Variations on a theorem by Alan Camina on conjugacy class sizes

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Abstract

Let $G$ be a finite group. We extend Alan Camina’s theorem on conjugacy class sizes which asserts that if the conjugacy class sizes of $G$ are exactly $\{1, p^a, q^b, p^aq^b\}$ for two primes $p$ and $q$, then $G$ is nilpotent. If we assume that $G$ is solvable, we show that when the set of conjugacy class sizes of $G$ is $\{1, m, n, mn\}$ with $m$ and $n$ arbitrary positive integers such that $(m, n) = 1$, then $G$ is nilpotent and $m = p^a$ and $n = q^b$ for two primes $p$ and $q$.

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1. Introduction

We will assume that any group is finite. It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist several results studying the solvability or the nilpotence of a group under some arithmetical conditions on its conjugacy class sizes. N. Itô shows in [12] that if the sizes of the conjugacy classes of a group $G$ are $\{1, m\}$, then $G$ is nilpotent, $m = p^a$ for some prime $p$ and $G = P \times A$, with $P$ a Sylow $p$-subgroup of $G$ and $A \subseteq Z(G)$. There exist other deeper

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results. For instance, in [13], Itô shows that if the conjugacy class sizes of $G$ are $\{1, n, m\}$, then $G$ is solvable. D. Chillag and M. Herzog prove in [7] that if 4 does not divide any conjugacy class size of $G$, then $G$ is solvable. Later, A.R. Camina and R.D. Camina gave a proof of that result independent of the Classification of Simple Finite Groups in [5]. On the other hand, A.R. Camina proves in [6] that if the conjugacy class sizes of $G$ are $\{1, p^a, q^b, p^aq^b\}$, with $p$ and $q$ two distinct primes, then $G$ is nilpotent. Notice that the hypotheses of Camina’s theorem imply the solvability of $G$ just by using Burnside’s $p^aq^b$-theorem.

In the introduction of [6], Camina asserts that it seems extremely likely that a group whose conjugacy class sizes satisfy the following property is solvable: If $m$ and $n$ are the cardinals of two distinct conjugacy classes of $G$ with $m \leq n$, then either $m$ divides $n$ and $(n/m, m) = 1$, or $(m, n) = 1$ and there is a class of size $mn$. One particular case of this property is when the set of such cardinals is exactly $\{1, n, m, nm\}$ with $(n, m) = 1$, however it seems difficult to prove the solvability of such groups. In this paper, we prove the following.

**Theorem A.** Let $G$ be a solvable group and suppose that the conjugacy class sizes of $G$ are $\{1, n, m, nm\}$ with $(m, n) = 1$. Then $G$ is nilpotent and $n = p^a$ and $m = q^b$ for some distinct primes $p$ and $q$.

In order to show Theorem A, we will first prove a particular case which is also an extension of Camina’s theorem. We also present a new proof of it, without making use of some results due to I.M. Isaacs and D.S. Passman in [11] on primitive permutation groups which appeared in the original proof. Such an extension is the following.

**Theorem B.** Let $G$ be a solvable group and suppose that the conjugacy class sizes of $G$ are $\{1, p^a, n, p^an\}$ with $(p, n) = 1$ and $a \geq 0$. Then $G$ is nilpotent and $n = q^b$ for some prime $q$.

Recently, there have appeared some papers analyzing the $p$-structure of $p$-solvable groups when some arithmetical conditions on the sizes of the conjugacy classes of $p'$-elements are imposed (see, for instance, [2,3] or [14]). More precisely, in the proofs of Theorems A and B we will use the main result of [3]. We believe that it is remarkable how we use these results related to local information of a group to obtain global information on the structure of the group.

We will denote by $x^G$ the conjugacy class of $x$ in $G$ and we call $|x^G|$ the index of $x$ in $G$. The rest of the notation is standard.

2. Preliminary results

We will need the following elementary results on conjugacy classes of $\pi$-elements where $\pi$ is an arbitrary set of primes.

**Lemma 1.** Let $G$ be a $\pi$-separable group.
(a) The conjugacy class size of any $\pi'$-element of $G$ is a $\pi$-number if and only if $G$ has abelian Hall $\pi'$-subgroups.

(b) The conjugacy class size of every $\pi$-element of $G$ is a $\pi$-number if and only if $G = H \times K$, where $H$ and $K$ are a Hall $\pi$-subgroup and a $\pi$-complement of $G$, respectively.

Proof. (a) is easy to prove by arguing on induction on the order of $G$, and (b) is exactly [4, Lemma 8].

We stress that Lemma 1 implies that if $G$ is $p$-solvable and $p$ does not divide any conjugacy class size, then $G$ has a central Sylow $p$-subgroup. In fact, the hypothesis of $p$-solvability is not needed, as the following result shows.

Lemma 2. Let $G$ be a group. A prime $p$ does not divide any conjugacy class size of $G$ if and only if $G$ has a central Sylow $p$-subgroup.

Proof. See, for instance, [9, Theorem 33.4].

We will use the following result due to N. Itô, which characterizes the structure of those groups which possess only two conjugacy class sizes.

Theorem 3. Suppose that $1$ and $m > 1$ are the only lengths of conjugacy classes of a group $G$. Then $G = P \times A$, where $P \in \text{Syl}_p(G)$ and $A$ is abelian. In particular, then $m$ is a power of $p$.

Proof. See [9, Theorem 33.6].

The authors obtained in [3] the following generalization of Itô’s theorem for $p$-regular conjugacy classes in $p$-solvable groups.

