On the zero-divisor graph of a commutative ring

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Abstract

Let \( R \) be a commutative ring and \( \Gamma(R) \) be its zero-divisor graph. In this paper it is shown that for any finite commutative ring \( R \), the edge chromatic number of \( \Gamma(R) \) is equal to the maximum degree of \( \Gamma(R) \), unless \( \Gamma(R) \) is a complete graph of odd order. In [D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, in: Lecture Notes in Pure and Appl. Math., Vol. 220, Marcel Dekker, New York, 2001, pp. 61–72] it has been proved that if \( R \) and \( S \) are finite reduced rings which are not fields, then \( \Gamma(R) \cong \Gamma(S) \) if and only if \( R \cong S \). Here we generalize this result and prove that if \( R \) is a finite reduced ring which is not isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) or to \( \mathbb{Z}_6 \) and \( S \) is a ring such that \( \Gamma(R) \cong \Gamma(S) \), then \( R \cong S \).

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Introduction

The concept of zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [6]. He let all elements of the ring be vertices of the graph and was interested mainly in colorings. In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures.
of rings. The zero-divisor graph of a commutative ring has been studied extensively by Anderson, Frazier, Lauve, Levy, Livingston and Shapiro, see [2–4]. The zero-divisor graph concept has recently been extended to non-commutative rings, see [7].

Throughout the paper, all rings are assumed to be commutative with unity $1 \neq 0$. If $R$ is a ring, $Z(R)$ denotes its set of zero-divisors. A ring $R$ is said to be reduced if $R$ has no non-zero nilpotent element. A ring $R$ is said to be decomposable if $R$ can be written as $R_1 \times R_2$, where $R_1$ and $R_2$ are rings; otherwise $R$ is said to be indecomposable. If $X$ is either an element or a subset of $R$, then $\text{Ann}(X)$ denotes the annihilator of $X$ in $R$. For any subset $X$ of $R$, we define $X^* = X \setminus \{0\}$. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^*$ in which two vertices $x$ and $y$ are adjacent if and only if $x \neq y$ and $xy = 0$.

For a graph $G$, the degree $d(v)$ of a vertex $v$ in $G$ is the number of edges incident to $v$. We denote the minimum and maximum degree of vertices of $G$ by $\delta(G)$ and $\Delta(G)$, respectively. A graph $G$ is regular if the degrees of all vertices of $G$ are the same. We denote the complete graph with $n$ vertices and complete bipartite graph with two parts of sizes $m$ and $n$, by $K_n$ and $K_{m,n}$, respectively. The complete bipartite graph $K_{1,n}$, is called a star. A Hamiltonian cycle of $G$ is a cycle that contains every vertex of $G$. A graph is Hamiltonian if it contains a Hamiltonian cycle. A subset $X$ of the vertices of $G$ is called a clique if the induced subgraph on $X$ is a complete graph. A $k$-vertex coloring of a graph $G$ is an assignment of $k$ colors $\{1, \ldots, k\}$ to the vertices of $G$ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(G)$ of a graph $G$, is the minimum $k$ for which $G$ has a $k$-vertex coloring. A $k$-edge coloring of a graph $G$ is an assignment of $k$ colors $\{1, \ldots, k\}$ to the edges of $G$ such that no two adjacent edges have the same color. The edge chromatic number $\chi'(G)$ of a graph $G$, is the minimum $k$ for which $G$ has a $k$-edge coloring. A graph $G$ is said to be critical if $G$ is connected and $\chi'(G) = \Delta(G) + 1$ and for any edge $e$ of $G$, we have $\chi'(G \setminus \{e\}) < \chi'(G)$.

