Ping-Pong on Negatively Curved Groups

Rita Gitik*

Department of Mathematics, California Institute of Technology, Pasadena, California 91125
E-mail: ritagtk@math.lsa.umich.edu

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INTRODUCTION

The following result for classical Schottky groups of rank 2 goes back to Blaschke, Klein, Schottky, and Poincaré. It can be viewed as a special case of the Klein–Mas combination theorem ([Mas, p. 135]).

Lemma. Let $h$ and $k$ be hyperbolic isometries of hyperbolic space such that their axes have disjoint endpoints. There exists a constant $C$ which depends only on $h$ and $k$, such that if $m > C$ and $n > C$, then the subgroup $\langle h^m, k^n \rangle$ is isomorphic to the free product $\langle h^m \rangle \ast \langle k^n \rangle$.

The well-known proof of this result can be abstracted into the following simple form (Cf. [L-S, p. 168]). Let $H$ and $K$ be subgroups of a group $G$, and let $\langle H, K \rangle$ be the smallest subgroup of $G$ containing $H$ and $K$.

The Ping-Pong Lemma. Let $G$ be a group acting on a set $S$. Let $H$ and $K$ be subgroups of $G$ such that $|K : G_0| > 2$ or $|H : G_0| > 2$, where $G_0 = H \cap K$. If there exist disjoint nonempty subsets $S_H$ and $S_K$ of $S$ such that $K \setminus G_0$ maps $S_K$ into $S_H$ and $H \setminus G_0$ maps $S_H$ into $S_K$, then the subgroup $\langle H, K \rangle$ is isomorphic to the amalgamated free product $\langle H, K \rangle = H \ast_{G_0} K$.

Proof. Let $h_1k_1 \cdots h_n$ be an element of $\langle H, K \rangle \setminus G_0$, where all $h_i$ and $k_i$ are not in $G_0$. Then for any $x \in S_H$, $(h_1k_1 \cdots h_n)(x) \in S_K$, so $(h_1k_1 \cdots h_n)(x) \neq x$; hence any element of odd syllable length in $\langle H, K \rangle$ is non-trivial. Consider an element of even length $h_1k_1 \cdots h_2k_2$. Without loss of generality, $|H : G_0| > 2$; hence there exists $h \in H$ such that $h \notin G_0$.

* Research partially supported by NSF Grant DMS 9022140 at MSRI. Current address: A and H Consultants, Ann Arbor, MI.
and \( h^{-1}h_1 \not\in G_0 \). But then the element \( h^{-1}h_1k_1 \cdots h_nh_nh \) has odd syllable length; hence it is nontrivial. Therefore \( h_1k_1 \cdots h_nk_n \) is nontrivial.

Note that there exist counterexamples to the Ping-Pong lemma when \( |K:G_0| = |H:G_0| = 2 \), even if the action of \( G \) on \( S \) is free.

The Ping-Pong Lemma allows us to describe the subgroup \( \langle H, K \rangle \). In general, even in the simple case when \( H = \langle h \rangle \) and \( K = \langle k \rangle \) are infinite cyclic groups with trivial intersection, we do not have much information about the group \( \langle h, k \rangle \). In particular, this group need not be isomorphic to the free group of rank 2.

In this paper we describe the subgroup \( \langle H, K \rangle \) when \( H \) and \( K \) are quasiconvex subgroups of a negatively curved group \( G \). We also give a condition for \( \langle H, K \rangle \) to be quasiconvex in \( G \). In general, even if both \( H = \langle h \rangle \) and \( K = \langle k \rangle \) are infinite cyclic (hence quasiconvex in \( G \); [Gr, p. 210]), the group \( \langle h, k \rangle \) might be non-quasiconvex in \( G \). This might happen even if \( \langle h, k \rangle \) is isomorphic to the free group of rank 2.

We prove the following results.

