A, \beta\}$. Now, (5.40)–(5.42) imply (5.39). Note that for $j = r$, (5.39) follows from (5.40) and (5.42) and hence no restrictions are imposed on γ_∞.

Assertion (b) follows from (5.38) and

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\,\|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 1, \dots, r, \tag{5.43}$$

which are verified just similarly as (5.18). Indeed, by (5.19) we have

$$\|G^{(\ell)}\|_{p(-\infty,a)} \le c\left(\|\chi^{1-1/p}(\chi^{1-i}\tilde{g})'\|_{p(0,A)} + \|\chi^{r-1/p}(\chi^{1-i}\tilde{g})^{(r)}\|_{p(0,A)}\right), \tag{5.44}$$

where $A = \mathrm{e}^a$. By (5.21) there holds for $m = 1, \dots, r$

$$x^{m-1/p}(x^{1-i}\tilde{g}(x))^{(m)} = \frac{x^{1-i-1/p}}{(i-2)!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(i+m-k-2)!\,x^k\tilde{g}^{(k)}(x). \tag{5.45}$$

Next, by (5.22) we have the estimate

$$\left\|\chi^{1-i-1/p}\sum_{k=0}^{\min\{i-1,m\}}(-1)^{m-k}\binom{m}{k}(i+m-k-2)!\,\chi^k\tilde{g}^{(k)}\right\|_{p(0,A)} \le c\,\|\chi^{1-1/p}\tilde{g}^{(i)}\|_{p(0,A)}, \quad m = 1, \dots, r. \tag{5.46}$$

Further, by Theorem 3.4 with $\mu = i-1$ and $r = \nu = j$ we have for $k = 0, \dots, j-1$

$$\begin{aligned} x^k g^{(k)}(x) = &\sum_{n=\max\{i-1,k\}}^{j-1}\frac{x^n}{(n-k)!}b_{j,n}(g, A) \\ &+ \sum_{n=j-i+1}^{j-k-1}\frac{(-1)^n x^{j-n-1}}{n!(j-k-n-1)!}\int_0^x y^n g^{(j)}(y)\,\mathrm{d}y \\ &+ \sum_{n=0}^{j-\max\{i-1,k\}-1}\frac{(-1)^n x^{j-n-1}}{n!(j-k-n-1)!}\int_A^x y^n g^{(j)}(y)\,\mathrm{d}y. \end{aligned} \tag{5.47}$$

Using this formula with $k = 0$ and properties (ii) and (iii) of $\mathscr{L}_{i,j-1}$ we get

$$(\mathscr{L}_{i,j-1}g)(x) = \sum_{n=i}^{j-1}\frac{x^n}{n!}b_{j,n}(g, A) + (\mathscr{L}_{i,j-1}\bar{R}g)(x),$$

where we have set

$$\begin{aligned} (\bar{R}g)(x) = &\sum_{n=j-i+1}^{j-1}\frac{(-1)^n x^{j-n-1}}{n!(j-n-1)!}\int_0^x y^n g^{(j)}(y)\,\mathrm{d}y \\ &+ \sum_{n=0}^{j-i}\frac{(-1)^n x^{j-n-1}}{n!(j-n-1)!}\int_A^x y^n g^{(j)}(y)\,\mathrm{d}y. \end{aligned}$$

Hence for $k = i, \dots, j-1$ we have

$$x^k(\mathscr{L}_{i,j-1}g)^{(k)}(x) = \sum_{n=k}^{j-1}\frac{x^n}{(n-k)!}b_{j,n}(g, A) + \sum_{n=k}^{j-1}a_n(\bar{R}g)\frac{n!\,x^n}{(n-k)!}.$$

From (5.47) and the last relation we get for $k = i, \dots, j-1$

$$x^k \tilde{g}^{(k)}(x) = \sum_{n=0}^{j-k-1} \frac{(-1)^n x^{j-n-1}}{n!(j-k-n-1)!} \int_A^x y^n g^{(j)}(y)\, \mathrm{d}y - \sum_{n=k}^{j-1} a_n(\bar{R}g) \frac{n!\, x^n}{(n-k)!}. \tag{5.48}$$

Hardy's inequality implies for $n \le j-i-1$

$$\|\chi^{\gamma_0} \psi_{j,n}(A, \cdot; \cdot)\|_{p(0,A)} \le c\, \|\chi^{\gamma_0+j} g^{(j)}\|_{p(0,A)}, \tag{5.49}$$

where $\psi_{j,n}$ is defined in (4.4). By property (i) of $\mathscr{L}_{i,j-1}$ and Hölder's inequality we have

$$|a_n(\bar{R}g)| \le c\, \|\bar{R}g\|_{1(\alpha,\beta)} \le c\, \|\bar{R}g\|_{\infty(\alpha,\beta)} \le c\, \|\chi^{\gamma_0+j} g^{(j)}\|_{p(0,\bar{\beta})}, \tag{5.50}$$

where $\bar{\beta} = \max\{A, \beta\}$.

Relations (5.48)–(5.50) imply

$$\|\chi^{\gamma_0+k} \tilde{g}^{(k)}\|_{p(0,A)} \le c\, \|\chi^{\gamma_0+j} g^{(j)}\|_{p(0,\bar{\beta})}, \quad k = i, \dots, j-1. \tag{5.51}$$

Now, if $j = r$, (5.44)–(5.46) and (5.51) imply (5.43) for any γ_∞. For $j < r$ we have γ_0, $\gamma_\infty > -j - 1/p$ and hence Proposition 4.2(d) with $k = j$ gives

$$\|\chi^{\gamma_0+k} g^{(k)}\|_{p(0,\bar{\beta})} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad k = j, \dots, r-1. \tag{5.52}$$

Relations (5.44)–(5.46), (5.51) and (5.52) imply (5.43) for $j < r$.

For the proof of assertions (c) and (d) we set $G = (\chi^{\gamma_\infty+1/p} \tilde{g}) \circ \mathscr{E}$. Just as above we get

$$\|F_\infty - G\|_{p(-a,\infty)} \le c\, \|\mathrm{w}(f-g)\|_{p(0,\infty)}. \tag{5.53}$$

Assertion (c) follows from (5.53) and

$$\|G^{(\ell)}\|_{p(-a,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 0, 1, \dots, r.$$

They are verified as in the proof of assertion (a) as the estimates

$$\|\chi^{\gamma_\infty} \tilde{g}\|_{p(1/A,\infty)} \le c\, \|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(\bar{\alpha},\infty)}, \quad i = 0,$$
$$\|\chi^{\gamma_\infty} \tilde{g}\|_{p(1/A,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad i > 0,$$

where $\bar{\alpha} = \min\{1/A, \alpha\}$, follow respectively from Proposition 4.8(c) and Proposition 4.8(d) with $k = 0$. Note that in the case $i = 0$ no restrictions are imposed on γ_0.