Theorem 4. Suppose that $G$ is a finite $p$-solvable group and that $\{1, m\}$ are the $p$-regular conjugacy class sizes of $G$. Then $m = p^a q^b$, with $q$ a prime distinct from $p$ and $a, b \geq 0$. If $b = 0$ the $G$ has abelian $p$-complement. If $b \neq 0$, then $G = P Q \times A$, with $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$ and $A \subseteq \mathbb{Z}(G)$. Furthermore, if $a = 0$ then $G = P \times Q \times A$.

Proof. This is exactly [3, Theorem A].

We will also make use of the classic Thompson’s $A \times B$-Lemma.


Proof. See, for instance, [1, 24.2].

We will prove the following result on conjugacy class sizes.
Lemma 6. Suppose that the three smallest non-trivial indices of elements of a group $G$ are $a < b < c$, with $(a, b) = 1$ and $a^2 < c$. Then the set \{ $g \in G$: $|g^G| = 1$ or $a$ \} is a normal subgroup of $G$.

Proof. Let $C_1, C_2, \ldots , C_s$ be the distinct conjugacy classes of $G$, and write $K_i$ for the class sum of the elements of $C_i$. It is well known that

$$K_i K_j = \sum_{r=1}^{s} a_{ijr} K_r$$

where $a_{ijr}$ is a non-negative integer for all $i, j, r = 1, \ldots , s$. In addition, by [8, 87.4], for instance, for $1 \leq i, j, r \leq s$ there exists a non-negative integer $l$ such that

$$a_{ijr} = \frac{|C_j|}{|C_r|} l.$$ 

Now, assume that $C_i$ and $C_j$ are two classes of size $a$ and notice that

$$a^2 = \sum_{r=1}^{s} a_{ijr} |C_r|.$$ 

Since $a^2 < c$, this shows that if $|C_r| \geq c$, then $a_{ijr} = 0$. Moreover, if $|C_r| = b$ then $a_{ijr} = al/b$ for some $l \geq 0$, so in particular, $b$ divides $l$ and $a$ divides $a_{ijr}$. Thus $ab$ must divide $a_{ijr} |C_r|$, which forces $a_{ijr} = 0$. From these facts we deduce that \{ $g \in G$: $|g^G| = 1$ or $a$ \} is a (normal) subgroup of $G$. \(\Box\)

Notice that if the solvability hypothesis of Theorem A is eliminated, then by Lemma 6, it follows in the thesis of the theorem that $G$ is not simple. Finally, we will use the following result due to A. Camina.

Lemma 7. Let $G$ be a group such that $p^a$ is the highest power of the prime $p$ which divides the index of an element of $G$. Assume that there is a $p$-element in $G$ whose index is precisely $p^a$. Then $G$ has a normal $p$-complement.

Proof. This is [6, Theorem 1]. \(\Box\)

3. Proof of Theorem B

As we have pointed out in the introduction, we will also give a new proof of Camina’s theorem in the proof of Theorem B.

Proof of Theorem B. The proof has been divided into several steps.

Step 1. If $G$ is $p$-nilpotent then the theorem is proved.
Suppose that $G$ is $p$-nilpotent and let $H$ be the normal $p$-complement of $G$. For every $x \in H$ we have


If $|x^G| = 1$ or $p^a$, then $H \subseteq C_G(x)$ and thus $|x^H| = 1$. If $|x^G| = n$ or $p^a n$, then the above equality along with the fact that $|x^H|$ divides $|x^G|$ imply that $|x^H| = n$. Therefore, any conjugacy class of $p'$-elements of $G$ has size 1 or $n$, and by Theorem 4, we have that $n = p^a q^b$ for some prime $q \neq p$. Since $(n, p) = 1$, then $n = q^b$ and again by Theorem 4, we conclude that $G$ is nilpotent, so the theorem is proved.

**Step 2.** We may assume that there are no $p'$-elements of index $p^a$. Consequently, there exists some $p'$-element of index $p^a$.

If $G$ has a $p'$-element of index $p^a$ then by Lemma 7, $G$ is $p$-nilpotent and the theorem is proved by Step 1. In order to see the consequence in this step it is enough to consider the decomposition of any element of index $p^a$ as a product of a $p$-element by a $p'$-element.

**Step 3.** We may assume that there are no $p'$-elements of index $n$. Consequently, there exist $p$-elements of index $n$.

Suppose that $y$ is a $p'$-element of index $n$. Notice that the Sylow $p$-subgroups of $G$ cannot be central. Thus, we can choose some non-central $p$-element $x \in C_G(y)$ and then $C_G(xy) = C_G(x) \cap C_G(y)$ and $|C_G(y) : C_G(x) \cap C_G(y)|$ must be equal to 1 or $p^a$. Hence, any $p$-element of $C_G(y)$ has index 1 or $p^a$ in $C_G(y)$. By Lemma 1(b), we can write $C_G(y) = P_x \times V_y$, with $P_x \in Syl_p(G)$ and $V_y$ a $p'$-group. Now, choose $H$ a $p'$-complement of $G$ such that $V_y \subseteq H$. By Step 2, there exists some $p'$-element, say $t$, of index $p^a$ and up to conjugacy we may assume that $H \subseteq C_G(t)$. Therefore, $y \in V_y \subseteq C_G(t)$, so $t \in C_G(y)$ and thus, $t \in V_y$. In particular, $P_y \subseteq C_G(t)$, contradicting the fact that $t$ has index $p^a$.

The consequence in the statement follows as in the above step.