**Theorem 1.** Let \( H \) and \( K \) be \( \mu \)-quasiconvex subgroups of a \( \delta \)-negatively curved group \( G \). There exists a constant \( C_0 \), which depends only on \( G, \delta, \) and \( \mu \), with the following property. For any subgroups \( H_1 < H \) and \( K_1 < K \) with \( H_1 \cap K_1 = H \cap K = G_0 \), if all the elements in \( H_1 \) and in \( K_1 \) which are shorter than \( C_0 \) belong to \( G_0 \), then \( \langle H_1, K_1 \rangle = H_1*_{G_0} K_1 \). If, in addition, \( H_1 \) and \( K_1 \) are quasiconvex in \( G \), then the subgroup \( \langle H_1, K_1 \rangle \) is quasiconvex in \( G \).

Recall that a subgroup \( H \) is malnormal in \( G \) if for any \( g \not\in H \) the intersection of \( H \) and \( gHg^{-1} \) is trivial.

**Theorem 2.** Let \( H \) and \( K \) be \( \mu \)-quasiconvex subgroups of a \( \delta \)-negatively curved group \( G \). Assume that \( H \) is malnormal in \( G \). There exists a constant \( C_1 \), which depends only on \( G, \delta, \) and \( \mu \), with the following property. For any subgroup \( H_1 < H \) with \( H_1 \cap K = H \cap K = G_0 \), if all the elements in \( H_1 \) which are shorter than \( C_1 \) belong to \( G_0 \), then \( \langle H_1, K \rangle = H_1*_{G_0} K \). If, in addition, \( H_1 \) is quasiconvex in \( G \), then the subgroup \( \langle H_1, K \rangle \) is quasiconvex in \( G \).

Of course, Theorems 1 and 2 might be vacuously true, because there might be no subgroups \( H_1 \) and \( K_1 \) with the required properties. However, these theorems have interesting applications in the following frequently encountered case. Recall that a group \( K \) is residually finite if for any finite set of nontrivial elements in \( K \), there exists a finite index subgroup of \( K \) which does not contain this set. The class of residually finite groups is very rich; it contains all finitely generated linear groups and all fundamental groups of geometric 3-manifolds.
If $K$ is a quasiconvex subgroup of a negatively curved group $G$, then $K$ is finitely generated, so for any positive integer $n$, $K$ has only finitely many elements shorter than $n$. Hence if such $K$ is infinite and residually finite, it has an infinite family of distinct finite index subgroups $K_n$, such that $K_n$ does not contain nontrivial elements shorter than $n$. As $K_n$ is a finite index subgroup of a quasiconvex subgroup $K$, it is quasiconvex in $G$; hence we have the following results.

**Corollary 3.** Let $H$ and $K$ be infinite quasiconvex subgroups of a negatively curved group $G$. Assume that $H$ and $K$ are residually finite. If the intersection of $H$ and $K$ is trivial, then there exist infinite families of distinct finite index subgroups $H_m$ of $H$ and $K_n$ of $K$, such that the subgroups $\langle H_m, K_n \rangle$ are quasiconvex in $G$ and $\langle H_m, K_n \rangle = H_m * K_n$.

**Corollary 4.** Let $H$ and $K$ be infinite quasiconvex subgroups of a negatively curved group $G$. Assume that $H$ is malnormal in $G$ and residually finite. If the intersection of $H$ and $K$ is trivial, then there exists an infinite family of distinct finite index subgroups $H_m$ of $H$ such that the subgroup $\langle H_m, K \rangle$ is quasiconvex in $G$ and $\langle H_m, K \rangle = H_m * K$.

The above corollaries apply when the intersection of $K$ and $H$ is trivial. In general, recall that a group $K$ is LERF (locally extended residually finite) if for any finitely generated subgroup $K_0$ of $K$ and for any finite set $S$ of elements in $K$, but not in $K_0$, there exists a finite index subgroup of $K$ which contains $K_0$, but does not contain $S$. The class of LERF groups is much smaller than the class of residually finite groups, but it still contains many interesting groups. For example, free groups, surface groups, and nilpotent groups are LERF.

Let $K$ be an infinite quasiconvex subgroup of a negatively curved group $G$, and let $G_0$ be a finitely generated subgroup of $K$. If $K$ is LERF, then it has an infinite family of distinct finite index subgroups $K_n$ containing $G_0$, such that all the elements in $K_n$ which are shorter than $n$ belong to $G_0$. Hence we have the following results.

