An Index Formula for the Relative Class Number of an Abelian Number Field

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Let \( n \) be the conductor of an imaginary abelian number field \( K \), \( \mathcal{O} \) the ring of algebraic integers of \( K \), and \( \mathbb{Q}_n \) the \( n \)th cyclotomic field. We describe the index of the additive group generated by the conjugate elements of the trace \( \text{Tr}_{\mathbb{Q}_n/K}(i \cdot \cot(n/n)) \) in the group \( \mathcal{O} \cap i \cdot \mathbb{R} \) (if \( n = p^m \) is a prime power, one has to take \( \text{Tr}_{\mathbb{Q}_n/K}(ip \cdot \cot(n/n)) \) instead). This index equals the relative class number of \( K \), multiplied with a factor that is explicitly given in terms of the ramification of \( K \) over \( \mathbb{Q} \).

In some respect this result is an analogue of the representation of the class number of \( K \) as an index of circular units, rather than the hitherto known Stickelberger index formulas of Iwasawa et al. We also show that the higher derivatives \( i^m \cot^{m-1}(n/n), m \geq 2 \), yield index formulas analogous to the higher Stickelberger formulas of Kubert and Lang.

INTRODUCTION AND MAIN RESULT

Several attempts have been made to interpret the analytical class number formula of an (absolutely) abelian number field \( K \) in an arithmetical way. First, consider \( K^+ = K \cap \mathbb{R} \). In a number of cases formulas like

\[
[E^+ : U] = h_K^+ \cdot c_K^+ \tag{1}
\]

are known. Here \( E^+ \) is the unit group of the field \( K^+ \), \( h_K^+ \) its class number, \( U \) a suitably defined group of "circular units," and \( c_K^+ \) a rational number that is (more or less) explicitly known. Beginning with Kummer, this theory has been developed by various authors (cf. [3, 6, 11]). Iwasawa was the first to give an index formula analogous to (1) for the relative class number of the \( p' \)th cyclotomic field (cf. [4]). A series of more general formulas for imaginary abelian fields \( K \) followed Iwasawa's result (cf. [9, 10, 11]). Typically, their shape is

\[
[R^- : S] = h_K^- \cdot c_K^-, \tag{2}
\]

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where $R^-$ is the "imaginary part" of the integral group ring $R = \mathbb{Z}G$ over the Galois group $G = G_K = \text{Gal}(K/\mathbb{Q})$, $S$ the Stickelberger ideal in $R$ (appropriately defined), $h_K$ the relative class number of $K$, and $c_K$ a rational factor as in (1).

The following problems are frequently connected with formulas of type (2): Depending on the definition of $S$, it may happen that $c_K$ vanishes for certain fields $K$ and $[R^+ : S]$ becomes infinite (cf. [9]); or, the other extreme, (2) is always true but it is very difficult to compute $c_K$ in many cases [cf. [11]].

In the present paper $h_K^-$ is shown to be the index of two additive subgroups of the ring $\mathcal{O} = \mathcal{O}_K$ of algebraic integers of $K$, multiplied with a rational factor. The factor is explicitly given in terms of the ramification of $K$ over $\mathbb{Q}$. All quantities occurring in this index formula come from $K$ directly (not "via group ring"). In this respect our result, rather than the Stickelberger formulas (2), is a very analogue of (1).