**Step 4.** If $x$ is a $p$-element of index $p^a n$, then $C_G(x) = P_x \times V_x$ with $P_x$ a $p$-group and $V_x$ an abelian $p'$-group such that $V_x \not\subseteq Z(G)$. If $y$ is a $p'$-element of index $p^a n$, then $C_G(y) = P_y \times V_y$ with $P_y$ an abelian $p$-group such that $P_y \not\subseteq Z(G)$ and $V_y$ a $p'$-group.

Let $x$ be a $p$-element of index $p^a n$ and let $y$ be any $p'$-element of $C_G(x)$. Notice that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and since $p^a n$ is the largest class size of $G$, then $C_G(xy) = C_G(x)$, so $C_G(x) \subseteq C_G(y)$. This implies that $y \in Z(C_G(x))$, so we can write $C_G(x) = P_x \times V_x$ with $P_x$ a $p$-group and $V_x$ an abelian $p'$-group. It remains to show that $V_x$ cannot be central in $G$.

Suppose that $V_x \subseteq Z(G)$, and notice that then $V_x = Z(G)_{p'}$ and $|G : Z(G)|_{p'} = n$. Choose $z$ a non-central $p$-element, which must have index $n$ or $p^a n$ by Step 2. In every case, notice that $Z(G)_{p'}$ is a $p'$-complement of $C_G(z)$. This implies that if we choose any non-central $p'$-element $w$ of $G$, then any $p$-element of $C_G(w)$ must be central in $G$. 

Thus $Z(G)_p$ is a Sylow $p$-subgroup of $C_G(w)$. Since $w$ has index $p^a$ or $p^a n$, then $|G : Z(G)|_p = p^a$. This yields

$$|G : Z(G)| = |G : Z(G)|_p |G : Z(G)|_{p'} = p^a n,$$

which contradicts the existence in $G$ of elements of index $p^a n$. Thus, the first assertion of the step is proved.

The second part of this step can be proved by reasoning in a similar way with a $p'$-element of index $p^a n$.

**Step 5.** $n = q^b$ or $n = q^b r^c$ for some primes $q$ and $r$ distinct from $p$. Consequently, in the first case we can assume that $G$ is a $\{p, q\}$-group, and in the second one that $G$ is a $\{p, q, r\}$-group.

By Step 2, we can choose a $p'$-element, say $y$, of index $p^a$. Furthermore, if we consider the primary decomposition of $y$ as a product of elements of prime power order, it is immediate that we can assume $y$ to be a $q$-element for some prime $q \neq p$. Now if we take a $q'$-element $w$ of $C_G(y)$, we have $C_G(wy) = C_G(w) \cap C_G(y)$ and $|C_G(y) : C_G(w) \cap C_G(y)|$ must be 1 or $n$. This proves that any $q'$-element of $C_G(y)$ has index 1 or $n$ in $C_G(y)$, so we can apply Theorem 4 to conclude that $n = q^b r^c$, with $b, c \geq 0$ and $r$ some prime distinct from $q$ and $p$ (since $(n, p) = 1$). Therefore, the first assertion of this step follows. The second assertion follows by applying Lemma 2.

**Step 6.** If $p^a > n$, then the set

$$L_p := \{ x : x \text{ is } p\text{-element and } |x^G| = 1 \text{ or } n \}$$

is an abelian normal $p$-subgroup of $G$. If $p^a < n$, then the set

$$L_{p'} := \{ x : x \text{ is } p'\text{-element and } |x^G| = 1 \text{ or } p^a \}$$

is an abelian normal $p'$-subgroup of $G$.

It is enough to apply Lemma 6 to obtain that if $p^a > n$ then the set $W := \{ x : |x^G| = 1 \text{ or } n \}$ is a normal subgroup of $G$. Analogously, if $p^a < n$, then the set $W' := \{ x : |x^G| = 1 \text{ or } p^a \}$ is a normal subgroup of $G$.

Now, if $x$ is any element of index $n$ and factorize $x = x_p x_{p'}$, with $x_p$ and $x_{p'}$ a $p$-element and a $p'$-element, respectively, it follows that $x_{p'}$ must be central by Step 2, whence $x \in L_p \times Z(G)_{p'}$. Therefore, $W = L_p \times Z(G)_{p'}$ and $L_p$ is also a normal $p$-subgroup of $G$. The argument for $L_{p'}$ is similar.

Finally, we see for instance that $L_p$ is abelian, as the argument for $L_{p'}$ is the same. If we take any $y \in L_p$ then $|L_p : C_{L_p}(y)|$ divides $(|L_p|, n) = 1$. Consequently, $L_p$ is abelian.

For the rest of the proof we fix the following notation. If $p^a < n$, we define

$$L_s := \{ x : x \text{ is an } s\text{-element and } |x^G| = 1 \text{ or } p^a \}$$
for any prime $s$ dividing $n$. Notice that $L_s$ is an abelian normal subgroup of $G$ by Step 6. Moreover, by Step 5, we have $n = q^b$ or $n = q^b r^c$ for two primes $q$ and $r$ distinct from $p$, so $s \in \{q, r\}$. We are going to distinguish three cases: $p^a > n$, $n = q^b > p^a$ and $n = q^b r^c > p^a$ with $b, c > 0$.

**Step 7.**

(7.1) If $p^a > n$, then $L_p$ is an abelian normal Sylow $p$-subgroup of $G$.

(7.2) If $n = q^b > p^a$, then $G$ is $p$-nilpotent and the theorem is proved.

