Lyubeznik’s invariants
for cohomologically isolated singularities

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Received 28 January 2006
Available online 12 July 2006
Communicated by Paul Roberts

Abstract
In this note I give a description of Lyubeznik’s local cohomology invariants for a certain natural class of local rings, namely the ones which have the same local cohomology vanishing as one expects from an isolated singularity. This strengthens our results of [Manuel Blickle, Raphael Bondu, Local cohomology multiplicities in terms of étale cohomology, Ann. Inst. Fourier 55 (7) (2005)] while at the same time somewhat simplifying the proofs. Through examples I further point out the bad behavior of these invariants under reduction mod $p$.

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Keywords: Local cohomology; Characteristic $p$; Perverse sheaves

1. Introduction

Let $A = R/I$ for $I$ an ideal in a regular (local) ring $(R, m)$ of dimension $n$ and containing a field $k$ which we assume to be separably closed. The main results of [Lyu93,HS93] state that the local cohomology module $H^m_m(H^n_{I^{-1}}(R))$ is injective and supported at $m$. Therefore it is a finite direct sum of $e = e(H^m_m(H^n_{I^{-1}}(R)))$ many copies of the injective hull $E_{R/m}$ of the residue field of $R$. Lyubeznik shows in [Lyu93] that this number

$$\lambda_{a,i}(A) \overset{\text{def}}{=} e(H^m_m(H^n_{I^{-1}}(R)))$$
does not depend on the auxiliary choice of $R$ and $I$. If $A$ is a complete intersection, these invariants are essentially trivial (all are zero except $\lambda_{d,d} = 1$ where $d = \dim A$). The goal of this paper is to describe these invariants for a large class of rings, including those which are a complete intersection away from the closed point. Alternatively this class of rings can be viewed as consisting of the rings which behave cohomologically like an isolated singularity.

**Theorem 1.1.** Let $k$ be a separably closed field and let $A = \mathcal{O}_{Y,x}$ for $Y$ a closed $d$-dimensional $k$-subvariety of a smooth variety $X$. If for $i \neq d$ the modules $H^{d-i}_{[Y]}(\mathcal{O}_X)$ are supported in the point $x$ then:

1. For $2 \leq a \leq d$ one has
   \[
   \lambda_{a,d}(A) = \lambda_{0,d-a+1}(A) \quad \text{for } a \neq d \quad \text{and} \quad \lambda_{d,d}(A) = \lambda_{0,1}(A) + 1
   \]
   and all other $\lambda_{a,i}(A)$ vanish.

2. \[
   \lambda_{a,d}(A) - \delta_{a,d} = \begin{cases} \dim_{\mathbb{F}_p} H^{d-a+1}_{[Y]}(Y_{\text{ét}}, \mathbb{F}_p) & \text{if } \text{char } k = p, \\ \dim_{\mathbb{C}} H^{d-a+1}_{[Y]}(Y_{\text{an}}, \mathbb{C}) & \text{if } k = \mathbb{C}, \end{cases}
   \]
   where $\delta_{a,d}$ is the Kroneker delta function.

In the case that $A$ has only an isolated singularity and $k = \mathbb{C}$, this was shown by Garcia Lopez and Sabbah in [GLS98]. In [BB05, Theorems 1.1 and 1.2] Bondu and myself proved this result under an additional assumption, namely that $Y$ is close to $F$-rational in positive characteristic (respectively an intersection homology manifold in char. 0) away from the point $x$.

The purpose of this note is to show that this additional assumption is unnecessary—an observation which also makes the proof much clearer. The techniques and part of the proof given here is similar to the one in [BB05]² such that—without repeating it here—we heavily rely on the setup of [BB05]. Particularly, we suggest that a reader not familiar with the Emerton–Kisin correspondence first consult [BB05, Section 3.1] (better yet [EK03]) for a brief introduction to unit $\mathcal{O}_{F,X}$-modules and a summary of the results needed below.