To be able to enounce the main theorem we need the following notations: Let $n$ be the conductor of the abelian field $K$, $\mathbb{Q}_n = \mathbb{Q}(\zeta)$, $\zeta = e^{2\pi i/n}$, the $n$th cyclotomic field, and $\eta = i \cdot \cot(\pi/n) = (1 + \zeta)/(1 - \zeta) \in \mathbb{Q}_n$. If $n$ is a composite (i.e., not a prime power), we put $n^* = n$ and $\eta^* = \eta$; if $n = p^r$, $p$ prime, $r \geq 1$, $n^* = pn$, and $\eta^* = pn$. In any case the number $\eta^*$ is in $\mathcal{O}_n - \mathbb{Z}[\zeta]$, and hence its trace relative to $K$, i.e., $\eta_K^* = \text{Tr} \mathcal{O}_n K(\eta^*)$, is in $\mathcal{O} = \mathcal{O}_K$. Let $\tau \in G$ be the complex conjugation. The sets $\mathcal{O}^+ = \mathcal{O} \cap \mathbb{R} = \{ a \in \mathcal{O}; \tau(a) = a \}$ and $\mathcal{O}^- = \mathcal{O} \cap i\mathbb{R} = \{ a \in \mathcal{O}; \tau(a) = -a \}$ are $R$-submodules of $\mathcal{O}$. For the time being, let $K$ be imaginary. Then each of $\mathcal{O}^+$ and $\mathcal{O}^-$ is a free $\mathbb{Z}$-module of rank $d/2$, $d = [K : \mathbb{Q}]$ denoting the degree of $K$ over $\mathbb{Q}$. The module $\mathcal{O}^-$ contains $(1 - \tau) \mathcal{O} = \{ a - \tau(a); a \in \mathcal{O} \}$. Its index in $\mathcal{O}^-$ is a power of 2 that will be computed below. The module $(1 - \tau) \mathcal{O}$, in turn, contains $R\eta_K^*$, and the corresponding index is the subject of our (main) Theorem 1.

As usual, $Q$ means the unit index of $K$ ($Q = 1$ or 2), and $w$ the number of roots of unity in $K$. In addition, let $\delta = |D_K/D_K^+|$ be the absolute value of the quotient of the discriminants of $K$ and $K^+$. By $e_p$, $d_p$, $c_p$ we denote the ramification index, the residue class degree, and the number of prime divisors in $K$, respectively, of a rational prime number $p$ (in particular, $e_p d_p c_p = d$). The corresponding quantities for $K^+$ are denoted by $e_p^+, d_p^+, c_p^+$. Theorem 1.

Let the above notations hold. If $K$ is imaginary, then

$$[(1 - \tau) \mathcal{O} : R\eta_K^*] = \frac{(2n^*)^{d/2} Q \cdot w \cdot \delta}{\prod_{p \in \mathcal{Z}} (1 + p^{-d_p/2})^{c_p}} \times \prod_{p \in \mathcal{Z}} (1 - p^{d_p})^{c_p/2} \cdot h_K^-.$$  

Here $\mathcal{Z} = \{ p; p \mid n, d_p = 2d_p^+ \}$ and $\mathcal{F} = \{ p; p \mid n, c_p = 2c_p^+ \}$.  


Note that the group ring $R$ is not essential for the formulation of Theorem 1. Indeed, if $\mathcal{V}$ is a complete set of representatives of $G/\langle \tau \rangle$, then $R\mathcal{V}_K = \bigoplus \{ \mathbb{Z}\sigma \mathcal{V}_K; \sigma \in \mathcal{V} \}$. The sets $\mathcal{F}$ and $\mathcal{Z}$ consist of all primes $p$ ramified in $K^+$ such that the prime divisors of $K^+$ lying above $p$ are, respectively, inert in $K$ or decomposed in $K$.

We briefly consider the special case $K = \mathbb{Q}_n$. For $k \in \mathbb{Z}$, $(k, n) = 1$, let $\sigma_k : \mathbb{Q}_n \to \mathbb{Q}_n ; \zeta \mapsto \zeta^k$ be the Artin automorphism attached to $k$. Then $G_n = \text{Gal}(\mathbb{Q}_n/\mathbb{Q}) = \{ \sigma_k; 1 \leq k \leq n, \ (k, n) = 1 \}$, and $\sigma_k(i \cdot \cot(\pi/n)) = i \cdot \cot(\pi k/n)$. In particular, the numbers $i \cdot \cot(\pi k/n)$ (or $i p \cdot \cot(p n/k)$), $1 \leq k \leq n/2$, $(k, n) = 1$, form a $\mathbb{Z}$-basis of $R\mathcal{V}_K$ if $n$ is a composite (or $n = p^r$, respectively). The explicit values of $Q$, $\eta$ are well known (cf., e.g., [12, pp. 39, 43]). If $n$ is a prime power, both sets $\mathcal{F}$ and $\mathcal{Z}$ are empty. In the composite case, write $n = p^{r_1} \cdot n_p$, $p \mid n_p$, for each prime divisor $p$ of $n$. Then $\mathcal{F} = \{ p; p \mid n, -1 \notin \langle \tilde{p} \rangle \}$ and $\mathcal{Z} = \{ p; p \mid n, -1 \notin \langle \tilde{p} \rangle \}$, where $\langle \tilde{p} \rangle$ is the subgroup of $(\mathbb{Z}/n_p \mathbb{Z})^\times$ generated by $\tilde{p}$. Finally, $d_p = |\langle \tilde{p} \rangle|$ and $c_p = \phi(n_p)/d_p$.