Beck in [6] proved several interesting theorems for the vertex chromatic number of a zero-divisor graph. For example, he showed that for any commutative ring $R$, if $R$ is a direct product of finitely many reduced rings and principal ideal rings, then $\chi(\Gamma(R))$ equals to the size of maximum clique of $\Gamma(R)$. Although Beck used a different graph, his results apply to the current setting. There are many interesting questions about zero-divisor graphs. For instance, Anderson, Frazier, Lauve and Livingston asked in [2]: “For which finite commutative rings $R$, is $\Gamma(R)$ planar?” In [1] it was proved that if $R$ is a finite local ring such that $\Gamma(R)$ has at least 33 vertices, then $\Gamma(R)$ is not a planar graph.

**Results**

The vertex chromatic number of zero-divisor graphs has been studied extensively by Beck in [6]. Here we will study the edge chromatic number of zero-divisor graphs and prove that if $R$ is a finite commutative ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.

If $G$ is a graph, clearly in any edge coloring of $G$, the edges incident with one vertex should be colored with different colors. This observation implies that $\chi'(G) \geq \Delta(G)$. An important theorem due to Vizing is the following.
Vizing’s Theorem [8, p. 16]. If $G$ is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.

Also the following lemma is a key to our proof.

Vizing’s Adjacency Lemma [8, p. 24]. If $G$ is a critical graph, then $G$ has at least $\Delta(G) - \delta(G) + 2$ vertices of maximum degree.

Remark 1. We note that if $G$ is a graph and $\chi'(G) = \Delta(G) + 1$, then there exists a subgraph of $G$, say $G_1$, such that $\chi'(G_1) = \Delta(G) + 1$ and for any edge $e$ of $G_1$ we have $\chi'(G_1 \setminus e) = \Delta(G)$. Clearly $G_1$ has a connected subgraph, say $H$, such that $\chi'(H) = \Delta(G) + 1$. The graph $H$ is a critical graph with maximum degree $\Delta(G)$. If $x$ is a vertex of $H$ with degree $\Delta(G)$, then by Vizing’s Adjacency Lemma, $H$ has at least $\Delta(G) - d_H(v) + 2$ vertices of degree $\Delta(G)$, for any vertex $v$ which is adjacent to $x$. Therefore if $G$ is a graph such that for every vertex $u$ of maximum degree there exists an edge $uv$ such that $\Delta(G) - d(u) + 2$ is more than the number of vertices with maximum degree in $G$, then by the above argument and Vizing’s Theorem, we have $\chi'(G) = \Delta(G)$.

It is not hard to see that if $R$ is an Artinian local ring, then the Jacobson radical of $R$ equals $Z(R)$. Thus $Z(R)$ is a nilpotent ideal and this implies that if $R$ is not a field, then $\text{Ann}(Z(R)) \neq \{0\}$. Moreover, each element of $\text{Ann}(Z(R))^*$ is adjacent to each other vertex of $\Gamma(R)$.

Theorem 1. If $R$ is a finite local ring which is not a field, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.

Proof. Since $R$ is a finite local ring, $\text{Ann}(Z(R)) \neq \{0\}$. If $\Gamma(R)$ is a complete graph, then by [8, Theorem 1.2, p. 12], we are done. Thus suppose that $\Gamma(R)$ is not a complete graph and so $\text{Ann}(Z(R)) \neq Z(R)$. If $x \in Z(R) \setminus \text{Ann}(Z(R))$, then there is an element $a \in Z(R)$ such that $ax \neq 0$. This implies that $x$ is adjacent to no vertices of $a + \text{Ann}(Z(R))$. Therefore $d(x) \leq |Z(R)| - |\text{Ann}(Z(R))|$. Hence $\Delta(\Gamma(R)) - d(x) + 2 \geq |\text{Ann}(Z(R))| + 1$. Clearly, $\text{Ann}(Z(R))^*$ is the set of all vertices of maximum degree in $\Gamma(R)$. So, by Remark 1, we have $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. $\Box$

Now using König’s Theorem, we show that the previous theorem is true for any finite commutative ring.

König’s Theorem [8, p. 11]. For any bipartite graph $G$, we have $\chi'(G) = \Delta(G)$.