**Corollary 5.** Let $H$ and $K$ be infinite quasiconvex subgroups of a negatively curved group $G$ with $H \cap K = G_0$. Assume that $H$ and $K$ are LERF. Then there exist infinite families of distinct finite index subgroups $H_m$ of $H$ and $K_n$ of $K$ containing $G_0$, such that the subgroups $\langle H_m, K_n \rangle$ are quasiconvex in $G$ and $\langle H_m, K_n \rangle = H_m *_{G_0} K_n$.

**Corollary 6.** Let $H$ and $K$ be infinite quasiconvex subgroups of a negatively curved group $G$ with $H \cap K = G_0$. Assume that $H$ is malnormal in $G$ and LERF. Then there exists an infinite family of distinct finite index subgroups $H_m$ of $H$ containing $G_0$, such that the subgroup $\langle H_m, K \rangle$ is quasiconvex in $G$ and $\langle H_m, K \rangle = H_m *_{G_0} K$. 
PROOFS OF THE RESULTS

Let $X$ be a set and let $X^* = \{x, x^{-1} | x \in X\}$, where $(x^{-1})^{-1} = x$ for $x \in X$. Denote the set of all words in $X^*$ by $W(X^*)$, and denote the equality of two words by "\(\equiv\)". Let $G$ be a group generated by the set $X^*$, and let Cayley($G$) be the Cayley graph of $G$ with respect to the generating set $X^*$. The set of vertices of Cayley($G$) is $G$, the set of edges of Cayley($G$) is $G \times X^*$, and the edge $(g, x)$ joins the vertex $g$ to $gx$.

Definition. The label of the path $p = (g, x_1)(gx_1, x_2) \cdots (gx_1x_2 \cdots x_{n-1}, x_n)$ in Cayley($G$) is the word Lab$(p) = x_1 \cdots x_n \in W(X^*)$. As usual, we identify the word Lab$(p)$ with the corresponding element in $G$. We denote the initial and the terminal vertices of $p$ by $\iota(p)$ and $\tau(p)$, respectively, and the inverse of $p$ by $\bar{p}$. Denote the length of the path $p$ by $|p|$, where $|g, x_1)(gx_1, x_2) \cdots (gx_1x_2 \cdots x_{n-1}, x_n)| = n$.

A geodesic in the Cayley graph is a shortest path joining two vertices. A group $G$ is $\delta$-negatively curved if any side of any geodesic triangle in Cayley($G$) belongs to the $\delta$-neighborhood of the union of two other sides (see [Gr, C-D-P]). Let $\lambda \leq 1$, $L > 0$, and $\epsilon > 0$. A path $p$ in the Cayley graph is a $(\lambda, \epsilon)$-quasigeodesic if for any subpath $p'$ of $p$ and for any geodesic $\gamma$ with the same endpoints as $p'$, $|\gamma| > \lambda |p'| - \epsilon$. A path $p$ is a local $(\lambda, \epsilon, L)$-quasigeodesic if for any subpath $p'$ of $p$ that is shorter than $L$ and for any geodesic $\gamma$ with the same endpoints as $p'$, $|\gamma| > \lambda |p'| - \epsilon$ (cf. [C-D-P, p. 24]).

Theorem 1.4 (p. 25) of [C-D-P] (see also [Gr, p. 187]) states that for any $\lambda_0 \leq 1$ and for any $\epsilon_0 > 0$ there exist constants $(L, \lambda, \epsilon)$ that depend only on $(\lambda_0, \epsilon_0)$ and $\delta$, such that any local $(\lambda_0, \epsilon_0, L)$-quasigeodesic in $G$ is a global $(\lambda, \epsilon)$-quasigeodesic in $G$.

Recall that $H$ is a $\mu$-quasiconvex subgroup of $G$ if any geodesic in Cayley($G$) which has its endpoints in $H$ belongs to the $\mu$-neighborhood of $H$.