(7.3) If $n = q^b r^c > p^a$ with $b, c > 0$ and if $L_s \subseteq Z(G)$ for some $s \in \{q, r\}$ (observe that by Step 2 such a prime $s$ must exist), then $L_s$ is an abelian normal Sylow $s$-subgroup of $G$.

(7.1) In order to prove that $L_p$ is a Sylow $p$-subgroup of $G$ it is enough to show, by taking into account Step 2, that there are no $p$-elements of index $p^a n$. Suppose that $z$ is a $p$-element of index $p^a n$ and by Step 4, write $C_G(z) = P_z \times V_z$, with $V_z$ a non-central abelian $p'$-group and $P_z$ a $p$-group. If $t \in V_z$, it is clear that $C_G(z) \subseteq C_G(t)$, so in particular $C_{L_p}(z) \subseteq C_{L_p}(t)$. By applying Theorem 5, we get $t \in M := C_G(L_p)$ and therefore, $V_z \subseteq M$. On the other hand, by Step 3, we know that $t$ has index $p^a$ or $p^a n$, so $|C_G(t) : C_G(z)|$ must be equal to $1$ or $n$. This proves that $L_p \subseteq C_G(z)$ and we conclude that $L_p$ centralizes every $p$-element of index $p^a n$. But on the other hand, any $p$-element of index $n$ trivially centralizes $L_p$ as it is abelian. Therefore, we conclude that any $p$-element of $G$ lies in $M$, whence $|G : M|$ is a $p'$-number. Furthermore, since $L_p \subseteq M \subseteq C_G(k)$ for any $k$ non-central element of $L_p$, which has index $n$, then $n$ must divide $|G : M|$. Now, if we consider the equality

$$|G : M| |M : V_z| = |G : C_G(z)| |C_G(z) : V_z|,$$

then all the properties remarked above imply that $V_z$ is a $p$-complement of $M$.

Let $x$ be a $p$-element of $G$, which we know lies in $M$. If $x$ has index $1$ or $n$, then it certainly follows that $x \in Z(M)$. If $x$ has index $p^a n$, then by Step 4, we write $C_G(x) = P_x \times V_x$ with $V_x$ a non-central abelian $p'$-group and $P_x$ a $p$-group. As we have seen above, $V_x$ is a $p$-complement of $M$, and in particular $V_x \subseteq C_M(x)$ and $|M : C_M(x)|$ is a $p$-number. Therefore, we have shown that the index of any $p$-element of $M$ is a $p$-number. Thus, by applying Lemma 1(b), we can factor $M = P \times T$, where $P \in \text{Syl}_p(G)$ and $T$ is a $p'$-group, which must be equal to $V_x$. In particular, $P$ is normal in $G$. But now, if we choose some non-central $y \in V_x$, then $P \subseteq C_G(y)$, which contradicts Step 3.

(7.2) If $n = q^b > p^a$, we can argue as in case (7.1) to show that $L_q$ is a normal Sylow $q$-subgroup of $G$. But in this case we know that $G$ is a $\{p, q\}$-group by Step 5, so $G$ is $p$-nilpotent and the theorem is proved by Step 1.

(7.3) In this case, by Step 5, $G$ can be assumed to be a $\{p, q, r\}$-group. Moreover, we can assume without loss of generality that the fixed prime $s$ of the statement is, for instance, $q$ and we will prove that $L_q$ is a Sylow $q$-subgroup of $G$.

To prove this, since we know that $L_q$ is an abelian normal subgroup of $G$, it is sufficient to show that $G$ does not possess $q$-elements of index $p^a n$. Suppose that $w$ is such an
element and write, by Step 4, $C_G(w) = P_w \times V_w$, with $V_w$ a $p'$-group and $P_w$ a non-central abelian $p$-group. If $u \in P_w$, it is clear that $C_G(w) \subseteq C_G(u)$, so in particular $CL_q(w) \subseteq C_L_q(u)$. By applying Theorem 5, we get $u \in N \coloneqq C_G(L_q)$. Consequently, $L_q$ centralizes any $p$-element of $C_G(w)$, that is, $P_w \subseteq N$. On the other hand, if we take some non-central $u \in P_w$, then it has index $n$ or $p^a n$ by Step 2, so $|C_G(u) : C_G(w)|$ must be 1 or $p^a$. This proves that $L_q \subseteq C_G(w)$ and hence, $L_q$ centralizes every $q$-element of index $p^a n$. But also, any $q$-element of index $p^a$ trivially centralizes $L_q$ as it is abelian. Therefore, any $q$-element of $G$ lies in $N$, so in particular, $N \subseteq C_G(y)$ for any $y \in L_q$ and $p^a$ divides $|G : N|$. Now, if we consider the equality

$$|G : N||N : P_w| = |G : C_G(w)||C_G(w) : P_w|,$$

then all the above properties imply that $P_w \in Syl_{p}(N)$.

We claim that the index in $N$ of any $q$-element (which lies in $N$) is either 1 or a fixed $p'$-number. Let $y$ be a $q$-element of $G$, which we know that has index 1, $p^a$ or $p^a n$. If $y$ has index 1 or $p^a$, then certainly $y \in Z(N)$ and the claim is proved. Assume then that $y$ has index $p^a n$. As in the above paragraph, we can write $C_G(y) = P_y \times V_y$ with $V_y$ a $p'$-group and $P_y$ a non-central abelian $p$-group. However, we have seen that $P_y \in Syl_{p}(N)$, so in particular, $P_y \subseteq C_N(y)$ and $|N : C_N(y)|$ is a $p'$-number. Let $t$ be a $q'$-element of $C_G(y)$ and notice that $C_G(yt) = C_G(y) \cap C_G(t) \subseteq C_G(y)$. Hence, $C_G(yt) = C_G(y)$ and $C_G(y) \subseteq C_G(t)$. Therefore, $t \in Z(C_G(y))$ and we may write $C_G(y) = Q_y \times T_y$ with $Q_y$ a $q$-group and $T_y$ an abelian $q'$-group, which moreover cannot be central in $G$ since $P_y \subseteq T_y$.