At this point a few words about the relation of Theorem 1 to the Stickelberger index formulas are in place. The number

$$\eta_1 = (i \cdot \cot(\pi/n) + 1)/2 = n^{-1} \sum_{k=1}^{n} k \zeta^{-k}$$

may be regarded as a formal analogue in $\mathbb{Q}_n$ of the Stickelberger element

$$\Theta_1 = \sum \{ k \sigma_k^{-1}; 1 \leq k \leq n, \ (k, n) = 1 \}$$

in the group ring $\mathbb{Q}G_n$. Clearly $\eta = (1 - \tau) \eta_1$, $\tau = \sigma_{-1}$. The reader will find further analogies; for example, the character coordinates of $\eta_1$ (or $\eta$, cf. Section 1) play a role in the proof of Theorem 1 similar to that of the character values of $\Theta_1$ in Stickelberger index theory.

Sections 1 and 2 of this paper are devoted to the proof of Theorem 1. In Section 3 we give formulas corresponding to the “higher” Stickelberger index formulas of Kubert and Lang (cf., e.g., [5]). The numbers in $\mathbb{Q}_n$ that replace the higher Stickelberger elements in $\mathbb{Q}G_n$ are, in substance, values of the higher cotangent derivatives.

The present work is closely connected with [2], where the reader can find additional references and historical remarks.

1. APPLICATION OF CHARACTER COORDINATES

Let the above notations hold, with $K$ not necessarily imaginary. As usual, we interpret the character group $X_K = \{ \chi : G_K \to \mathbb{C}^\times; \chi \text{ a group}$
homomorphism} as a group of Dirichlet characters mod \( n \). Indeed, for \( \chi \in \chi_K \) and \( k \in \mathbb{Z} \), \( (k, n) = 1 \), we put \( \chi(k) = \chi(\sigma_k) \), when \( \sigma_k \) denotes the restriction of \( \sigma_k \in G_n \) to \( K \). In the sequel \( f_\chi \) will stand for the conductor of \( \chi \), and \( \chi_f \) for the primitive Dirichlet character mod \( f_\chi \), belonging to \( \chi \).

One of our main tools are the character coordinates introduced by Leopoldt (cf. [8]). For each \( a \in K \) and each \( \chi \in \chi_K \), the \( \chi \)-coordinate \( y_K(\chi | a) \) is defined by

\[
y_K(\chi | a) \cdot T(\tilde{\chi}) = \sum_{\sigma \in G_K} \tilde{\chi}(\sigma) \cdot \sigma(a).
\]

Here \( \chi = \chi^{-1} \) is the complex-conjugate character of \( \chi \), and \( T(\chi) \) is the Gauss sum

\[
T(\tilde{\chi}) = \sum_{k=1}^{f_\chi} \tilde{\chi}_f(k) e^{-2\pi i k / f_\chi}.
\]

Let us keep in mind two important invariance properties of the character coordinates [8, Sect. 1, formulas (5), (7)]. On the one hand, for each \( \sigma \in G = G_K \),

\[
y_K(\chi | \sigma(a)) = \chi(\sigma) y_K(\chi | a).
\]

On the other hand, if \( L \) is a subfield of \( K \) and \( \chi \in \chi_L (\subseteq \chi_K) \), then

\[
y_K(\chi | a) = y_L(\chi | \text{Tr}_{K/L}(a)).
\]

The following theorem about the discriminant \( D(\sigma(a); \sigma \in G) \) of the set of conjugates of \( a \) is also contained in [8].