Assertion (d) follows from (5.53) and

$$\|G^{(\ell)}\|_{p(-a,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad \ell = 1, \dots, r.$$

Following the proof of Theorem 5.8(d) with $\tilde{g}$ instead of g in (5.26)–(5.28) we see that for the validity of the above inequalities it is enough to prove the estimates

$$\left\| \chi^{-j-1/p} \sum_{k=0}^{\min\{j,m\}} (-1)^{m-k} \binom{m}{k} (j+m-k-1)!\, \chi^k \tilde{g}^{(k)} \right\|_{p(1/A,\infty)}$$
$$\le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad m = 1, \dots, r, \tag{5.54}$$

with $1 \le j < r$.

By Theorem 3.4 with $\mu = i$ and $\nu = j + 1$ we have for $k = 0, \ldots, j$

$$\begin{aligned} x^k g^{(k)}(x) = & \sum_{n=\max\{i,k\}}^{j} \frac{x^n}{(n-k)!} b_{r,n}(g, 1/A) \\ & + \sum_{n=r-i}^{r-k-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=r-j-1}^{r-\max\{i,k\}-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_{1/A}^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=0}^{r-j-2} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y. \end{aligned}$$

By means of this formula for $k = 0$ and properties (ii) and (iv) of $\mathscr{L}_{i,j-1}$, we get

$$(\mathscr{L}_{i,j-1} g)(x) = \sum_{n=i}^{j-1} \frac{x^n}{n!} b_{r,n}(g, 1/A) + (\mathscr{L}_{i,j-1} \check{R} g)(x),$$

where we have set

$$\begin{aligned} (\check{R} g)(x) = & \sum_{n=r-i}^{r-1} \frac{(-1)^n x^{r-n-1}}{n!(r-n-1)!} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=r-j-1}^{r-i-1} \frac{(-1)^n x^{r-n-1}}{n!(r-n-1)!} \int_{1/A}^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=0}^{r-j-2} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y. \end{aligned}$$

Hence for $k = 0, \ldots, j$ there holds

$$x^k (\mathscr{L}_{i,j-1} g)^{(k)}(x) = \sum_{n=\max\{i,k\}}^{j-1} \frac{x^n}{(n-k)!} b_{r,n}(g, 1/A) + \sum_{n=\max\{i,k\}}^{j-1} a_n(\check{R} g) \frac{n!\, x^n}{(n-k)!}.$$

Consequently, we have for $k = 0, \ldots, j$

$$x^k \tilde{g}^{(k)}(x) = (Qg)(x) \frac{x^j}{(j-k)!} + (\check{R}_k g)(x) - \sum_{n=\max\{i,k\}}^{j-1} a_n(\check{R} g) \frac{n!\, x^n}{(n-k)!},$$

where Qg is defined in (5.30) and

$$\begin{aligned} (\check{R}_k g)(x) = & \sum_{n=r-i}^{r-k-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=r-j}^{r-\max\{i,k\}-1} \frac{(-1)^n x^{r-n-1}}{n!(r-k-n-1)!} \int_{1/A}^x y^n g^{(r)}(y)\,\mathrm{d}y \\ & + \sum_{n=0}^{r-j-2} \frac{(-1)^{n+1} x^{r-n-1}}{n!(r-k-n-1)!} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y. \end{aligned}$$

Hence, taking into consideration also Lemma 5.6 with $n = j$, we get

$$
\begin{aligned}
x^{-j-1/p} \sum_{k=0}^{\min\{j,m\}} (-1)^{m-k} \binom{m}{k} (j+m-k-1)!\, x^k \tilde{g}^{(k)}(x) \\
= \sum_{n=r-i}^{r-1} \rho'_{j,m,n} x^{r-j-n-1-1/p} \int_0^x y^n g^{(r)}(y)\,\mathrm{d}y \\
+ \sum_{n=r-j}^{r-i-1} \rho'_{j,m,n} x^{r-j-n-1-1/p} \int_{1/A}^x y^n g^{(r)}(y)\,\mathrm{d}y \\
+ \sum_{n=0}^{r-j-2} \rho''_{j,m,n} x^{r-j-n-1-1/p} \int_x^\infty y^n g^{(r)}(y)\,\mathrm{d}y \\
+ \sum_{n=i}^{j-1} \rho'''_{j,m,n}\, a_n(\check{R}g)\, x^{n-j-1/p},
\end{aligned}
\tag{5.55}
$$

where $\rho'_{j,m,n}$ and $\rho''_{j,m,n}$ are given in (5.32) and (5.33), respectively, and

$$
\begin{aligned}
\rho'''_{j,m,n} &= n! \sum_{k=0}^{\min\{n,m\}} (-1)^{m-k} \binom{m}{k} \frac{(j+m-k-1)!}{(n-k)!} \\
&= (-1)^m \frac{(j+m-n-1)!(j-1)!}{(j-n-1)!},
\end{aligned}
\tag{5.56}
$$

as to calculate $\rho'''_{j,m,n}$ we used [10, Ch. 1, (5c)].

Let us observe that (5.35) and (5.36) are valid. Next, as in the proof of (5.34), we get

$$
\|\chi^{\gamma_\infty} \psi_{r,n}(0, 1/A; \cdot)\|_{p(1/A,\infty)} \le c\, \|\chi^{\gamma_0+r} g^{(r)}\|_{p(0,1/A)}, \quad n \ge r-i,\ i > 0,
$$

which together with (5.35) implies

$$
\|\chi^{\gamma_\infty} \psi_{r,n}(0, \cdot; \cdot)\|_{p(1/A,\infty)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}, \quad n \ge r-i, i > 0. \tag{5.57}
$$

Further, property (i) of $\mathscr{L}_{i,j-1}$, and the inequalities of Minkowski and Hölder imply for $n = i, \ldots, j-1$

$$
|a_n(\check{R}g)| \le c\, \|\check{R}g\|_{1(\alpha,\beta)} \le c\, \|\check{R}g\|_{\infty(\alpha,\beta)} \le c\, \|\mathrm{w}\chi^r g^{(r)}\|_{p(0,\infty)}. \tag{5.58}
$$