Remark 2. Assume that $R = R_1 \times \cdots \times R_n$ is a finite decomposable commutative ring. We note that if $x = (x_1, \ldots, x_n)$ has maximum degree in $\Gamma(R)$, then $x$ has exactly one non-zero component, say $x_1$. Now suppose that $R_1$ is a local ring. We consider two cases: If $R_1$ is a field, then $\Delta(\Gamma(R)) = d(x) = |R_2| \cdots |R_n| - 1$; If $R_1$ is not a field, then we have $x_1 \in \text{Ann}(Z(R_1))^*$ and $\Delta(\Gamma(R)) = d(x) = |Z(R_1)||R_2| \cdots |R_n| - 2$. 
Theorem 2. If $R$ is a finite decomposable ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$.

Proof. It is well known that every commutative Artinian ring is isomorphic to the direct product of finitely many local rings, see [5, p. 90]. Suppose that $R = R_1 \times \cdots \times R_n$, where $n \geq 2$ and each $R_i$ is a local ring. By Remark 2, without loss of generality suppose that the non-zero components of the vertices with maximum degree in $\Gamma(R)$ occur in $R_1, \ldots, R_k$.

First we claim that all of the rings $R_1, \ldots, R_k$ are fields or none of them are fields. Working towards a contradiction suppose that $R_1$ is a field and $R_2$ is not a field. Now, every vertex with maximum degree in $R_1 \times \{0\} \times \cdots \times \{0\}$ has degree $|R_1||Z(R_2)||R_3| \cdots |R_n| - 2$. Thus we have $|Z(R_2)||R_3| \cdots |R_n|(|R_1| - |R_2/Z(R_2)|) = 1$, a contradiction. Therefore by Remark 2, for any $i$, $1 \leq i \leq k$, $\Delta(\Gamma(R)) = |R_1| \cdots |R_{i-1}||Z(R_i)||R_{i+1}||R_{i+2}||R_n| - 1$, where $r = 1$ or $2$. Hence, we have $|R_1/Z(R_1)| = \cdots = |R_k/Z(R_k)|$. Moreover, since for each $j$, $k + 1 \leq j \leq n$, the degree of any vertex in $\{0\} \times \cdots \times \{0\} \times R_j \times \{0\} \times \cdots \times \{0\}$ is less than $\Delta(\Gamma(R))$, we have

$$|R_j/Z(R_j)| \geq |R_1/Z(R_1)|.$$  

For any $t$, $1 \leq t \leq n$, suppose that $e_t$ is the element whose $t$th component is one and other components are zero. First, suppose that the rings $R_1, \ldots, R_k$ are not fields. Then $\Gamma(R)$ has $\sum_{i=1}^k |\text{Ann}(Z(R_i))|$ vertices of maximum degree. Clearly, every vertex of maximum degree in $\Gamma(R)$ is adjacent to at least one of the $e_t$’s. Now for any $i$, $1 \leq i \leq n$, we have

$$\Delta(\Gamma(R)) - d(e_t) + 2 \geq (|R_1| \cdots |R_{i-1}||Z(R_i)||R_{i+1}| \cdots |R_n| - 2)$$

$$- (|R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1) + 2$$

$$= |R_1| \cdots |R_{i-1}||Z(R_i)| - 1||R_{i+1}| \cdots |R_n| + 1$$

$$> \sum_{i=1}^k |\text{Ann}(Z(R_i))|.$$  

Hence by Remark 1, we conclude that $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. Next, suppose that the rings $R_1, \ldots, R_n$ are fields. Then $\Gamma(R)$ has $\sum_{i=1}^k |R_i^*|$ vertices of maximum degree. If $n > 2$, then every vertex of maximum degree in $\Gamma(R)$ is adjacent to $1 - e_i$, for some $i$, $1 \leq i \leq k$. Note that in this case $|R_1| = \cdots = |R_k|$ and if we set $|R_1| = a$, then by $(\ast)$ we have $|R_j| \geq a$, for any $j$, $j > k$. Now since $a^{n-1} - a + 2 > n(a-1)$, for any $i$, $1 \leq i \leq k$, we have