In addition, if we choose any non-central $t \in T_y$, we get $C_G(y) \subseteq C_G(t)$. In particular, $CL_q(y) \subseteq CL_q(t)$, and by Theorem 5, we obtain that $L_q \subseteq C_G(t)$, whence $T_y \subseteq N$. Since we know that any $q$-element lies in $N$, we conclude that $C_G(y) \subseteq N$. Now the following equality

$$|G : N||N : C_G(y)| = p^a n,$$

with the fact that $p^a$ divides $|G : N|$, force $|N : C_G(y)|$ to be a fixed $p'$-number, $m := p^a n / |G : N|$ for every $q$-element $w$ of index $p^a n$. Thus, the claim of this paragraph is proved.

Now, we will show that any $p$-element of $N$ has also index 1 or $m$ in $N$. Let $x$ be a non-central $p$-element of $N$. Up to conjugacy, we can assume, for instance, that $x \in P_w$ where $P_w \times V_w$ is the decomposition of $C_G(w)$ and $w$ is a fixed $q$-element of index $p^a n$, given at the beginning of this case. It is clear that $C_G(w) \subseteq C_G(x)$ and then $|x^G| = n$ or $p^a$. We also know that $C_G(w) \subseteq N$ by the above paragraph, so $P_w \subseteq C_G(w) \subseteq C_N(x)$. Also, as $P_w$ is a Sylow $p$-subgroup of $N$, then $|C_N(x) : C_G(w)|$ is a $p'$-number. If $|x^G| = n$, then the following equalities

$$|G : N||N : C_N(x)||C_N(x) : C_G(w)| = p^a n = |G : C_G(x)||C_G(x) : C_G(w)|$$

imply that $C_G(w) = C_N(x)$ and $|N : C_N(x)| = m$. In the other case, that is, when $|x^G| = p^a$ then $C_G(w) = C_N(x)$ and $|N : C_N(x)| = m$, as we wanted to prove.
Finally, we will show that any \( \{p, q\} \)-element of \( N \) has also index 1 or \( m \). Let \( x \) be a non-central \( \{p, q\} \)-element of \( N \) and write \( x = x_p x_q \), where \( x_p \) and \( x_q \) are the \( p \)-part and the \( q \)-part of \( x \). We have \( C_G(x) = C_G(x_p) \cap C_G(x_q) \) and we distinguish three possibilities for the index of \( x_q \) in \( G \). If \( x_q \) is central in \( G \), then \( G_G(x) = C_G(x_p) \) and \( C_N(x) = C_N(x_p) \), so \( |x^{N}| = |x_p^{N}| = 1 \) or \( m \) according to the above paragraph. If \( |x_q^{G}| = p^a \) then \( x_q \in L_q \), so \( x_q \in Z(N) \), so \( C_N(x) = C_N(x_p) \) and again by the above paragraph we get \( |x^{N}| = 1 \) or \( m \). Finally, if \( |x_q^{G}| = p^a m \), it follows that \( C_G(x) = C_G(x_q) \) and \( C_N(x) = C_N(x_q) \), so \( |x^{N}| = |x_q^{N}| = 1 \) or \( m \), since we have proved above that all \( q \)-elements in \( N \) also have index 1 or \( m \) in \( N \).

Now we are able to apply Theorem 4 and obtain that \( N = R Q \times A \), with \( R \in \text{Syl}_q(N) \), \( Q \in \text{Syl}_p(G) \) and \( A \) abelian. In particular, the non-central \( p \)-elements of \( N \), which exist because \( P_w \subseteq N \), have index not divisible by \( q \), which is a contradiction with Step 2.

**Step 8.**

(8.1) If \( p^a > n \), then the \( p \)-complements of \( G \) are abelian.

(8.2) If \( n = q^b r^c > p^a \), with \( b, c > 0 \), then the Sylow \( p \)-subgroups of \( G \) are abelian.

(8.1) Let \( H \) be a \( p \)-complement of \( G \) and assume that it is not abelian. By Lemma 1(a) and Step 3, there exist \( p \)-elements in \( H \) of index \( p^a n \). Let \( w \) be any such element. By Step 4, we write \( C_G(w) = P_w \times V_w \) with \( P_w \) an abelian \( p \)-group such that \( P_w \not\subseteq Z(G) \) and \( V_w \) a \( p \)-group. We will prove that \( V_w \) is abelian too. We may choose a non-central \( p \)-element \( u \in C_G(w) \), which certainly satisfies \( C_G(w) \subseteq C_G(u) \). By (7.1), we know that \( |u^G| = n \), so \( |C_G(u) : C_G(w)| = p^a \). Therefore, \( V_w \) is a \( p \)-Hall subgroup of \( C_G(u) \). On the other hand, if \( v \) is a \( p \)-element of \( C_G(u) \), then \( |C_G(u) : C_G(uv)| = |C_G(u) : C_G(u) \cap C_G(v)| \) is a power of \( p \). Thus, by Lemma 1(b), \( C_G(u) \) has abelian Hall \( p \)-subgroups. So \( V_w \) is abelian as we wanted to show and consequently, \( C_G(w) \) is abelian too.