**Theorem 2 [8].** Let \( D_K \) be the discriminant of \( K \) and \( a \in K \). Then

\[
D(\sigma(a); \sigma \in G) = \prod_{\chi \in \chi_K} y_K(\chi | a)^2 \cdot D_K.
\]

The proof of Theorem 2 given in [8] seems to be somewhat "ad hoc." However, the theorem is an almost immediate consequence of the well-known formula for the abelian group determinant. In our setting, this formula reads as (cf. [12, p. 71])

\[
\det(\sigma^{-1}(a))_{\sigma, \sigma' \in G} = \prod_{\chi \in \chi_K} \left( \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma(a) \right).
\]

Hence (3) yields \( D(\sigma(a); \sigma \in G) = \prod_{\chi \in \chi_K} (y_K(\chi | a) T(\tilde{\chi}))^2 = \prod_{\chi \in \chi_K} y_K(\chi | a)^2 \times \prod_{\chi \in \chi_K} T(\tilde{\chi}) T(\tilde{\chi}). \) Observing that \( T(\tilde{\chi}) \cdot T(\tilde{\chi}) = \chi(-1) \cdot f_\chi \), we obtain the theorem from the conductor-discriminant formula.
Due to Theorem 2, the module $Ra = \sum_{\sigma \in G} \mathbb{Z}\sigma(a)$ has the (maximal) rank $d$ if and only if all coordinates $y_k(\chi | a)$, $\chi \in X_K$, are different from zero. If this is the case and, in addition, $a$ is in $\mathcal{O}$, Theorem 2 implies
\[
[\mathcal{O} : Ra] = \pm \prod_{\chi \in X_K} y_k(\chi | a).
\]

One can therefore compute the index of $Ra$ in $\mathcal{O}$ with the aid of the character coordinates of $a$.

Next we prove a modification of formula (6) that is more adequate to the situation of Theorem 1 (and Theorem 6 below). We write $X_K^+ = \{ \chi \in X_K; \chi(-1) = 1 \}$ for the character group of $K^+$ and $X_K^- = X_K \setminus X_K^+$ for the set of odd characters of $K$.

**Theorem 3.** Let $K$ be imaginary and $M$ an $R$-submodule of $\mathcal{O} = \mathcal{O} \cap i\mathbb{R}$ of finite index $[\mathcal{O}^+ : M]$. Suppose that $a$ is in $M$ and that $y_k(\chi | a) \neq 0$ for all characters $\chi \in X_K^+$. Then $[M : Ra]$ is finite, and
\[
\lambda [M : Ra] = \pm \prod_{\chi \in X_K^+} y_k(\chi | a),
\]
with $\lambda = [\mathcal{O} : \mathcal{O}^+ \oplus M]$.

**Proof.** Take an element $\tilde{a} \in \mathcal{O}^+$ such that $[\mathcal{O}^+ : R\tilde{a}] < \infty$. For each $\chi \in X_K^-$ we get, according to (4), $y_k(\chi | \tilde{a}) = y_k(\chi | \tau(\tilde{a})) = -y_k(\chi | \tilde{a}) = 0$. Moreover, (5) shows that $y_k(\chi | \tilde{a}) = 2 \cdot y_k(\chi | \tilde{a})$ for each $\chi \in X_K^+$. Therefore the element $b = \tilde{a} + a \in \mathcal{O}$ has the coordinates
\[
y_k(\chi | b) = \begin{cases} y_k(\chi | a), & \chi \in X_K^-, \\ 2 \cdot y_k(\chi | \tilde{a}), & \chi \in X_K^+. \end{cases}
\]
None of them vanishes, so (6) gives
\[
[\mathcal{O} : Rb] = \pm \prod_{\chi \in X_K^+} y_k(\chi | a) \cdot 2^{d/2} \cdot [\mathcal{O}^+ : R\tilde{a}].
\]

From the chain $Rb \subseteq R\tilde{a} \oplus Ra \subseteq \mathcal{O}^+ \oplus M \subseteq \mathcal{O}$ we learn that the left side of (7) equals $\lambda \cdot [\mathcal{O}^+ : R\tilde{a}] \cdot [R\tilde{a} \oplus Ra : Rb]$. But the intermediate factor is $[\mathcal{O}^+ : R\tilde{a}] \cdot [M : Ra]$, and the number $[\mathcal{O}^+ : R\tilde{a}]$ can be cancelled on either side of (7). Hence the only thing still to be shown is the identity $[R\tilde{a} \oplus Ra : Rb] = 2^{d/2}$.