### 2. Proof of the result

Let us first fix some notation. Let $Y \subseteq X$ be a closed subscheme of $X$. Let $X$ be smooth of dimension $n$ and $Y$ of dimension $d$. Let $x \in Y$ be a point. We take the freedom to shrink $X$ (and $Y$) since everything is local at $x$. We denote the inclusions as follows:

\[
\begin{array}{ccc}
Y & \xrightarrow{i'} & X \\
\downarrow^j & & \downarrow^{j'} \\
Y - x & \xrightarrow{i} & X - x \\
\end{array}
\]

\[\text{In [BB05] a necessary assumption that } k \text{ is separably closed was missing as pointed out to me by Brian Conrad. However in determining } \lambda_{a,i}(A) \text{ one can always reduce to the case that } k \text{ is separably closed since } \lambda_{a,i}(A) = \lambda_{a,i}(A \otimes_k k^{\text{sep}}).\]
To ease notation we carry out the argument in the case when char \(k\) is positive.\(^3\) The proof of the characteristic zero result is exactly the same, replacing the Emerton–Kisin correspondence by the Riemann–Hilbert correspondence.

**Proof of Theorem 1.1.** Part (1) is already proved in [BB05, Section 2.1] (for this part, the assumption close to \(F\)-rational of [BB05] was not used). Hence we only have to show part (2), that is that the equality

\[
\lambda_{a,d}(A) - \delta_{a,d} = \dim H^{d-a+1}(\mathbb{F}_p)
\]

holds for \(2 \leq i \leq d\).

Recall from [BB05, Section 3.1] that the functor \(\mathrm{Sol}\), which can be thought of as (a dual of) taking Frobenius fixed points, satisfies by [EK04, Example 9.3.1] that \(\mathrm{Sol}(\mathcal{O}_X) = \mathbb{F}_p[n]\) (the single term complex with \(\mathbb{F}_p\) in degree \(-n\)). By [BB05, Proposition 3.3(2)] applied to the closed immersion \(k:x \hookrightarrow X\), one computes

\[
\mathrm{Sol}(H^n_{[x]}(\mathcal{O}_X)) = \mathrm{Sol}(H^n_{[x]}R\Gamma_{[x]}(\mathcal{O}_X))
\]

\[
\cong H^{-n}k!k^{-1} \mathrm{Sol}(\mathcal{O}_X)[n]
\]

\[
\cong H^{-n}k!k^{-1}\mathbb{F}_p[n] = \mathbb{F}_p|_X.
\]

By definition, \(\lambda_{a,i}(A)\) is the unique integer \(e\) such that \(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X)) \cong (H^a_{[x]}(\mathcal{O}_X))^\oplus e\) and it follows from the preceding computation that

\[
\lambda_{a,i}(A) = e(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X))) = \dim_{\mathbb{F}_p} \mathrm{Sol}(H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X))).
\]

The main trick in the following proof is to replace \(H^{n-i}_{[Y]}(\mathcal{O}_X)\) by something more accessible—that is by something whose solutions \(\mathrm{Sol}\) can be computed easily. The rest is mere computation.

By definition of the intermediate extension\(^4\) one has an exact sequence

\[
0 \to j^!_{\eta}H^{n-i}_{[Y]}(\mathcal{O}_X)|_{X-x} \to H^{n-i}_{[Y]}(\mathcal{O}_X) \to C \to 0,
\]

where the cokernel \(C\) is supported on \(x\).\(^5\) Using the long exact sequence for \(\Gamma_{[x]}(\_\_\_)\) we get that for \(a \geq 2\)

\[
H^a_{[x]}(H^{n-i}_{[Y]}(\mathcal{O}_X)) \cong H^a_{[x]}(j^!_{\eta}(H^{n-i}_{[Y]}(\mathcal{O}_X)|_{X-x})).
\]

\(^3\) In this proof we refer to Lemmas 2.3 and 2.5 of [BB05] which are stated there for \(\mathcal{D}_X\)-modules in characteristic zero. However, as the discussion preceding Remark 3.4 of [BB05] points out, these are valid in our setting of unit \(\mathcal{O}_{X,F}\) modules in characteristic \(p\) as well.