If \( Z(H) = Z(G) \), then there would not be \( p \)-elements of index \( p^a \), and this yields a contradiction with Step 2. Thus there exist non-central elements in \( Z(H) \). For any such element, say \( y \), note that \( y \in C_G(w) \) and as \( C_G(w) \) is abelian, we have \( C_G(w) \subseteq C_G(y) = C_{L_p}^{(y)} H \). Moreover, since \( L_p \subseteq G \), we have \( C_{L_p}^{(y)} \subseteq C_G(y) \). Since \( H \subseteq C_G(y) \) and \( L_p \) is abelian, it follows that \( T := C_{L_p}^{(y)} \subseteq G \). Furthermore, as \( |C_G(y) : C_G(w)| = n \), it follows that \( T \) is the Sylow \( p \)-subgroup of \( C_G(w) \), so \( T = P_w \) and in particular, \( T \) is not central in \( G \). Notice that we have also proved that \( T \) centralizes any \( p \)-element in \( H \) of index \( p^a n \) and any element in \( Z(H) \).

Now, if we take \( v \in H \) of index \( p^a \), then there exists some \( g \in G \) such that \( H^g \subseteq C_G(v) \), whence \( v^{-1} g^{-1} \in Z(H) \). By the above paragraph, \( T \subseteq C_G(v^{-1} g^{-1}) \) and as \( T \) is normal in \( G \), we get that \( T \) also centralizes \( v \). Then \( T \subseteq C_G(H) \) and as \( L_p \) is abelian, we conclude that \( T \subseteq Z(G) \), a contradiction.

(8.2) In this case, we know that \( G \) is a \( \{p, q, r\} \)-group by Step 5. For one prime in \( \{q, r\} \), say \( q \), we can assume without loss that \( L_q \) is non-central in \( G \), so by (7.3), \( L_q \) is an abelian normal Sylow \( q \)-subgroup of \( G \). If \( L_r \) is also non-central, then \( L_r \) is, again by (7.3), an abelian normal Sylow \( r \)-subgroup of \( G \). Consequently, \( L_p = L_q \times L_r \) would be an abelian normal \( p \)-complement of \( G \), so by Step 1, the theorem is proved. Accordingly, we
may assume that \( L_r \subseteq Z(G) \), that is to assume that every non-central \( r \)-element of \( G \) has index \( p^a n \).

Let \( P \in \text{Syl}_p(G) \) and suppose that \( P \) is not abelian. We will work to get a contradiction. By Step 2, there exist \( p \)-elements of index \( p^a n \). Let \( z \in P \) be any such element. By Step 4, we write \( C_G(z) = P_z \times V_z \), where \( V_z \) is a abelian \( p' \)-group with \( V_z \not\subseteq Z(G) \) and \( P_z \) is a \( p \)-group. Let \( R_0 \) be a Sylow \( r \)-subgroup of \( C_G(z) \) and let \( R_0 \cap R \) where \( R \) is a Sylow \( r \)-subgroup of \( G \). If \( R_0 \) is central, there can be no \( r \)-elements of index \( p^a n \) as \( |R : R_0| = r^e \). But then \( R_0 \) contains \( r \)-elements, say \( w \), of index \( p^a n \). Since \( w \in V_z \), then \( C_G(z) \subseteq C_G(w) \), so \( C_G(w) = C_G(z) \). By applying Step 4, we conclude that \( C_G(z) \) is abelian. This is true for all \( z \in P \) of index \( p^a n \).

On the other hand, notice that there must exist \( p \)-elements in \( Z(P) - Z(G)_p \), otherwise there would not exist \( p \)-elements of index \( n \), a contradiction with Step 3. For any \( x \in Z(P) - Z(G)_p \) and for any \( z \in P \) of index \( p^a n \), we have \( C_G(z) \subseteq C_G(x) \). Since \( L_q \subseteq G \), then \( C_{L_q}(x) \subseteq C_G(x) \) and as \( |C_G(x) : C_G(z)| = p^a \), it follows that \( T := C_{L_q}(x) \) is the Sylow \( q \)-subgroup of \( C_G(z) \) for all \( z \in P \) of index \( p^a n \). Furthermore, we observe that \( T \) does not depend on the choice of \( x \).

Now, let \( R \) be a Sylow \( r \)-subgroup of \( G \) and note that \( G = R P L_q \). Let \( y \) be a non-central element of \( R \), which we know that has index \( p^a n \) by the first paragraph (and this element exists because \( r \) divides \( n \)). Again by Step 4, we write \( C_G(y) = P_y \times V_y \), with \( P_y \) an abelian \( p' \)-group such that \( P_y \not\subseteq Z(G) \) and \( V_y \) a \( p' \)-group. If we take \( k \in P_y \), then \( C_G(y) \subseteq C_G(k) \). We distinguish two possibilities for the index of \( k \). If \( |k^G| = p^a n \), then \( C_G(y) = C_G(k) \) and there exists \( g \in L_q R \) such that \( P_y \subseteq P^g \). Also, by the above paragraph we observe that \( T^g \) must be the Sylow \( q \)-subgroup of \( C_G(k) = C_G(y) \). If \( |k^G| = n \), we may choose \( g \in L_q R \) such that \( P_y \subseteq P^g \subseteq C_G(k) \). Then \( C_{L_q}(k) \) is the Sylow \( q \)-subgroup of \( C_G(k) \) and we know that \( C_{L_q}(k) \) is the Sylow \( q \)-subgroup of \( C_G(u) \) for all \( u \in P^g \) of index \( p^a n \). Hence, \( C_{L_q}(k) = T^g \), for some \( g \in L_q R \). Since \( |C_G(k) : C_G(y)| = p^a \), we have \( T^g \subseteq C_G(y) \). Therefore, we have proved that for any non-central \( y \in R \) there exists some \( g \in L_q R \) such that \( T^g \) is a Sylow \( q \)-subgroup of \( C_G(y) \). This yields