$$\Delta(\Gamma(R)) - d(1 - e_i) + 2 \geq (|R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1) - (|R_1| - 1) + 2$$

$$\geq a^{n-1} - a + 2 > \sum_{i=1}^k |R_i^*|.$$  

Thus by Remark 1, we conclude that $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. So assume that $n = 2$. If $k = 1$ and $R_2$ is not a field, then by $(\ast)$ we have $|R_2| \geq 2|R_1|$. Since in this case
any vertex of maximum degree in $\Gamma(R)$ is adjacent to $e_2$ and $\Delta(\Gamma(R)) - d(e_2) + 2 = (|R_2| - 1) - (|R_1| - 1) + 2 > |R_1|$, by Remark 1, we obtain $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. If either $k = 1$ and $R_2$ is a field or $k = 2$, then $\Gamma(R)$ is a complete bipartite graph. Hence, by König’s Theorem, we have $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$ and the proof is complete. \hfill $\square$

Now we are in a position to assert our main theorem.

**Theorem 3.** If $R$ is a finite ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.

The question of when $\Gamma(R) \cong \Gamma(S)$ implies that $R \cong S$ is very interesting and this question has been investigated in [2] and [3]. In [3] it is shown that for any commutative ring $R$, $\Gamma(\Gamma(R))$ and $\Gamma(R)$ are isomorphic, where $T(R)$ is the ring of fractions of $R$ with respect to the multiplicatively closed subset $R \setminus Z(R)$ of $R$.

**Theorem 4.** If $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ are finite local rings, then the following hold:

(i) For $n \geq 2$, $\Gamma(R_1 \times \cdots \times R_n) \cong \Gamma(S_1)$ if and only if $n = 2$ and either $R_1 \times R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R_1 \times R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. In the first case either $S_1 \cong \mathbb{Z}_3[x]/(x^2)$ and in the later case $S_1$ is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$.

(ii) For $n, m \geq 2$, $\Gamma(R_1 \times \cdots \times R_n) \cong \Gamma(S_1 \times \cdots \times S_m)$ if and only if $n = m$ and there exists a permutation $\pi$ over $\{1, \ldots, n\}$ such that for any $i$, $1 \leq i \leq n$, $|R_i| = |S_{\pi(i)}|$ and $\Gamma(R_i) \cong \Gamma(S_{\pi(i)})$.

**Proof.** (i) Since $n \geq 2$, we have $\Gamma(R_1 \times \cdots \times R_n) \cong \Gamma(S_1)$ is not empty and thus $S_1$ is not a field. Since $\Gamma(S_1)$ has a vertex which is adjacent to every other vertex in $\Gamma(S_1)$, by [4, Theorem 2.5], we have $R_1 \times \cdots \times R_n \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field. Thus $n = 2$. On the other hand, since $\Gamma(S_1) \cong \Gamma(\mathbb{Z}_2 \times F)$ is a star, by [4, Theorem 2.13], we conclude that $\Gamma(\mathbb{Z}_2 \times F)$ has fewer than four vertices. Hence $|F| \leq 3$, and $F \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Now, by [2, Example 2.1(a)], the proof is complete. The other direction of the theorem is proved by direct verification.

(ii) First suppose that $n = m$ and $|R_i| = |S_i|$ and $\Gamma(R_i) \cong \Gamma(S_i)$ for any $i$, $1 \leq i \leq n$. Define the function $f_i : R_i \rightarrow S_i$, by $f_i(0) = 0$, and $f_i$ is one to one correspondence between $R_i \setminus Z(R_i)$ and $S_i \setminus Z(S_i)$ and the restriction of $f_i$ to $Z(R_i)^*$ is a graph isomorphism between $\Gamma(R_i)$ and $\Gamma(S_i)$. Now, it is easy to see that the function $f : \Gamma(R_1 \times \cdots \times R_n) \rightarrow \Gamma(S_1 \times \cdots \times S_m)$ defined by $f((x_1, \ldots, x_n)) = (f_1(x_1), \ldots, f_n(x_n))$ is a graph isomorphism.