For this purpose let $\mathcal{Y} \subseteq G$ be a complete set of representatives of $G/\langle \tau \rangle$. Then $R\tilde{a} \oplus Ra$ has the $\mathbb{Z}$-basis $(..., \sigma(\tilde{a}), \sigma(a), ...)_{\sigma \in \mathcal{Y}}$, and $Rb$ has
the respective basis \((..., \sigma(b), \sigma(\tau(b)), ...)_{\sigma \in \chi}\). The latter arises from the former by means of a matrix with determinant
\[
\begin{vmatrix}
1 & 1 \\
1 & -1
\end{vmatrix}^{n-1} = (-2)^{d/2}.
\]
This concludes the proof.

Theorem 3 will be applied in the proof of Theorem 1, with \(M = (1 - \tau) \mathcal{O}\) and \(a = \eta_K^*\). In fact, \(\eta_K^*\) is in \((1 - \tau) \mathcal{O}\). To see this, consider \(\eta_1 = (i \cdot \cot(\pi/n) + 1)/2 = 1/(1 - \zeta)\). For \(n\) composite, \(\eta_1\) is in \(\mathcal{O}_n\), since \(1 - \zeta\) is a unit of \(\mathcal{O}_n\). Then \(\eta^* = (1 - \tau) \eta_1\) is in \((1 - \tau) \mathcal{O}_n\) and \(\eta_K^*\) in \((1 - \tau) \mathcal{O}_K\). In the case \(n = p^r\) one has to take into account that \(1 - \zeta\) divides \(p\).

Let \(\chi \in X_K\). Due to (5), \(\eta^*_K(\chi|\eta_K^*) = \eta_K(\chi|\eta^*)\), where \(\eta_K(\chi|\eta^*)\) means the \(\chi\)-coordinate with respect to the field \(\mathcal{Q}_n\). These coordinates have been computed in [2]. We quote the result for higher derivatives of the cotangent, too, because this case will be needed in Section 3.

**Theorem 4** [2]. *Let \(\chi\) be a character mod \(n\). For each integer \(m \geq 1\) the \(\chi\)-coordinate \(y_n(\chi|\eta^*_n) = y_n(\chi|\eta^{m-1}_n(\chi/n))\) vanishes if \(\chi(-1) = (-1)^{m-1}\). If, however, \(\chi(-1) = (-1)^m\), this coordinate takes the value*
\[
\left(\frac{2n}{f_\chi}\right)^m \prod_{p | n} (1 - \overline{\chi}_f(p)/p^m) \cdot B_{m,\chi_f}/m.
\]
*Here \(B_{m,\chi_f}\) means the generalized Bernoulli number in Leopoldt's sense (cf. [7]).*

Now we are in a position to prove most of Theorem 1. On combining Theorems 3 and 4, we obtain for an imaginary field \(K\)
\[
\pm \lambda[(1 - \tau) \mathcal{O} : R\eta_K] = (2n^*)^{d/2} \prod_{\chi} f_{\chi}^{-1}
\]
\[
\times \prod_{\chi} \left(\prod_{p | n} (1 - \overline{\chi}_f(p)/p)\right) \prod_{\chi} B_{1,\chi_f},
\]
with \(\lambda = [\mathcal{O} : \mathcal{O}^+ \oplus (1 - \tau) \mathcal{O}]\) and \(\chi\) running through \(X_K\). By the conductor-discriminant formula, \(\prod_{\chi} f_{\chi} = \delta\), and the classical relative class number formula (cf., e.g., [12, p. 421]) yields
\[
\pm \prod_{\chi} B_{1,\chi_f} = 2^{d/2} \cdot h_K/(Qw).
\]
The evaluation of the Euler factors \(\prod_{\chi} (1 - \overline{\chi}_f(p)/p), p | n\), is well known
(cf., e.g., [3, p. 19]). More generally, if $Y$ is a variable and $p$ an arbitrary prime number,

$$
\prod_{\chi \in \chi_K} (Y - \chi_f(p)) = Y^{d(1 - 1/c_p)} (Y^{d_p} - 1)^{c_p}. \quad (8)
$$

On inserting $p$ for $Y$, one gets the $p$-factors of Theorem 1. So the proof of this theorem is complete, provided that

$$
\lambda = 2^{d/2}.
$$

This identity is the subject of Section 2.