\[
R \subseteq \bigcup_{g \in L_q R} C_{L_q R}(T^g)
\]

and hence,

\[
L_q R = \bigcup_{g \in L_q R} C_{L_q R}(T^g) L_q,
\]

which implies that \( L_q R = C_{L_q R}(T)L_q \) and then \( R \subseteq C_G(T^g) \) for some \( g \in L_q R \). We define \( H := R^{g^{-1}} L_q \) and observe that since \( L_q \) is abelian then \( H \subseteq C_G(T) \).

We will prove now that \( P \subseteq C_G(T) \). We have seen above that \( T \subseteq C_G(z) \) for all \( z \in P \) of index \( p^a n \), so we only have to show that \( T \) also centralizes any element in \( P \) of index \( n \). Let \( x \in P \) such that \( |x^G| = n \). Then there exists some \( g \in H \) such that \( P^g \subseteq C_G(x) \), whence, \( x^{g^{-1}} \in Z(P) \). We know then that \( C_{L_q}(x^{g^{-1}}) \) is a Sylow \( q \)-subgroup of \( C_G(z) \) for all \( z \in P \) of index \( p^a n \) too, so \( T = C_{L_q}(x^{g^{-1}}) \). As \( g \) centralizes \( T \), we obtain \( T = T^g \subseteq C_G(x) \). We conclude that \( P \subseteq C_G(T) \), as required.
The above paragraphs show that $T \subseteq Z(G)$. But now, if $y$ is a non-central element of $R$ of index $p^an$, then, as we have seen above, there exists some $g \in G$ such that $T^g$ is a central Sylow $q$-subgroup of $C_G(y)$. As $L_q$ is non-central, we can take some $v$ of index $p^a$ and we have that $C_G(v)$ must contain some Sylow $r$-subgroup. This contradicts the above assertion.

**Step 9 (Conclusion).** (9.1) Assume first that $p^a > n$. We claim that each prime divisor $s$ of $n$ satisfies $|G : Z(G)| = n_s$, so we will get $|G : Z(G)|_{p'} = n$. Suppose that this is proved. Let $z$ be an element of index $p^an$ and write $z = z_p'z_{p''}$, with $z_p'$ and $z_{p''}$ the $p$-part and $p'$-part of $z$, respectively. If $z_p \notin Z(G)$, then by Step 7, $|z_p'| = n$ and $Z(G)_{p'}$ is a Hall $p'$-subgroup of $C_G(z)$, so $z_{p''} \in Z(G)$, which is a contradiction since $z$ has index $p^an$. If $z_p \in Z(G)$, then $|z_p'| = p^an$, so $z_{p''} \in Z(G)$ and this is a contradiction too.

We will prove the above claim. Let $s$ be a prime divisor of $n$ and let $S$ be a Sylow $s$-subgroup of $G$. By applying Brodkey’s theorem (see, for instance, [10, 5.28]) and taking into account Step (8.1), we deduce that there exists some $p$-element $y \in L_p$ such that $S \cap S' = O_s(G)$. But notice that $[O_s(G), L_p] = 1$ and as the $p$-complements of $G$ are abelian by (8.1), it follows that $O_s(G) = Z(G)_s$. Furthermore, $y$ cannot be central in $L_p$, otherwise $S$ would be central in $G$ contradicting the fact that $s$ divides $n$. Consequently, $y$ must have index $n$ in $G$. If we choose a $p$-complement $H$ of $G$ with $S \subseteq H$, then $C_G(y) = L_pC_H(y)$. If $w$ is a $p$-element of $C_H(y)$, then $w = y^{-1}wy \in S \cap S' = Z(G)_s$. Thus, we deduce that every $s$-element of $C_G(y)$ lies in $Z(G)$ and hence, $|G : C_G(y)|_s = |G : Z(G)|_s = n_s$, as required.

(9.2) Suppose now that $p^a < n = q^br^c$, with $b, c > 0$. Arguing the same as at the beginning of (8.2), we can assume that $L_q$ is non-central, and thus, $L_q$ is an abelian normal Sylow $q$-subgroup of $G$. We can also assume that $L_r \subseteq Z(G)$. Therefore, there exists a non-central $r$-element in $G$ of index $p^an$. Take $y \in L_q$ of index $p^a$ and let $w$ be a $q'$-element of $C_G(y)$. Then $|C_G(y) : C_G(yw)| = |C_G(y) : C_G(y) \cap C_G(w)|$ is equal to 1 or $n = q^br^c$. By applying Theorem 4, we obtain that $C_G(y) = QR \times A$, where $Q$ and $R$ are $q$ and $r$-Sylow subgroups of $G$ and $A$ is an abelian $p$-subgroup. By (8.2), the Sylow $p$-subgroups of $G$ are abelian, so we have $A \subseteq Z(G)$ and as a consequence, $p^a = |G : Z(G)|$. But if we take $g \in G$ an $r$-element of index $p^an$, then by Step 4, $C_G(g) = P_g \times V_g$, with $P_g$ a non-central abelian $p$-subgroup, and this is the final contradiction.