Now, (5.55), (5.57), (5.35), (5.36) and (5.58) imply (5.54). Let us note that for $i = 0$, (5.57) is not used and in (5.58) we actually have

$$
|a_n(\check{R}g)| \le c\, \|\chi^{\gamma_\infty+r} g^{(r)}\|_{p(\bar{\alpha},\infty)},
$$

where $\bar{\alpha} = \min\{1/A, \alpha\}$. Hence no restrictions are imposed on γ_0. Thus the proof of (d) is completed. □

Now we are ready to prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. The theorem follows from Theorem 5.2 with $q = \mathscr{L}_{i,j-1} f$, Theorem 5.7 with $\ell = 0$ and $\ell = r$, $F = \mathscr{A}_{i,j-1}(\chi^{1/p}\mathrm{w}) f$ for both theorems and (5.1). □

Proof of Theorem 1.5. The upper bound for $K^r_{\rm w}(f,t^r)_p$ is implied by Theorem 5.3 with $q_0 = \mathscr{L}_{i,j'-1}f$, $F_0 = \mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f$, $q_\infty = \mathscr{L}_{i',j-1}f$, $F_\infty = \mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f$ and (5.1). The lower bound for $K^r_{\rm w}(f,t^r)_p$ follows from Theorems 5.8 and 5.10 with $F_0 = \mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f$, $F_\infty = \mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f$ and (5.1). The proof of the lower bound branches to four cases corresponding to $\ell_0 = 0$ or 1 and $\ell_\infty = 0$ or 1.

Let us consider, for example, $\ell_0 = 0, \ell_\infty = 1$. Then $\gamma_0 \in \mathscr{T}_i(p)$ and $\gamma_\infty = -j-1/p$, $0 \le i' \le i$ and $j+1 \le j' \le r$. If $j' \le i$ (which is possible only if $j < i$) we apply Theorem 5.8(a) and if $i < j'$ we apply Theorem 5.10(a), in both cases with j' instead of j and $F_0 = \mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f$, and get

$$t^{r-\ell}K^\ell(\mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f, t^\ell)_{p(-\infty,a)} \le c\,K^r_{\rm w}(f,t^r)_p, \quad \ell = 0,1,\dots,r. \tag{5.59}$$

If $j \le i'$ (which is possible only if $j \le i$) we apply Theorem 5.8(d) and if $i' < j$ we apply Theorem 5.10(d), in both cases with i' instead of i and $F_\infty = \mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f$, and get

$$t^{r-\ell}K^\ell(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f, t^\ell)_{p(-a,\infty)} \le c\,K^r_{\rm w}(f,t^r)_p, \quad \ell = 1,2,\dots,r. \tag{5.60}$$

Combining (5.59) with $\ell = 0$ and $\ell = r$ and (5.60) with $\ell = 1$ and $\ell = r$ we get the lower bound for $K^r_{\rm w}(f,t^r)_p$ in (1.11). □

Remark 5.11. The *exact* ranges of the integer parameters i' and j' under which the assertion of Theorem 1.5 is valid are as follows:

$$0 \le i' (\le r) \quad \text{if } j \le i - (1-[1/p])\ell_0, \text{ or} \tag{5.61}$$

$$0 \le i' \le i - (1-[1/p])\ell_0 \quad \text{if } i - (1-[1/p])\ell_0 < j; \quad \text{and} \tag{5.62}$$

$$(0 \le) j' \le r \quad \text{if } j + (1-[1/p])\ell_\infty \le i, \quad \text{or}$$

$$j + (1-[1/p])\ell_\infty \le j' \le r \quad \text{if } i < j + (1-[1/p])\ell_\infty,$$

where $[\xi]$ denotes the integer part of the real number ξ. Below we give the arguments for i' as the considerations for j' are similar.

(a) For $p = 1$ and $\gamma_0 = -i$, $i \in \{1,\dots,r\}$, relation (1.11) holds with $i' = i$ as well. This is verified analogously to the assertion of the theorem as we take into consideration the case $p = 1$ in Theorem 3.4. If we combine the case $i' = i$ with the cases $0 \le i' \le i - \ell_0$ considered in Theorem 1.5 we verify that the theorem is true for the range of i' given in (5.62).

(b) If $j \le i-\ell_0$, then for every $i' \ge i-\ell_0$ we have $\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f = (\chi^{\gamma_\infty+1/p}f)\circ\mathscr{E}$, as the case $i' = i-\ell_0$ is considered in Theorem 1.5. Hence the restriction $i' \le i-\ell_0$ is redundant and Theorem 1.5 holds for every i'. Thus (5.61) is verified in all cases except $\ell_0 = 1$, $p = 1$ and $j = i$. In the latter case for every $i' \ge i$ we have $\mathscr{A}_{i',i-1}(\chi^{\gamma_\infty+1})f = (\chi^{\gamma_\infty+1}f)\circ\mathscr{E}$, as the case $i' = i$ is considered in (a).

(c) Let $1 \le p \le \infty$, $\gamma_0 \in \mathscr{T}_i(p)$, $\gamma_\infty \in \mathscr{T}_j(p) \cup \{-j-1/p\}$, $i < j$ or $p = 1$, $\gamma_0 = -i$, $i \in \{1,\dots,r\}$, $\gamma_\infty \in \mathscr{T}_j(1) \cup \{-j-1\}$, $i < j$. Then for $f = \chi^i \in L_p({\rm w})(0,\infty)$ we have $K^r_{\rm w}(f,t^r)_p \equiv 0$ but $\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f$ is not an algebraic polynomial for $i' > i$ and hence $\omega_k(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f,t)_{p(-a,\infty)}$ does not vanish for any $k \in \mathbb{N}_0$.

(d) Let $1 < p \le \infty$, $\gamma_0 = 1-i-1/p$, $i \in \{1,\dots,r\}$, $\gamma_\infty \in \mathscr{T}_j(p) \cup \{-j-1/p\}$, $i-1 < j$. For $0 < \delta < 1$ and $b = \min\{{\rm e}^{-a},\alpha\}$ we set $f_\delta(x) = b^{-\delta}x^{i-1+\delta}$ for $x \in (0,b)$, and $f_\delta(x) = \sum_{k=0}^{i-1}\binom{i-1+\delta}{k} b^{i-k-1}(x-b)^k$ for $x \in [b,\infty)$. Thus, $f_\delta \in AC^{i-1}_{loc}(0,\infty)$. Then, on the one hand, we have by Theorem 5.15 $K^r_{\mathrm{w}}(f_\delta, t^r)_p \le c\, K^i_{\mathrm{w}}(f_\delta, t^i)_p \le c\, t^i \|\mathrm{w}\chi^i f_\delta^{(i)}\|_{p(0,\infty)} \le c\,\delta^{1-1/p}\, t^i$ with c independent of δ. And, on the other hand, in view of $|f_\delta(x) - x^{i-1}| \le c\delta$ for $x \in [b, \beta]$ we have $\omega_k(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p}) f_\delta, t)_{p(-a,\infty)} \ge c\, t^k$ for $i' \ge i$ and any $k \in \mathbb{N}_0$ with c independent of δ.