\(^4\) The intermediate extension \(j^!_{\eta}\) of a module \((\mathcal{D}_{X-x})_\cdot\) or \(\mathcal{O}_{F,(X-x)}\)-module if in characteristic 0 or \(p\), respectively) is uniquely defined as the smallest submodule \(M'\) of \(H^0j_!(M)\) such that \(j^!M' \cong M\), where \(j:X-x \to X\) is the open inclusion of the complement of \(x\) into \(X\). The intermediate extension appears here as a substitute for \(j_!\) which does not exist in the characteristic \(p\) context. It was this realization (replace \(j_!\) with \(j^!_{\eta}\)) which made it possible to make the argument work in all characteristics.

\(^5\) The functors we call \(j_{\eta}\) and \(j^!_{\eta}\) are denoted by \(j_{\pm}\) and \(j^!_{\pm}\) in [EK04]. In order to adjust to the notation used in the Riemann–Hilbert correspondence I changed this notation here.
This first substitution, combined with [BB05, Lemma 2.3], we record as a lemma:

**Lemma 2.1.** With notation as above, and without any assumptions on the singularities of $A = \text{Spec} \mathcal{O}_{Y,x}$

$$\lambda_{a,d}(A) = \dim(H^{-a}j_!^*\text{Sol }H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}))_x$$

for $2 \leq a \leq d$.

**Proof.** By definition we have

$$\lambda_{a,d}(A) = e(H^n_{[x]}H^{n-d}_{[Y]}(\mathcal{O}_X)) = \dim(\text{Sol }H^n_{[x]}H^{n-d}_{[Y]}(\mathcal{O}_X)) = \dim(k_!k^{-1}H^{-a}\text{Sol }H^{n-d}_{[Y]}(\mathcal{O}_X)) \quad [\text{BB05, Lemma 2.3}]$$

$$= \dim(H^{-a}\text{Sol }j_!^*(H^{n-d}_{[Y]}(\mathcal{O}_X)|_{X-x}))_x,$$

where $k : x \to X$ is the inclusion of the point. The subscript $(\_)_x$ denotes the fiber at $x$ and is just an alternative way to express $k_!k^{-1}$. The commutation of Sol with $j_!^*$ [BB05, Section 3.2] now finishes the argument. \(\square\)

The assumption that $H^{n-i}_{[Y]}(\mathcal{O}_X)$ is supported at $x$ for $i \neq d$ we rephrase by saying that one has a quasi-isomorphism of complexes

$$H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}) \cong \mathcal{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d]$$

and the solutions of the latter can be computed easily\(^6\) (and is perverse!) as done in [BB05, Proof of Lemma 2.5]:

$$\text{Sol}((\mathcal{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d])) = i_!(\mathbb{F}_p)_{Y-x}[d].$$

Since $j$ is just the inclusion of the complement of a point we have that

$$j_!^*(\_)_x \cong \tau_{\leq d-1}\mathcal{R}j_!^*(\_)_x$$

by [Bor84, V.2.2 (2)] or [BBD82, Proposition 1.4.23]. Continuing the computation of Lemma 2.1 using these observations we get for $a \geq 1$ that

$$\lambda_{a,i}(A) = \dim(H^{-a}j_!^*\text{Sol }H^{n-d}_{[Y-x]}(\mathcal{O}_{X-x}))_x$$

$$= \dim(H^{-a}j_!^*\text{Sol}(\mathcal{R}\Gamma_{[Y-x]}(\mathcal{O}_{X-x})[n-d]))_x$$

$$= \dim(H^{-a}j_!^*i_!(\mathbb{F}_p)_{Y-x}[d])_x$$

$$= \dim(H^{-a}i_!^*j_!^*(\mathbb{F}_p)_{Y-x}[d])_x.$$\(^6\) In the case $k = \mathbb{C}$ this computation yields $i_!\mathbb{C}_{Y-x}[d]$ instead.
\[ = \dim ( H^{d-a} \tau \leq d-1 R f_*(\mathbb{F}_p) y^{-x})_x \]
\[ = \dim ( H^{d-a} R f_*(\mathbb{F}_p) y^{-x})_x. \]

To leave out the truncation in the last equation is allowed since \( a \geq 2 \) and hence \( d - a \leq d - 2 \). Now we can evoke [BB05, Lemma 2.7] to conclude that the last line is equal to \( H^{d-a+1} (Y_{\text{ét}}, \mathbb{F}_p) + \delta_{a,d} \) as required. \( \square \)

**Remark 2.2.** A slight refinement of the same techniques yield also to a more general statement. Namely, if one requires vanishing of \( H^a H^{n-i} (R|_{\text{Spec} R - x}) \) below the diagonal \( a = i + m \) then the result remains true in the range \( d - m + 2 \leq a \leq d \).