Before concluding this section, let us examine an imaginary quadratic field $K$ with conductor $n$ and (nontrivial) character $\chi: \mathbb{Z} \to \{0, \pm 1\}$. In this case we can put together Theorem 4, formulas (3), (5), and the fact that $T(\chi) = -i \sqrt{n}$ to enunciate a somewhat sharper version of Theorem 1. Indeed,

$$
\sum_{k=1}^{n} \chi(k) \cdot \cot(\pi k/n) = 4 \sqrt{n} h_K/w,
$$

$h_K = h_K^{-1}$ being the class number of $K$. For $n$ prime, $n \equiv 3 \mod 4$, $n \neq 3$, this may be written as

$$
\sum_{k=1}^{(n-1)/2} \cot(\pi k^2/n) = \sqrt{n} h_K.
$$

2. AN INDEX CALCULUS

For the time being, $K$ is not necessarily imaginary, and all notations are as above. We are going to compute $\lambda = [\mathcal{O}: \mathcal{O}^+(1 - \tau) \mathcal{O}]$ and, without any extra effort, $\lambda = [\mathcal{O}: \mathcal{O}^+ \oplus \mathcal{O}^-]$. The index $\lambda$ will be applied in Section 3. In our computation several results of Leopoldt concerning the $\mathcal{R}$-module structure of $\mathcal{O}$ will be used (cf. [8]). For this reason, let us first summarize the matters we need from [8].

According to the main theorem of [8], the $\mathcal{R}$-module $\mathcal{O}$ is isomorphic to a certain submodule $A$ of the rational group ring $\mathbb{Q}G$. In consequence of this isomorphism, $\lambda = [A : A^+ \oplus (1 - \tau) A]$ and, without any extra effort, $\lambda = [\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-]$, with $A^+ = \{x \in A; \tau x = x\}$, $A^- = \{x \in A; \tau x = -x\}$. Let $H$ be a subgroup of $G$. Two characters $\chi$, $\chi'$ are said to be in the same $H$-class if and only if the groups $\langle \chi \rangle H$ and $\langle \chi' \rangle H$ are identical. To each $H$-class $Z \subseteq X_K$ we attach the idempotent element $e_Z = \sum_{\chi \in Z} \varepsilon_\chi$ in $\mathbb{Q}G$, with $\varepsilon_\chi = |G|^{-1} \sum_{\sigma \in G} \chi(\sigma^{-1}) \sigma$. The $\mathcal{R}$-module $A_Z = \mathcal{R} e_Z \subseteq \mathbb{Q}G$ has the $\mathbb{Z}$-rank $|Z|$. Modules of this kind
will constitute \(A\). We collect some important properties of \(H\)-classes of \(Z\) and of \(\varepsilon_Z, A_Z\) in the subsequent lemma.

**Lemma 1.** Let \(H\) be a subgroup of \(G\) and \(Z \subseteq X_K\) an \(H\)-class.

(a) There exists a set \(\mathcal{V}_Z \subseteq H\) with the following property: If \(\mathcal{V}\) is an arbitrary complete set of representatives of \(G/H\), the elements \(\sigma \varepsilon_Z, \sigma \in \mathcal{V}\), \(\sigma' \in \mathcal{V}_Z\), form a \(Z\)-basis of \(A_Z\).

(b) Suppose that \(\tau \in H\). Then \(Z \subseteq X_K^+\) or \(Z \subseteq X_K^-\). Accordingly, either \(\tau \varepsilon_Z = \varepsilon_Z\) or \(\tau \varepsilon_Z = -\varepsilon_Z\).

Assertion (a) is part of a more precise description of a \(Z\)-basis of \(A_Z\) given in [8], and assertion (b) is left as an exercise. In the situation of (b) we say that \(Z\) is an even or odd \(H\)-class, respectively.

For a prime number \(p\) let \(V_j^{(p)}, j = 0, 1, 2, \ldots, \) be the \(j\)th ramification group of \(p\) in \(K\). The subgroups \(H\) of \(G\) we are concerned with are

\[H_1 = \prod_{p \mid n} V_j^{(p)}\]  
\[H_2 = V_2^{(2)} \times \prod_{p \mid n \text{ odd}} V_j^{(p)} (\subset H_1).\]