Items (c) and (d) (with $k = r$ and $\delta < ct^{(r-i)p/(p-1)}$) above show that (1.11) cannot be true for i' outside of the range given in (5.62) for $1 \le p \le \infty$, $\ell_0 = 0$; $p = 1$, $\ell_0 = 1$ and $1 < p \le \infty$, $\ell_0 = 1$ respectively.

Remark 5.12. The indices i of $\mathscr{A}_{i,j'-1}$ and j of $\mathscr{A}_{i',j-1}$ are, in general, the only possible choices in (1.11). The only exception for the first operator is the case $p = \infty$ and $\gamma_0 = 1 - i$, $i \in \{1, \ldots, r\}$, when (1.11) is also valid with $\mathscr{A}_{i-1,j'-1}(\chi^{\gamma_0}) f$ instead of $\mathscr{A}_{i,j'-1}(\chi^{\gamma_0}) f$ as $\mathscr{L}_{i-1,j'-1}$ (in the definition of $\mathscr{A}_{i-1,j'-1}(\chi^{\gamma_0})$) satisfies conditions (i) and (ii) (with $i-1$ in the place of i and $j = j'$) but not necessarily (iii). Indeed, let $(\mathscr{L}_{i-1,j'-1} f)(x) = \sum_{n=i-1}^{j'-1} a_n(f)\, x^n$ satisfy (i)–(ii) with $i-1$ in the place of i and $j = j'$. Then the linear operator $(\mathscr{L}_{i,j'-1} f)(x) = \sum_{n=i}^{j'-1} a_n(f)\, x^n$ satisfies (i)–(iii) and hence $\mathscr{A}_{i,j'-1}$ defined through it satisfies (1.11). On the other hand, we have

$$\mathscr{A}_{i,j'-1}(\chi^{\gamma_0}) f - \mathscr{A}_{i-1,j'-1}(\chi^{\gamma_0}) f = a_{i-1}(f) \in L_\infty(\mathbb{R})$$

and the right-hand side of (1.11) remains the same under this replacement. Note that $\gamma_0 \in \mathscr{T}_{exc}(\infty)$ and, thus, $\ell_0 = 1$.

Similarly, the only exception for the index j of $\mathscr{A}_{i',j-1}$ is in the case $p = \infty$, $\gamma_\infty = -j$, $j \in \{0, \ldots, r-1\}$. Here $\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty}) f$ can be replaced by $\mathscr{A}_{i',j}(\chi^{\gamma_\infty}) f$ in (1.11) as $\mathscr{L}_{i',j}$ satisfies conditions (i) and (ii) but not necessarily (iv).

5.3. *Characterization of $K^r_{\chi^\gamma}(f, t^r)_{p(0,a)}$ and $K^r_{\chi^\gamma}(f, t^r)_{p(a,\infty)}$*

Similar characterization is valid for the analogues of $K^r_{\mathrm{w}}(f, t^r)_p$ on the intervals $(0, a)$ and (a, ∞), where $a > 0$.

Theorem 5.13. *Let $r \in \mathbb{N}$, $i \in \mathbb{N}_0$, $i \le r$, $1 \le p \le \infty$, $\gamma \in \mathbb{R}$, $a, t_0 > 0$ and $0 < t \le t_0$. Let also $f \in L_p(\chi^\gamma)(0, a)$ and $\mathscr{A}_{i,r-1}$ be given by (1.9) as $\mathscr{L}_{i,r-1}$ satisfies conditions (i) with $\beta \le a$ and (ii). Then we have:*

(a) *For $\gamma \in \mathscr{T}_i(p)$ there holds*

$$K^r_{\chi^\gamma}(f, t^r)_{p(0,a)} \sim \omega_r(\mathscr{A}_{i,r-1}(\chi^{\gamma+1/p}) f, t)_{p(-\infty,\log a)} + t^r \|\mathscr{A}_{i,r-1}(\chi^{\gamma+1/p}) f\|_{p(-\infty,\log a)}.$$

(b) *For $\gamma = 1 - i - 1/p$, $i > 0$, if $\mathscr{L}_{i,r-1}$ also satisfies (iii), there holds*

$$K^r_{\chi^\gamma}(f, t^r)_{p(0,a)} \sim \omega_r(\mathscr{A}_{i,r-1}(\chi^{\gamma+1/p}) f, t)_{p(-\infty,\log a)} + t^{r-1} \omega_1(\mathscr{A}_{i,r-1}(\chi^{\gamma+1/p}) f, t)_{p(-\infty,\log a)}.$$

Proof. The upper estimates of $K^r_{\chi^\gamma}(f, t^r)_{p(0,a)}$ by moduli on $(-\infty, \log a)$ follow from (5.7) with $\mathscr{L}_{i,r-1} f$, a and $\log a$ in the place of q_0, A and a respectively and (5.1). The lower estimate in (a) for $i < r$ is verified as in the proof of Theorem 5.10(a) in the case $j = r$, whereas for $i = r$ it is verified as in the proof of Theorem 5.8(a). The lower estimate in (b) for $i < r$ is verified as in the proof of Theorem 5.10(b) in the case $j = r$, whereas for $i = r$ it is verified as in the proof of Theorem 5.8(b). □

Theorem 5.14. *Let $r \in \mathbb{N}$, $j \in \mathbb{N}_0$, $j \le r$, $1 \le p \le \infty$, $\gamma \in \mathbb{R}$, $a, t_0 > 0$ and $0 < t \le t_0$. Let also $f \in L_p(\chi^\gamma)(a,\infty)$ and $\mathscr{A}_{0,j-1}$ be given by (1.9) as $\mathscr{L}_{0,j-1}$ satisfies conditions (i) with $\alpha \ge a$ and (ii). Then we have:*

(a) *For $\gamma \in \mathscr{T}_j(p)$ there holds*

$$K^r_{\chi^\gamma}(f,t^r)_{p(a,\infty)} \sim \omega_r(\mathscr{A}_{0,j-1}(\chi^{\gamma+1/p})f,t)_{p(\log a,\infty)} + t^r \|\mathscr{A}_{0,j-1}(\chi^{\gamma+1/p})f\|_{p(\log a,\infty)}.$$