3. Examples

This section is to provide some examples of the bad behavior of the invariants \( \lambda_{a,i} \) under reduction to positive characteristic. The uniformity of the theorem for all characteristics seems, on the first sight, to suggest that one can expect a good behavior of the invariants under reduction mod \( p \). This impression is however quickly shattered, essentially for reasons that local cohomology is well known to behave poorly under reduction. Alternatively one also can observe that the cohomology theory corresponding to \( H^i (Y, \mathbb{C}) \) under reduction is not \( H^i (Y_{\text{ét}}, \mathbb{F}_p) \) but rather \( p \)-adic rigid cohomology or crystalline cohomology, of which \( H^i (Y_{\text{ét}}, \mathbb{F}_p) \) is only a small part, namely the slope zero part.

The examples that follow are standard examples for the bad behavior of local cohomology under reduction mod \( p \). I learned them from a talk by Anurag Singh at Oberwofach in March 2005. Our general setup is as follows: Let \( A = R/I \) where \( R \) is a polynomial ring over \( \mathbb{Z} \) and \( I \) is a homogeneous ideal. We denote by \( A_0 = A \otimes \mathbb{C} \) the generic characteristic zero model and by \( A_p = A \otimes \mathbb{F}_p \) for all \( p \) prime the positive characteristic models.

**Example 3.1.** Let \( R = \mathbb{Z}[u, v, w, x, y, z] \) and \( I = (\delta_1, \delta_2, \delta_3) \) be the ideal of \( 2 \times 2 \) minors of the displayed matrix of variables. Then \( A = R/I \) has a free resolution (as an \( R \)-module) given by

\[
0 \to R^2 \xrightarrow{(u \ y \ w \ z)} R^3 \xrightarrow{(\delta_1 \ \delta_2 \ \delta_3)} R \to 0.
\]

This shows that \( \text{Ext}^3 (R/I, R) = 0 \) and therefore, reducing mod \( p \), that \( \text{Ext}^3 (R/p I/p^e, R_p) = 0 \) for all \( e \) by the flatness of the Frobenius. This implies that \( H_1^3 (R_p) = 0 \) as well since in positive characteristic \( H_1^i (R_p) = \lim Ext^i (R/p I/p^{e-1}, R_p) \). On the other hand, it is well known that in zero characteristic, \( H_1^3 (R_{\mathbb{Q}}) \) is not zero. Hence this provides an example where for the characteristic zero model we have \( \lambda_{0,3} = \lambda_{2,4} \neq 0 \) whereas in all positive characteristics \( \lambda_{0,3} = \lambda_{2,4} = 0 \).

The next example even shows that the vanishing of \( \lambda_{a,i} \) can vary in an arithmetic progression:

**Example 3.2.** Let \( A \) be the homogeneous coordinate ring of \( \mathbb{P}^1 \times E \) where \( E \) is the elliptic curve \( \text{Proj} \mathbb{Z}[x,y,z]/x^3 + y^3 + z^3 \). Then \( A \) is given as the quotient of \( R \) (as above) by the ideal

\[
I = (\delta_1, \delta_2, \delta_3, x^3 + y^3 + z^3, ux^2 + vy^2 + zw^2, u^2 x + v^2 y + w^2 z, u^3 + v^3 + w^3).
\]
The resolution of $A = R/I$ can be computed to be equal to

$$0 \rightarrow R \rightarrow R^6 \rightarrow R^{11} \rightarrow R^7 \rightarrow R \rightarrow 0$$

and one verifies that $H^4_I(R_p) = 0$ if and only if $\text{char } k \equiv 2 \mod 3$ (this essentially follows from the fact that depending on the modulus of $p \mod 3$ the elliptic curve is supersingular or not). Hence we have that $\lambda_{0,2} = \lambda_{2,3} = 0$ if and only if $\text{char } k \equiv 2 \mod 3$.

References