Proof of Theorem 1. Let $H_1$ and $K_1$ be subgroups of $H$ and $K$, respectively, such that $H_1 \cap K_1 = H \cap K = G_0$. Consider an element $l$ of $\langle H_1, K_1 \rangle$ such that $l \notin G_0$. Then there exists a representation $l = h_1k_1 \cdots k_{m-1}h_m$, where $h_i \in H_1$, $k_i \in K_1$, and $h_i$ do not belong to $G_0$, $k_i$, and $h$ are geodesics in $G$, $h_1$ is a shortest representative of the coset $h_1G_0$, $h_m$ is a shortest representative of the coset $G_0h_m$, and for $1 < i < m$, $h_i$ is a shortest representative of the double coset $G_0h_iG_0$. (The elements $h_1$ and $h_m$ might be trivial). Let $p$ be the path in Cayley($G$) beginning at 1 with the decomposition of the form $p = p_1q_1 \cdots q_{m-1}p_m$, where Lab$(p_i) = h_i$ and Lab$(q_i) = k_i$. 

Let \( A \) be the number of words in \( G \) which are shorter than \( 2\mu + \delta \). As mentioned above, there exist constants \((L, \lambda, \epsilon)\) which depend only on \((\mu, A, \delta)\), such that any local \((\frac{1}{2}, (4\mu \cdot A + \delta), L)\)-quasigeodesic in \( G \) is a global \((\lambda, \epsilon)\)-quasigeodesic in \( G \).

Let \( C_0 = \max(L, \epsilon/\lambda) \). We claim that if all the elements in \( H_1 \) and in \( K_1 \) which are shorter than \( C_0 \) belong to \( G_0 \), then any path \( p \), as above, is a \((\lambda, \epsilon)\)-quasigeodesic in \( G \). Indeed, it is enough to show that \( p \) is a local \((\frac{1}{2}, (4\mu \cdot A + \delta), L)\)-quasigeodesic in \( G \).

As \( h_i \not\in G_0 \) and \( k_i \not\in G_0 \), it follows that \( |q_i| > C_0 \) and \( |p_i| > C_0 \). As \( L \leq C_0 \), any subpath \( t \) of \( p \) with \(|t| < L \) has a (unique) decomposition \( t_1 t_2 \), where without loss of generality, \( t_1 \) is a subpath of some \( p_i \) and \( t_2 \) is a subpath of \( q_i \). Let \( t_3 \) be a geodesic in \( G \) joining the endpoints of \( t \). We will show that \( |t_3| \geq |t|/3 - (4\mu \cdot A + \delta) \). Without loss of generality, assume that \( |t_3| \geq |t_3| \). If \( |t_3| \leq \frac{4}{3}|t| \), then \( |t_3| \geq |t_2| - |t_1| \geq \frac{1}{3}|t| \). So assume that \( |t_1| > \frac{1}{3}|t| \). As \( t_1 \) is a geodesic in \( G \) which is not in \( G_0 \), it follows that the decomposition \( t_1 = t_1' t_1'' \), where \( t_1' \) is the maximal subpath of \( t_1 \) which belongs to the \( \epsilon \)-neighborhood of \( t_2 \), and \( t_1'' \) belongs to the \( \delta \)-neighborhood of \( t_3 \).

It follows from Lemma 7 (below) that \( |t_1'| \leq A \cdot 4\mu \); hence \( |t_2| - |t_1'| \geq \frac{4}{3}|t| - A \cdot 4\mu \). Therefore, \( |t_3| \geq \frac{1}{3}|t| - (A \cdot 4\mu + \delta) \), proving that \( p \) is a local \((\frac{1}{2}, (4\mu \cdot A + \delta), L)\)-quasigeodesic in \( G \). So \( p \) is a global \((\lambda, \epsilon)\)-quasigeodesic in \( G \). The definition of \( C_0 \) implies that if a \((\lambda, \epsilon)\)-quasigeodesic \( p \) in \( G \) is longer than \( C_0 \), then \( \text{Lab}(p) \neq 1 \). As any element \( l \in \langle H_1, K_1 \rangle \) that is not in \( G_0 \) has a representative \( \text{Lab}(p) \) in \( G \), as above, with \(|p| > C_0 \), it follows that \( l \neq 1 \); hence \( \langle H_1, K_1 \rangle = H_1 *_{G_0} K_1 \).