(b) *For $\gamma = -j-1/p$, $j<r$, if $\mathscr{L}_{0,j-1}$ also satisfies (iv), there holds*

$$K^r_{\chi^\gamma}(f,t^r)_{p(a,\infty)} \sim \omega_r(\mathscr{A}_{0,j-1}(\chi^{\gamma+1/p})f,t)_{p(\log a,\infty)} + t^{r-1}\omega_1(\mathscr{A}_{0,j-1}(\chi^{\gamma+1/p})f,t)_{p(\log a,\infty)}.$$

Proof. The upper estimates of $K^r_{\chi^\gamma}(f,t^r)_{p(a,\infty)}$ by moduli on $(\log a,\infty)$ follow from (5.8) with $\mathscr{L}_{0,j-1}f$, a and $\log a$ in the place of q_0, $1/A$ and $-a$ respectively and (5.1). The lower estimate in (a) for $j=0$ is verified as in the proof of Theorem 5.8(c), whereas for $j>0$ it is verified as in the proof of Theorem 5.10(c) in the case $i=0$. The lower estimate in (b) for $j=0$ is verified as in the proof of Theorem 5.8(d), whereas for $j>0$ it is verified as in the proof of Theorem 5.10(d) in the case $i=0$. □

5.4. *K-functionals of continuous functions*

Consider the space

$$C(\mathrm{w})[0,\infty) = \{f : \mathrm{w}f \in C(0,\infty), \exists \lim_{x\to 0+0}(\mathrm{w}f)(x)\},$$

where $\mathrm{w}(x) = \mathrm{w}(\gamma_0,\gamma_\infty;x)$ is given in (1.8). For functions $f \in C(\mathrm{w})[0,\infty)$ we may define a slightly different functional than (1.4) imposing *the additional restriction* $g \in C(\mathrm{w})[0,\infty)$ on the functions g on which the infimum is taken. Denote this K-functional by

$$K(f,t^r;C(\mathrm{w})[0,\infty),AC^{r-1}_{loc},\chi^r D^r).$$

Let us note that Theorems 1.4 and 1.5 with $p=\infty$ hold for this K-functional too. This fact follows from the equivalence

$$K(f,t^r;C(\mathrm{w})(0,\infty),AC^{r-1}_{loc},\chi^r D^r) \le K(f,t^r;C(\mathrm{w})[0,\infty),AC^{r-1}_{loc},\chi^r D^r) \le c\,K(f,t^r;C(\mathrm{w})(0,\infty),AC^{r-1}_{loc},\chi^r D^r),$$

valid for $r \in \mathbb{N}$, $\gamma_0,\gamma_\infty \in \mathbb{R}$ and $f \in C(\mathrm{w})[0,\infty)$. The first inequality is obvious — an infimum on a narrower class is taken in the second K-functional. The second inequality follows from the results of Sections 5.1 and 5.2. First we observe that the modified Steklov function of F (used in the proof of (5.1)) has a limit at $-\infty$ provided F has a limit at $-\infty$. Hence Theorems 5.2 and 5.3 give the same upper bounds for $K(f,t^r;C(\mathrm{w})[0,\infty),AC^{r-1}_{loc},\chi^r D^r)$ as the quantities in Theorems 1.4 and 1.5.

The same observations are true if $\mathrm{w}f$ has a limit at ∞, or has simultaneously limits at 0 and at ∞.

5.5. Properties of $K^r_{\mathrm{w}}(f,t^r)_p$

Let us point out several properties of the weighted K-functional $K^r_{\mathrm{w}}(f,t^r)_p$ which follow from the estimates in Sections 5.1 and 5.2. The analogous properties of $K^r_{\chi^\gamma}(f,t^r)_{p(0,a)}$ and $K^r_{\chi^\gamma}(f,t^r)_{p(a,\infty)}$ can be verified in a similar way.

Theorem 5.15. *Let $r,m\in\mathbb{N}$, $m<r$, $1\le p\le\infty$, $t_0>0$ and $\mathrm{w}(x)=\mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with $\gamma_0,\gamma_\infty\in\mathbb{R}$. For $f\in L_p(\mathrm{w})(0,\infty)$ and $0<t\le t_0$ there holds*

$$K^r_{\mathrm{w}}(f,t^r)_p\le c\,K^m_{\mathrm{w}}(f,t^m)_p.$$

Proof. Let us set $F_0=(\chi^{\gamma_0+1/p}(f-\mathscr{L}_{i,m-1}f))\circ\mathscr{E}$ and $F_\infty=(\chi^{\gamma_\infty+1/p}(f-\mathscr{L}_{0,\min\{j,m\}-1}f))\circ\mathscr{E}$, where i and j are determined by $\mathscr{T}_i(p)\cup\{1-i-1/p\}\ni\gamma_0$ and $\mathscr{T}_j(p)\cup\{-j-1/p\}\ni\gamma_\infty$, and the operators $\mathscr{L}_{\mu,\nu}$ are defined by (1.10) and satisfy the conditions of Theorem 1.5 (with $r=m$). Let $\ell_0=1$ if $\gamma_0=1-m-1/p,\dots,-1/p$, and $\ell_0=0$ otherwise; let also $\ell_\infty=1$ if $\gamma_\infty=1-m-1/p,\dots,-1/p$, and $\ell_\infty=0$ otherwise. As is known,

$$\omega_r(F,t)_{p(J)}\le 2^{r-m}\omega_m(F,t)_{p(J)},\qquad F\in L_p(J),\tag{5.63}$$

where $J\subseteq\mathbb{R}$ is an interval. Then by Theorem 5.3 or Remark 5.4 (with $q_0=\mathscr{L}_{i,m-1}f$ and $q_\infty=\mathscr{L}_{0,\min\{j,m\}-1}f$), (5.1) and (5.63) we get

$$\begin{aligned}K^r_{\mathrm{w}}(f,t^r)_p\le{}& c\big(\omega_m(F_0,t)_{p(-\infty,a)}+t^{m-\ell_0}\omega_{\ell_0}(F_0,t)_{p(-\infty,a)}\\&+\omega_m(F_\infty,t)_{p(-a,\infty)}+t^{m-\ell_\infty}\omega_{\ell_\infty}(F_\infty,t)_{p(-a,\infty)}\big).\end{aligned}$$

The above inequality proves the theorem in view of Theorem 1.5 with $r=m$, $i'=0$, $j'=m$, i and j replaced respectively by $\min\{i,m\}$ and $\min\{j,m\}$. □

Similar considerations yield the following Marchaud-type inequality.