4. Proof of Theorem A

**Proof of Theorem A.** We will assume, for instance, that $m < n$ and we will denote by $\pi$ the set of primes dividing $n$. By Lemma 2, we can assume that the only primes dividing $|G|$ are the primes in $\pi$ and the prime divisors of $m$.

**Step 1.** If $G$ has a normal Hall $\pi$-subgroup, then the theorem is proved.

Suppose that $H$ is a normal Hall $\pi$-subgroup of $G$. For every $x \in H$ we have

If $|x^G| = 1$ or $m$, then $H \subseteq C_G(x)$ and thus $|x^H| = 1$. If $|x^G| = n$ or $mn$, then the above equality with the fact that $|x^H|$ divides $|x^G|$ imply that $|x^H| = 1$ or $m$. Therefore, any conjugacy class in $H$ has size 1 or $m$, so by Theorem 3 we get $m = p^a$ for some prime $p$. Then we can apply Theorem B to obtain that $G$ is nilpotent and $n = q^b$, so the theorem is proved.

**Step 2.** We may assume that there are no $\pi$-elements of index $n$ and that there are no $\pi'$-elements of index $m$.

Suppose that $x$ is a $\pi$-element of index $n$. By considering the primary decomposition of $x$ we can assume without loss that $x$ is a $p$-element for some prime $p \in \pi$. Now if $y$ is a $\pi'$-element of $C_G(x)$, then $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and this forces $|C_G(x) : C_G(x) \cap C_G(y)| = 1$ or $m$. By applying Theorem 4, we obtain that $m = p^c q^b$, but as $(n, m) = 1$, then $m = q^b$ and $G$ would be nilpotent by applying Theorem B. In this case we have necessarily $n = p^a$ for some $a > 0$.

The second assertion is also true since we can argue symmetrically with $m$ and $n$.

**Step 3.** If $x$ is a $\pi$-element of index $mn$, then $C_G(x) = H_x \times K_x$ with $H_x$ a $\pi$-group and $K_x$ an abelian $\pi'$-group such that $K_x \not\subseteq Z(G)$. Symmetrically, if $y$ is a $\pi'$-element of index $mn$, then $C_G(y) = H_y \times K_y$ with $H_y$ an abelian $\pi$-group such that $H_y \not\subseteq Z(G)$ and $K_y$ a $\pi'$-group.

This step follows arguing exactly as in Steps 4 and 5 of Theorem B.

**Step 4.** Write $L_{\pi} := \{x : x$ is $\pi$-element and $|x^G| = 1$ or $m\}$. Then $L_{\pi}$ is an abelian normal $\pi$-subgroup of $G$.

By applying Lemma 6, we obtain that the set $W := \{x : |x^G| = 1$ or $m\}$ is a normal subgroup of $G$. Now, if $x$ is any element of index $m$ and factorize $x = x_{\pi} x_{\pi'}$, with $x_{\pi}$ and $x_{\pi'}$ a $\pi$-element and a $\pi'$-element, respectively, it follows that $x_{\pi'}$ must be central by Step 2, whence $x \in L_{\pi} \times Z(G)_{\pi'}$. Therefore, $W = L_{\pi} \times Z(G)_{\pi'}$ and consequently, $L_{\pi}$ is a normal $\pi$-subgroup of $G$.

Finally, if we take any $y \in L_{\pi}$ then $|L_{\pi} : C_{L_{\pi}}(y)|$ divides ($|L_{\pi}|, m) = 1$, so $L_{\pi}$ is abelian.

**Step 5.** We may assume that $n = q^b r^c$ for some distinct primes $q$ and $r$.

As a consequence of Step 2, we may choose a $\pi$-element, say $x$, of index $m$. It is enough to consider the decomposition of any element of index $m$ as a product of a $\pi$-element by a $\pi'$-element. In addition, if we consider the primary decomposition of $x$ as a product of elements of prime power order, we can assume without loss that $x$ is a $q$-element for some prime $q \in \pi$. Now if we take a $q'$-element $w \in C_G(x)$, we have $C_G(wx) = C_G(w) \cap C_G(x)$ and $|C_G(x) : C_G(w) \cap C_G(x)|$ must be 1 or $n$. This proves that any $q'$-element of $C_G(x)$ has index 1 or $n$ in $C_G(x)$, so we can apply Theorem 2 to conclude that $n = q^b r^c$, with...
\( b, c \geq 0 \), as wanted. Moreover, we can assume \( b, c > 0 \) by Theorem B. Thus the step is proved.

We have seen that we can assume \( \pi = \{ q, r \} \) and since \( L_\pi \) is abelian we can certainly write \( L_\pi = L_q \times L_r \) where \( L_q \) and \( L_r \) are defined in the same way as \( L_\pi \) but for \( q \) and \( r \)-elements, respectively. Furthermore, as we know that there exist \( \pi \)-elements of index \( m \), we will assume without loss that one of these subgroups, say \( L_q \), is non-central in \( G \).

**Step 6.** \( L_q \) is a Sylow \( q \)-subgroup of \( G \).

This step can be proved by reasoning in the same way as in (7.3) of the proof of Theorem B.

**Step 7.** \( G \) has abelian Hall \( \pi' \)-subgroups.

If we take \( K \) a Hall \( \pi' \)-subgroup of \( G \), then \( G = KRL_q \), with \( R \in \text{Syl}_r(G) \) and one can prove, by following the same arguments as in (8.2) of the proof of Theorem B, that \( K \) is abelian.

**Step 8** (*Conclusion*). We can get a contradiction if we mimic the proof of (9.2) in Theorem B. □

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