To prove the second part of the theorem, for any element \( l \in \langle H_1, K_1 \rangle \), which is not in \( G_0 \), consider a geodesic \( \gamma \) in \( G \) joining the endpoints of the path \( p \), as above. As \( G \) is negatively curved, there exists a constant \( \alpha \) which depends only on \((\lambda, \epsilon)\) and \( \delta \), such that \( \gamma \) belongs to the \( \alpha \)-neighborhood of \( p \). Assume that \( H_1 \) and \( K_1 \) are \( \mu_1 \)-quasiconvex in \( G \); then any vertex \( v_i \) on \( p \) is in the \( \mu_1 \)-neighborhood of \( h_i k_1 \cdots k_{i-1} h_1 \), and any vertex \( w_i \) on \( q \) is in the \( \mu_1 \)-neighborhood of \( h_i k_1 \cdots h_i k_1 \), so the path \( p \) belongs to the \((\alpha + \mu_1)\)-neighborhood of \( \langle H_1, K_1 \rangle \).

If \( l \in G_0 \), then \( l \in H_2 \); so any geodesic \( \gamma \) labeled with \( l \) with \( \nu(\gamma) = 1 \) belongs to the \( \mu_1 \)-neighborhood of \( H_1 \). It follows that the subgroup \( \langle H_1, K_1 \rangle \) is \((\alpha + \mu_1)\)-quasiconvex in \( G \).

**Lemma 7.** Using the notation of the proof of Theorem 1, \( |t_1'| \leq A \cdot 4\mu \).

**Proof.** To simplify notation, we drop the subscript \( i \), so \( t_1 \) is a subpath of \( p_2 \) is a subpath of \( q \), \( \text{Lab}(p) = h \), and \( \text{Lab}(q) = k \). As \( H_1 < H \) and \( K_1 < K \), we consider \( h \) as an element of \( H \) and \( k \) as an element of \( K \).

Without loss of generality, assume that \( q \) begins at 1 (so it ends at \( k \)); then \( p \) begins at \( h^{-1} \) and ends at 1. As \( K \) and \( H \) are \( \mu \)-quasiconvex in \( G \), any
vertex \( v_i \) on \( p \) is in the \( \mu \)-neighborhood of \( H \), and any vertex \( w_i \) on \( q \) is in the \( \mu \)-neighborhood of \( K \). Hence we can find vertices \( v_1 \) and \( v_2 \) in \( t', \ w_1 \) and \( w_2 \) in \( t, h' \) and \( h'' \) in \( H \), and \( k' \) and \( k'' \) in \( K \) such that \( |v_i, w_i| < \delta, \)
\( |v_2, (h')^{-1}| < \mu, |v_2, (h'')^{-1}| < \mu, |w_1, k'| < \mu, \) and \( |w_2, k''| < \mu. \) Then \( |h'k'| < 2\mu + \delta \) and \( |h''k''| < 2\mu + \delta. \)

Assume that \( |t'_1| > A \cdot 4\mu. \) Then we can find vertices, as above, which, in addition, satisfy \( |v_2, v_2| > 4\mu \) and \( h'k' = h''k''. \) But then \( (h'')^{-1}h' = k''(k')^{-1}, \) so both products are in \( G_0. \) As \( h \) is a shortest element in the double coset \( G_0hG_0, \) it follows that \( |h| \leq |h(h'')^{-1}h'|. \) Let \( r \) be a geodesic joining \( (h'')^{-1} \) to \( v_2, \) let \( s' \) be a subpath of \( p \) joining \( v_2 \) to \( 1, \) and let \( s'' \) be a subpath of \( p \) joining \( h^{-1} \) to \( v_2. \) Then \( |h| = |p| = |s'| + |s''| \) and \( |h(h'')^{-1}h'| \leq |h(h'')^{-1}| + |h'| \leq |s''| + |r| + |h'|; \) hence \( |s'| + |s''| \leq |s''| + |r| + |h'|, \) \( s \) \( |s'| + |r| \leq 2|r| + |h'|. \) As \( |h''| \leq |s'| + |r|, \) and as \( |r| \leq \mu, \) it follows that \( |h''| \leq 2\mu + |h'|. \)