Theorem 5.16. *Let $r,m\in\mathbb{N}$, $m<r$, $1\le p\le\infty$, $t_0>0$ and $\mathrm{w}(x)=\mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with $\gamma_0,\gamma_\infty\in\mathbb{R}$. For $f\in L_p(\mathrm{w})(0,\infty)$ and $0<t\le t_0$ there holds*

$$K^m_{\mathrm{w}}(f,t^m)_p\le c\,t^m\left(\int_t^{t_0}\frac{K^r_{\mathrm{w}}(f,\tau^r)_p}{\tau^{m+1}}\,d\tau+\|\mathrm{w}f\|_{p(0,\infty)}\right).$$

Proof. By Theorem 5.3 with m in the place of r, $q_0=q_\infty=0$, $\ell_0=\ell_\infty=0$ (in view of Remark 5.4) and (5.1) we have

$$\begin{aligned}K^m_{\mathrm{w}}(f,t^m)_p\le{}& c\big(\omega_m((\chi^{\gamma_0+1/p}f)\circ\mathscr{E},t)_{p(-\infty,a)}\\&+\omega_m((\chi^{\gamma_\infty+1/p}f)\circ\mathscr{E},t)_{p(-a,\infty)}+t^m\|\mathrm{w}f\|_{p(0,\infty)}\big).\end{aligned}\tag{5.64}$$

Further, let i,j,i',j', $\mathscr{A}_{i,j'-1}$ and $\mathscr{A}_{i',j-1}$ satisfy the conditions of Theorem 1.5. Then by property (i) of $\mathscr{L}_{i,j'-1}$ and $\mathscr{L}_{i',j-1}$ we have

$$\begin{aligned}\omega_m((\chi^{\gamma_0+1/p}\mathscr{L}_{i,j'-1}f)\circ\mathscr{E},t)_{p(-\infty,a)}&\le c\,t^m\|((\chi^{\gamma_0+1/p}\mathscr{L}_{i,j'-1}f)\circ\mathscr{E})^{(m)}\|_{p(-\infty,a)}\\&\le c\,t^m\|f\|_{1(\alpha,\beta)}\le c\,t^m\|\mathrm{w}f\|_{p(0,\infty)}\end{aligned}$$

and, similarly,

$$\omega_m((\chi^{\gamma_\infty+1/p}\mathscr{L}_{i',j-1}f)\circ\mathscr{E},t)_{p(-a,\infty)}\le c\,t^m\|\mathrm{w}f\|_{p(0,\infty)}.$$

Consequently, by (5.64) we get

$$K_{\mathrm{w}}^m(f,t^m)_p \le c\big(\omega_m(\mathscr{A}_{i,j'-1}(\chi^{\gamma_0+1/p})f,t)_{p(-\infty,a)} + \omega_m(\mathscr{A}_{i',j-1}(\chi^{\gamma_\infty+1/p})f,t)_{p(-a,\infty)} + t^m\|\mathrm{w}f\|_{p(0,\infty)}\big). \tag{5.65}$$

Next, as is known for $F \in L_p(J)$, $J \subseteq \mathbb{R}$ is an interval, and $0 < t \le t_0$, the Marchaud inequality

$$\omega_m(F,t)_{p(J)} \le c\,t^m\left(\int_t^{t_0}\frac{\omega_r(F,\tau)_{p(J)}}{\tau^{m+1}}\,\mathrm{d}\tau + \|F\|_{p(J)}\right) \tag{5.66}$$

holds. Applying it to (5.65), we get by Theorem 1.5 and Proposition 5.5 the assertion of the theorem. □

As is well-known, for $p < \infty$ we have $\lim_{t\to 0} K^r(F,t)_p = 0$ for any $F \in L_p(\mathbb{R})$, whereas $\lim_{t\to 0} K^r(F,t)_\infty = 0$ for $F \in L_\infty(\mathbb{R})$ iff F is uniformly continuous on $\mathbb{R}$. Then Theorem 1.5 yields the following assertion.

Theorem 5.17. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with γ_0, $\gamma_\infty \in \mathbb{R}$ and $f \in L_p(\mathrm{w})(0,\infty)$.*

(a) *For $p < \infty$ we have $\lim_{t\to 0} K_{\mathrm{w}}^r(f,t)_p = 0$.*
(b) *We have $\lim_{t\to 0} K_{\mathrm{w}}^r(f,t)_\infty = 0$ iff $(\mathrm{w}f)\circ\mathscr{E}$ is uniformly continuous on $\mathbb{R}$.*

Also, by Theorem 1.5 we can derive the saturation class of $K_{\mathrm{w}}^r(f,t)_p$ from that of the unweighted fixed-step moduli. Let $J \subseteq \mathbb{R}$ be an interval and $BV(J)$ denote the set of all functions defined on J, which are equivalent to a function of bounded variation on J.

Theorem 5.18. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\mathrm{w}(x) = \mathrm{w}(\gamma_0,\gamma_\infty;x)$ be defined in (1.8) with $\gamma_0,\gamma_\infty \in \mathbb{R}$ and $f \in L_p(\mathrm{w})(0,\infty)$.*

(a) *For $p > 1$ we have $K_{\mathrm{w}}^r(f,t)_p = O(t)$ iff $f \in AC_{loc}^{r-1}(0,\infty)$ and $\mathrm{w}\chi^r f^{(r)} \in L_p(0,\infty)$.*
(b) *We have $K_{\mathrm{w}}^r(f,t)_1 = O(t)$ iff $f \in AC_{loc}^{r-2}(0,\infty)$ and $\mathrm{w}\chi^r f^{(r-1)} \in BV(0,\infty)$.*

Proof. We set $F_0 = (\chi^{\gamma_0+1/p}(f - \mathscr{L}_{i,r-1}f))\circ\mathscr{E}$ and $F_\infty = (\chi^{\gamma_\infty+1/p}(f - \mathscr{L}_{0,j-1}f))\circ\mathscr{E}$, where $\mathscr{L}_{i,r-1}$ and $\mathscr{L}_{0,j-1}$ satisfy the hypotheses of Theorem 1.5. In view of Proposition 5.5 we have $F_0 \in L_p(-\infty,a)$ and $F_\infty \in L_p(-a,\infty)$ with fixed $a > 0$.