A *branch class of the first kind* is an \(H_1\)-class whose elements \(\chi\) have a conductor \(f_\chi\) not divisible by 8. A *branch class of the second kind* is an \(H_2\)-class, all of whose elements have a conductor divisible by 8. Each character \(\chi \in X_K\) belongs to exactly one branch class (either of the first or of the second kind), and

\[A = \bigoplus \bigoplus_{\text{branch classes}} A_Z,\]

with \(Z\) running through the totality of all branch classes. Clearly \(\lambda = \prod_{Z} \lambda_Z, \bar{\lambda} = \prod_{Z} \bar{\lambda}_Z\), with \(\lambda_Z = [A_Z: A_2^+ \oplus (1 - \tau) A_Z], \bar{\lambda}_Z = [A_Z: A_2^+ \oplus A_Z^\tau]\), where the meaning of \(A_2^+\) and \(A_2^\tau\) is obvious. Suppose now that \(K\) is imaginary. In our computation of \(\lambda, \bar{\lambda}\) we distinguish three cases.

**Case 1.** \(\tau \notin H_1\). To begin with, let \(Z\) be a branch class of the first kind and \(\mathcal{V}_0\) a complete set of representatives of \(G/\langle \tau, H_1 \rangle\). Then \(\mathcal{V} = \mathcal{V}_0 \cup \tau \mathcal{V}_0\) is a complete set of representatives of \(G/H_1\). Due to Lemma 1(a) the module \(A_Z\) has a \(Z\)-basis of the shape \((\ldots, \sigma \varepsilon_Z, \tau \sigma \varepsilon_Z, \ldots, )\), with \(\sigma\) running through the product \(\mathcal{V}_0 \mathcal{V}_0^\tau\). This fact implies that \(A_Z^\tau = (1 + \tau) A_Z, A_Z = (1 - \tau) A_Z,\) and \(\lambda_Z = \bar{\lambda}_Z = 2^{|Z|/2}\) (cf. last part of proof of Theorem 3). If \(Z\) is a branch class of the second kind, let \(H_2\) play the role of \(H_1\) above an obtain the same result. Hence \(\lambda = \bar{\lambda} = \prod_{Z} 2^{|Z|/2} - 2d/2\).

**Case 2.** \(\tau \in H_1 \setminus H_2\). In particular, \(\tau \notin V_2^{(2)}\). But \(V_2^{(2)}\) is just the kernel of the group \(\{\chi \in X_K; 8 \mid f_\chi\}\), so there are odd characters in this group. This
means that \(| \bigcup \{ Z; \text{Z of the first kind, Z even} \} | = | \bigcup \{ Z; \text{Z of the first kind, Z odd} \} | = s/2, s = [G : V_2^{(2)}].\) By Lemma 1(b) the following holds for even classes of the first kind: \(A_Z = A_\infty, A_Z = 0,\) and thus \(\lambda_Z = \overline{\lambda}_Z = 1.\) For odd classes of the first kind the corresponding result is \(A_Z = A_\infty, (1 - \tau) A_Z = 2 A_\infty, \lambda_Z = 2^{[\text{Z}]}, \overline{\lambda}_Z = 1.\) Therefore the branch classes of the first kind contribute the respective factors \(2^{s/2}\) and 1 to the numbers \(\lambda\) and \(\overline{\lambda}.\)

For branch classes \(Z\) of the second kind there is no difference between Cases 1 and 2. We have \(\lambda_Z - \overline{\lambda}_Z = 2^{[\text{Z}]^2},\) and the total contribution of these classes to \(\lambda\) or \(\overline{\lambda}\) is \(2^{(d-s)/2}.\) Altogether, \(\lambda = 2^{d/2}, \overline{\lambda} = 2^{d-s)/2}.\) If we observe that \([V_0^{(2)} : V_2^{(2)}] = 2\) and \([G : V_0^{(2)}] = d/e_2,\) we may write \(\overline{\lambda} = 2^{d/2 - d/e_2}.

Case 3. \(\tau \in H_2.\) According to the lemma, \(\lambda_Z - \overline{\lambda}_Z = 1\) for even classes of either kind, and \(\lambda_Z = 2^{[\text{Z}]}, \lambda_Z = 1\) for odd ones. Since \(K\) is imaginary, \(|X^+| = |X^-| = d/2.\) This implies \(\lambda = 2^{d/2}, \overline{\lambda} = 1.\) We summarize the results of our calculations in the following theorem.