However, as \( |v_2, v_2| > 4\mu, \) the triangle inequality implies that \( |h''| = (h'')^{-1} \geq |s'| - |r| = |1, v_2, v_2| - |r| \geq |1, v_2| + 4\mu - \mu = |1, v_2| + 2\mu. \) Let \( a \) be a geodesic joining \( (h'')^{-1} \) to \( v_1. \) As \( |a| < \mu, \) the triangle inequality implies that \( |h'| = (h')^{-1} \leq |1, v_2| + |a| < |1, v_2| + \mu. \) Hence, \( |h''| > |h'| + 2\mu, \) a contradiction. Therefore, \( |t'_1| \leq A \cdot 4\mu. \)

We use the following property of malnormal quasiconvex subgroups of negatively curved groups (cf. [Gi]). The original proof of this fact for the special case when \( G \) is a free group is due to Rips (G-R).

Let \( H \) be a subgroup of \( G, \) and let \( G/H \) denote the set of right cosets of \( H \) in \( G. \) The relative Cayley graph of \( G \) with respect to \( H \) is an oriented graph whose vertices are the cosets \( G/H \) and the set of edges is \( (G/H) \times X^*, \) such that an edge \((Hg, x)\) begins at the vertex \( Hg \) and ends at the vertex \( Hgx. \) We denote it Cayley\((G, H). \) Note that for any path \( p \) in Cayley\((G, H) \) if \( \ell(p) = H \cdot 1, \) then \( \tau(p) = H \cdot \text{Lab}(p), \) so a path \( p \) beginning at \( H \cdot 1 \) is closed, if and only if \( \text{Lab}(p) \in H. \)

**Lemma 8.** Let \( H \) be a malnormal \( \mu \)-quasiconvex subgroup of a finitely generated group \( G, \) and let \( \delta \) be a nonnegative constant. (The group \( G \) does not have to be negatively curved.) Let \( \gamma_1, \gamma_2 \) be a path in Cayley\((G) \) such that \( \gamma_1 \) and \( \gamma_2 \) are geodesics in Cayley\((G), \) \( \text{Lab}(\gamma_1) \in H, \) \( \text{Lab}(\gamma_2) \in H, \) and \( \text{Lab}(r) \notin H. \) Let \( m \) be the number of vertices in the ball of radius \( \mu + 2\delta \) around \( H \cdot 1 \) in Cayley\((G, H), \) and let \( M = m^2 + 1. \) Then any subpath \( \alpha \) of \( \gamma_1 \) which belongs to the \( 2\delta \)-neighborhood of \( \gamma_2 \) in Cayley\((G) \) is shorter than \( M. \)

**Proof of Lemma 8.** Assume that \( |\alpha| > M. \) Without loss of generality, \( \ell(\gamma_2) = 1. \) Let \( \gamma'_1 \) be a geodesic in Cayley\((G) \) beginning at \( 1 \) such that \( \text{Lab}(\gamma'_1) = \text{Lab}(\gamma_1), \) and let \( \alpha' \) be the subpath of \( \gamma'_1 \) such that \( \text{Lab}(\alpha') = \text{Lab}(\gamma_2) \).
Lab(α). Let π: Cayley(G) → Cayley(G, H) be the projection map π(g) = Hg and π(g, x) = (Hg, x).

Note that γ2 and γ′ 1 are geodesics in Cayley(G) beginning at 1, Lab(γ′ 1) ∈ H and Lab(γ2) ∈ H, so as H is μ-quasiconvex in G, the projection π maps γ2 and γ′ 1 into a ball of radius μ around H · 1 in Cayley(G, H). As α ∈ N₂μ(H · 1) ⊂ Cayley(G, H), it follows that π(α) ∈ N₂μ(H · 1) ⊂ Cayley(G, H). Denote the vertices of π(α) by v₁, . . . , vₙ and the vertices of π(α′) by w₁, . . . , wₙ. As n ≥ M, and (v₁, . . . , vₙ, w₁, . . . , wₙ) ⊂ N₂μ(H · 1), there exist 1 ≤ i < j ≤ n such that (vᵢ, wᵢ) = (vⱼ, wⱼ). Let p₁ and p₂ be subpaths of π(γ₁) connecting π(γ₀) and vᵢ and vⱼ, respectively. Let q₁ and q₂ be subpaths of π(γ′ₐ) connecting H · 1 = π(γ′₁) and wᵢ and wⱼ, respectively. Let s = π(t) and let p denote the path p with the opposite orientation. Then sp₁p₂pᵢpⱼ and q₁q₂qᵢqⱼ are closed paths in Cayley(G, H) beginning at H · 1, so Lab(sp₁p₂pᵢpⱼ) = Lab(q₁q₂qᵢqⱼ) ∈ H.