Let $p > 1$. As is known, $\omega_r(F,t)_{p(J)} = O(t^r)$ iff $F \in AC_{loc}^{r-1}(J)$ and $F^{(r)} \in L_p(J)$. Using this fact, Theorem 1.5, Remark 5.4 and (5.1) we get that $K_{\mathrm{w}}^r(f,t)_p = O(t)$ iff $F_0 \in AC_{loc}^{r-1}(-\infty,a)$, $F_0^{(r)} \in L_p(-\infty,a)$ and $F_\infty \in AC_{loc}^{r-1}(-a,\infty)$, $F_\infty^{(r)} \in L_p(-a,\infty)$. Next, we have $F_0 \in AC_{loc}^{r-1}(-\infty,a)$ and $F_\infty \in AC_{loc}^{r-1}(-a,\infty)$ iff $f \in AC_{loc}^{r-1}(0,\infty)$. Also, as in the proof of (5.4)–(5.5) and (5.11)–(5.12) we verify that $F_0^{(r)} \in L_p(-\infty,a)$ and $F_\infty^{(r)} \in L_p(-a,\infty)$ iff $\mathrm{w}\chi^r f^{(r)} \in L_p(0,\infty)$. Thus assertion (a) is proved.

Let $p = 1$. As is known, $\omega_r(F,t)_{1(J)} = O(t^r)$ iff $F \in AC_{loc}^{r-2}(J)$ and $F^{(r-1)} \in BV(J)$. Hence by Theorem 1.5, Remark 5.4 and (5.1) we get that $K_{\mathrm{w}}^r(f,t)_1 = O(t)$ iff $F_0 \in AC_{loc}^{r-2}(-\infty,a)$, $F_0^{(r-1)} \in BV(-\infty,a)$ and $F_\infty \in AC_{loc}^{r-2}(-a,\infty)$, $F_\infty^{(r-1)} \in BV(-a,\infty)$. Again we have $F_0 \in AC_{loc}^{r-2}(-\infty,a)$ and $F_\infty \in AC_{loc}^{r-2}(-a,\infty)$ iff $f \in AC_{loc}^{r-2}(0,\infty)$. Further, since $\mathscr{E}^\delta \in BV(-\infty,a)$ for $\delta \ge 0$ and $((1+\mathscr{E})^\gamma)^{(k)} \in W_1^1(-\infty,a) \subset BV(-\infty,a)$ for every $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we have $F_0^{(r-1)} \in BV(-\infty,a)$ iff $((\chi^{\gamma_0+1}f)\circ\mathscr{E})^{(r-1)} \in BV(-\infty,a)$ iff $((\chi\mathrm{w}f)\circ\mathscr{E})^{(r-1)} \in BV(-\infty,a)$ iff $\mathrm{w}\chi^r f^{(r-1)} \in BV(0,\mathrm{e}^a)$. Just similarly, we get that $F_\infty^{(r-1)} \in BV(-a,\infty)$ iff $\mathrm{w}\chi^r f^{(r-1)} \in BV(\mathrm{e}^{-a},\infty)$. This proves (b). □

6. The linear operator $\mathscr{L}_{i,j-1}$

6.1. Operators $\mathscr{L}_{i,j-1}$ that satisfy conditions (i) and (ii)

Let $i, j \in \mathbb{N}_0$ as $i < j$ and $x_0, \dots, x_{j-i} \in (0, \infty)$ be fixed distinct points. We define the linear operator $\hat{\mathscr{L}}_{i,j-1} : L_{1,loc}(0, \infty) \to \pi_{i,j-1}$ by

$$(\hat{\mathscr{L}}_{i,j-1} f)(x) = (L_{i+1,j} I f)'(x),$$

where

$$(L_{i+1,j} F)(x) = \left[F(x_0) - \sum_{k=1}^{j-i} F(x_k) l_{i+1,j,k}(x_0) \right] \frac{1 - \sum\limits_{k=1}^{j-i} l_{i+1,j,k}(x)}{1 - \sum\limits_{k=1}^{j-i} l_{i+1,j,k}(x_0)} + \sum_{k=1}^{j-i} F(x_k) l_{i+1,j,k}(x), \tag{6.1}$$

$$l_{i+1,j,k}(x) = \frac{x^{i+1}(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_{j-i})}{x_k^{i+1}(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_{j-i})}$$

and

$$(If)(x) = (I_a f)(x) = \int_a^x f(y)\, dy, \quad a > 0.$$

The denominator in (6.1) $1 - \sum_{k=1}^{j-i} l_{i+1,j,k}(x_0)$ is not 0 as can be verified by assuming the contrary and applying Rolle's theorem.

The definition of $\hat{\mathscr{L}}_{i,j-1}$ directly implies that it satisfies condition (i) with $\alpha \le \min\{x_0, \dots, x_{j-1}\}$ and $\beta \ge \max\{x_0, \dots, x_{j-i}\}$. Next, let us observe that $L_{i+1,j} F$ is the only polynomial in $\mathbb{R} \oplus \pi_{i+1,j}$ which interpolates the function $F \in C_{loc}(0, \infty)$ at the $j - i + 1$ positive distinct nodes $x_0, \dots, x_{j-i}$. Hence $L_{i+1,j} F = F$ for any $F \in \mathbb{R} \oplus \pi_{i+1,j}$ and $\hat{\mathscr{L}}_{i,j-1} f = f$ for any $f \in \pi_{i,j-1}$. Thus the linear operator $\hat{\mathscr{L}}_{i,j-1}$ satisfies conditions (i) and (ii). Consequently, Theorem 1.4 holds with $\mathscr{L}_{i,j-1} = \hat{\mathscr{L}}_{i,j-1}$.

Let us also mention that for $p = \infty$ and $f \in C(\mathrm{w})(0, \infty)$ we can use in Theorem 1.4 (cf. Remark 1.2) the following modification of the Lagrange interpolation polynomials:

$$(\mathscr{L}_{i,j-1} f)(x) = \sum_{k=1}^{j-i} f(x_k)\, l_{i,j-1,k}(x).$$

6.2. Operators $\mathscr{L}_{i,j-1}$ that satisfy conditions (i)–(iv)

For $[\alpha, \beta] \subset (0, \infty)$ let $x_0, x_1, \dots, x_r \in [\alpha, \beta]$ be $r + 1$ fixed distinct points. The functionals $\{\int_{x_0}^{x_k} f(y)\, dy\}_{k=1}^r$ and the polynomials $\{\Phi'_\ell(x)/\Phi_\ell(x_\ell)\}_{\ell=1}^r$, where

$$\Phi_\ell(x) = \prod_{\substack{m=0 \\ m \ne \ell}}^{r} (x - x_m), \quad \ell = 1, 2, \dots, r,$$

form a normalized bi-orthogonal system in π_{r-1} because $\Phi'_\ell \in \pi_{r-1}$ and

$$\int_{x_0}^{x_k} \frac{\Phi'_\ell(y)}{\Phi_\ell(x_\ell)}\,\mathrm{d}y = \frac{\Phi_\ell(x_k) - \Phi_\ell(x_0)}{\Phi_\ell(x_\ell)} = \delta_{k,\ell}.$$