**Theorem 5.** Let \(K\) be an imaginary abelian number field of degree \(d.\)

(a) \([\mathcal{O} : \mathcal{O}^+ \oplus (1 - \tau) \mathcal{O}^-] = 2^{d/2}\)

(b) \([\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-] = \begin{cases} 2^{d/2}, & \tau \notin V_0^{(2)}, \\ 2^{d/2 - d/e_2}, & \tau \in V_0^{(2)} \setminus V_2^{(2)}, \\ 1, & \tau \in V_2^{(2)}. \end{cases}\)

3. **Index Formulas for Higher Cotangent Derivatives**

Let the above notations still apply. In particular, let \(n\) be the conductor of \(K, K\) not necessarily imaginary. In [5] Stickelberger elements in \(\mathbb{Q}G,\) of order \(m \geq 2\) have been introduced, viz.,

\[\Theta_m = (n^{m-1}/m) \sum \{ B_m(k/n) \sigma_k^{-1}; 1 \leq k \leq n, \ (k, n) = 1 \}\]

\((B_m(Y)\) means the \(m\)th Bernouilli polynomial). As a formal analogue of these elements, define for \(m \geq 2\)

\[\eta_m = (n^{m-1}/m) \sum_{k=1}^n B_m(k/n) \zeta^{-k} \in \mathbb{Q}_m.\]  \hspace{1cm} (9)

One essential difference between these two definitions is evident: In most cases it is not trivial to transform (9) into a representation of \(\eta_m\) by a \(\mathbb{Q}\)-basis. The numbers \(\eta_m\) may also be written as (cf. [1, 2])

\[\eta_m = (i/2)^m \cdot \cot((m-1)(\pi/n)), \quad m \geq 2.\]
If $n$ is a prime power, $\eta_m$ is not an algebraic integer. Hence we put

$$\eta_m^* = \begin{cases} 
\eta_m, & n \text{ composite}, \\
\pm \eta_m, & n = p^r, r \geqslant 2, \\
p^{r+1} \eta_m, & n = p, p^t | m, p^{t+1} \nmid m.
\end{cases}$$

Furthermore, let the number $\eta_m^*$ arise from $n$ by multiplication with $1$, $p$, $p^{t+1}$, in the respective cases. At any rate, $\eta_m^* \in \mathcal{O}_n$, due to a theorem of "von Staudt-Clausen" type shown in [1]. More precisely, $\eta_m^* \in \mathcal{O}_n^+$ for $m$ even, and $\eta_m^* \in \mathcal{O}_n^-$ for $m$ odd. Let $\eta_{m,K}^*$ be the trace $\eta_{m,K}^* = \text{Tr}_{\mathcal{Q}_{n,K}}(\eta_m^*)$. In the previous sections we have collected all the requisites needed for an easy computation of the indices $[\mathcal{O}: R\eta_{m,K}^*]$ ($K$ real, $m$ even) and $[\mathcal{O}^*: R\eta_{m,K}^*]$ ($K$ imaginary, $m$ odd). In fact, one has to apply Theorems 3–5 and formulas (6), (8). Here is the result.

**Theorem 6.** Let the notations of Theorem 1 hold.

(a) For $K$ real and $m$ even,

$$[\mathcal{O}: R\eta_{m,K}^*] = \pm \frac{(n_m^*)^{dm}}{D_{m,K}^{*}} \prod_{p|n} (1 - p^{-d_m})^{e_p} \prod_{\chi \in \chi_K} B_{m,x}. $$

(b) For $K$ imaginary and $m$ odd, $m \geqslant 3$,

$$[\mathcal{O}^*: R\eta_{m,K}^*] = \pm \frac{(n_m^*)^{dm/2}}{\delta^* \cdot m^{m/2}} \prod_{p \in \mathcal{P}} (1 + p^{-d_m})^{e_p} \prod_{\chi \in \chi_K} B_{m,x}. $$

In (b), $\mathcal{I}$ denotes the index $[\mathcal{O}: \mathcal{O}^+ \oplus \mathcal{O}^-]$ of Theorem 5.

The formulas of Theorem 6 can be considered as the equivalent of the index formulas of [5]. Possibly a suitable modification of the lattices $R\eta_{K}^*$ and $R\eta_{m,K}^*$ might cause a simplification of the right sides in Theorems 1 and 6.

**References**