But Lab(p₁) = Lab(q₁) and Lab(p₂) = Lab(q₂), so Lab(s) = Lab(p₁p₂pᵢpⱼ)Lab⁻¹(s) = H, and Lab(p₁p₂pᵢpⱼ) ⊂ H. Therefore the mal-normality of H in G implies that Lab(s) = H, contradicting the assumption that Lab(s) = Lab(t) ⊄ H. Hence |α| < M.

**Proof of Theorem 2.** Let H₁ be a subgroup of H such that H₁ ∩ K = H ∩ K = G₀. Let l be an element of (H₁, K) such that l ∉ G₀. Then there exists a representation l = h₁k₁ ... kₘ₋₁hₘ, where k₁ and h₁ are as in the proof of Theorem 1. Let p be a path in Cayley(G) with a decomposition of the form p = p₁q₁ ... qₘ₋₁pₘ, where Lab(p₁) = h₁ and Lab(qₙ) = h. Let M be as in Lemma 8, and let A be as in the proof of Theorem 1. As was mentioned above, there exist constants (L′, λ′, ε′) which depend only on (μ, A, δ), such that any local (λ, λ′, ε′)-quasigeodesic in G is a global (λ′, ε′)-quasigeodesic in G.

Let C₁ = max(L′, ε′/λ′). We claim that if all the elements in H₁ which are shorter than C₁ belong to G₀, then any path p, as above, is a (λ′, ε′)-quasigeodesic in G. Indeed, it is enough to show that p is a local (λ, λ′, ε′)-quasigeodesic in G.

As h₁ ∉ G₀, it follows that |p₁| > C₁. As L′ < C₁, any subpath t of p with |t| < L′ has a (unique) decomposition t₁t₂t₃, where t₁ and t₂ are subpaths of some p₁ and p₁₊₁, and t₃ is a subpath of q₁ (some of t₃ might be empty). Let t₄ be a geodesic in G connecting the endpoints of t.

If |t₃| > 2|t|/3, then |t₄| + |t₃| ≤ |t|/3, so |t₄| ≥ |t₃| − (|t₄| + |t₃|) ≥ 2|t₃|/3 − |t₃|/3 = |t₃|/3.

So assume that |t₃| ≤ 2|t|/3. Without loss of generality, assume that |t₃| ≥ |t₄|; hence |t₃| > |t|/6. As t₁t₂t₃t₄ is a geodesic 4-gon in a δ-negatively curved group G, there exists a decomposition t₄ = s₁s₂s₃s₄ such that
$s_2$ belongs to the $\delta$-neighborhood of $t_2$, $s_3$ belongs to the $2\delta$-neighborhood of $t_3$, and $s_4$ belongs to the $\delta$-neighborhood of $t_4$. According to Lemma 8, $|s_3| < M$, and according to Lemma 7, $|s_2| \leq 4\mu \cdot A$. But then $|t_4| + \delta \geq |s_4| = |r_1| - |s_2| - |s_3| \geq |r_1| - 4\mu \cdot A - M \geq |r|/6 - 4\mu \cdot A - M$.

Hence $|t_4| \geq |r|/6 - (M + \delta + 4\mu \cdot A)$, so the path $p$ is a local $(\lambda', (M + \delta + 4\mu \cdot A), L')$-quasigeodesic in $G$; hence it is a $(\lambda', \epsilon')$-quasigeodesic in $G$. Then we conclude the argument as in the proof of Theorem 1.

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REFERENCES