Hence the bi-orthogonal expansion $\tilde{\mathscr{L}} : L_1[\alpha, \beta] \to \pi_{r-1}$ given by

$$(\tilde{\mathscr{L}} f)(x) = \sum_{\ell=1}^{r} \frac{\Phi'_\ell(x)}{\Phi_\ell(x_\ell)} \int_{x_0}^{x_\ell} f(y)\,\mathrm{d}y$$

is a bounded linear operator and preserves the polynomials from π_{r-1}. Writing $\Phi'_\ell(x)$ as the Taylor polynomial of degree $r-1$ at 0 we get $(\tilde{\mathscr{L}} f)(x) = \sum_{n=0}^{r-1} \tilde{a}_n(f)x^n$, where

$$\tilde{a}_n(f) = \sum_{\ell=1}^{r} \frac{\Phi_\ell^{(n+1)}(0)}{n!\,\Phi_\ell(x_\ell)} \int_{x_0}^{x_\ell} f(y)\,\mathrm{d}y. \tag{6.2}$$

Because of the properties of $\tilde{\mathscr{L}}$ the linear functionals $\tilde{a}_n$ given by (6.2) satisfy

$$\tilde{a}_n(\chi^k) = \delta_{n,k}, \quad k, n = 0, 1, \ldots, r-1. \tag{6.3}$$

Now for $i, j \in \mathbb{N}_0$, $j \le r$, we define the linear operator $\tilde{\mathscr{L}}_{i,j-1} : L_1(\alpha, \beta) \to \pi_{i,j-1}$ by

$$(\tilde{\mathscr{L}}_{i,j-1} f)(x) = \sum_{n=i}^{j-1} \tilde{a}_n(f)x^n = \sum_{n=i}^{j-1} \left(\sum_{\ell=1}^{r} \frac{\Phi_\ell^{(n+1)}(0)}{n!\,\Phi_\ell(x_\ell)} \int_{x_0}^{x_\ell} f(y)\,\mathrm{d}y \right) x^n \tag{6.4}$$

with the convention that the sum in (6.4) is 0 if $j \le i$. The following lemma is an immediate consequence of (6.3).

Lemma 6.1. *We have:*

(a) $\tilde{\mathscr{L}}_{i,j-1} f = f$ *for any* $f \in \pi_{i,j-1}$*;*
(b) $\tilde{\mathscr{L}}_{i,j-1} f = 0$ *for any* $f \in \pi_{0,i-1} \oplus \pi_{j,r-1}$.

Obviously, $\tilde{\mathscr{L}}_{i,j-1}$ satisfies condition (i). Lemma 6.1 shows that it satisfies conditions (ii)–(iv) as well. Thus the linear operator $\tilde{\mathscr{L}}_{i,j-1}$ satisfies conditions (i)–(iv) and, consequently, Theorem 1.5 holds with $\mathscr{L}_{\mu,\nu} = \tilde{\mathscr{L}}_{\mu,\nu}$.

Let us note that in the characterization of the analogues of $K_{\mathrm{w}}(f, t^r)_p$ on the intervals $(0, a)$ or (a, ∞) we must fix the numbers $x_0, x_1, \ldots, x_r$ respectively in subintervals of $(0, a]$ or $[a, \infty)$.

Let us now explicitly give the operator $\tilde{\mathscr{L}}_{i,j-1}$ for $r = 1$ and $r = 2$. Let x_0, x_1, x_2 be fixed positive distinct numbers. For $r = 1$ we use the operator $\tilde{\mathscr{L}}$ only in the case $i = 0$, $j = 1$ and it is given by (see (6.4))

$$(\tilde{\mathscr{L}}_{0,0} f)(x) = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f(y)\,\mathrm{d}y.$$

For $r = 2$ there are three different operators of type $\tilde{\mathscr{L}}$, which are given by

$$(\tilde{\mathscr{L}}_{0,0} f)(x) = \tilde{a}_0(f), \qquad (\tilde{\mathscr{L}}_{0,1} f)(x) = \tilde{a}_0(f) + \tilde{a}_1(f)\,x, \qquad (\tilde{\mathscr{L}}_{1,1} f)(x) = \tilde{a}_1(f)\,x,$$

where

$$\tilde{a}_0(f) = -\frac{x_0 + x_2}{(x_1 - x_0)(x_1 - x_2)} \int_{x_0}^{x_1} f(y)\,dy - \frac{x_0 + x_1}{(x_2 - x_0)(x_2 - x_1)} \int_{x_0}^{x_2} f(y)\,dy,$$

$$\tilde{a}_1(f) = \frac{2}{(x_1 - x_0)(x_1 - x_2)} \int_{x_0}^{x_1} f(y)\,dy + \frac{2}{(x_2 - x_0)(x_2 - x_1)} \int_{x_0}^{x_2} f(y)\,dy.$$

The same pattern can be followed in constructing other operators of type $\mathscr{L}$. Let $\{q_\ell\}_{\ell=0}^{r-1}$ be the normalized Legendre polynomials for a given interval $[\alpha, \beta] \subset (0, \infty)$, i.e.

$$\int_\alpha^\beta q_k(y) q_\ell(y)\,dy = \delta_{k,\ell}, \quad k, \ell = 0, 1, \ldots, r-1.$$

Starting with the normalized bi-orthogonal system $\{\int_\alpha^\beta q_k(y) f(y)\,dy, q_\ell\}_{k,\ell=0}^{r-1}$ we get the operators

$$(\bar{\mathscr{L}}_{i,j-1} f)(x) = \sum_{n=i}^{j-1} \left(\sum_{\ell=0}^{r-1} \frac{q_\ell^{(n)}(0)}{n!} \int_\alpha^\beta q_\ell(y) f(y)\,dy \right) x^n$$

$$= \sum_{n=i}^{j-1} \left(\sum_{k=0}^{r-1} \sum_{\ell=0}^{r-1} \frac{q_\ell^{(k)}(0) q_\ell^{(n)}(0)}{k!n!} \int_\alpha^\beta y^k f(y)\,dy \right) x^n.$$

Then, Lemma 6.1 holds with $\bar{\mathscr{L}}_{i,j-1}$ in the place of $\tilde{\mathscr{L}}_{i,j-1}$ and, thus, $\bar{\mathscr{L}}_{i,j-1}$ satisfies conditions (i)–(iv).

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