Conversely suppose that $f : \Gamma(R_1 \times \cdots \times R_n) \rightarrow \Gamma(S_1 \times \cdots \times S_m)$ is a graph isomorphism. By Remark 2, without loss of generality we may assume that $x = (r, 0, \ldots, 0)$ is a vertex with maximum degree in $\Gamma(R_1 \times \cdots \times R_n)$. Thus $f(x)$ in $\Gamma(S_1 \times \cdots \times S_m)$ has maximum degree. By applying a permutation, we may assume that $y = f(x) = (s, 0, \ldots, 0)$. Now, we show that $|R_1| = |S_1|$ and $\Gamma(R_1) \cong \Gamma(S_1)$. First assume that $R_1 \cong \mathbb{Z}_2$. Toward a contradiction, suppose that $S_1$ is not isomorphic to $\mathbb{Z}_2$. If $B = (S_1 \setminus (Z(S_1) \cup \{s\})) \times \{0\} \times \cdots \times \{0\}$, then every vertex in $B$ has maximum degree.
among all vertices in $\Gamma(S_1 \times \cdots \times S_m)$ which are not adjacent to $y$. But among all vertices of $\Gamma'(R_1 \times \cdots \times R_n)$ which are not adjacent to $x$, those vertices having maximum degree are those whose first components are one and have just one non-zero component other than their first components. For instance, assume that $(1, t, 0, \ldots, 0)$ is one of these vertices. We know that $d((1, t, 0, \ldots, 0)) = |Z(R_2)||R_3| \cdots |R_n| - 1$ and the degree of each vertex in $B$ is $|S_2| \cdots |S_m| - 1$. This implies that $|Z(R_2)||R_3| \cdots |R_n| = |S_2| \cdots |S_m|$. Also, we have $d(x) = |S_2| \cdots |R_n| - 1$. If $S_1$ is a field, then we have $d(y) = |S_2| \cdots |S_m| - 1$. It follows that $|R_2| \cdots |R_n| - 1 = |S_2| \cdots |S_m| - 1$. Therefore $|R_2| = |Z(R_2)|$, which is a contradiction. Thus we conclude that $S_1$ is not a field. Hence we find that $d(y) = |Z(S_1)| |S_2| \cdots |S_m| - 2$. This yields $|R_2| \cdots |R_n| - 1 = |Z(S_1)| |S_2| \cdots |S_m| - 2$, hence $|Z(R_2)||R_3| \cdots R_n |(Z(S_1)| - |Z(R_2)/Z(R_2)|) = 1$. Therefore $n = 2$ and $|Z(R_2)| = 1$. It follows that $R_2$ is a field. Thus $x$ is adjacent to the all vertices of $\Gamma(R_1 \times \cdots \times R_n)$ and since $B \neq \emptyset$, it is a contradiction.

Thus we may assume that neither $R_1$ nor $S_1$ is isomorphic to $\mathbb{Z}_2$. If $A = (R_1 \setminus (Z(R_1) \cup \{r\})) \times \{0\} \times \cdots \times \{0\}$, then every vertex in $A$ has maximum degree among all vertices in $\Gamma'(R_1 \times \cdots \times R_n)$ which are not adjacent to $x$. The degree of any vertex in $A$ is equal to $|R_2| \cdots |R_n| - 1$. Also, since $S_1 \not\subset \mathbb{Z}_2$, $B$ is the set of all vertices in $\Gamma'(S_1 \times \cdots \times S_m)$ with maximum degree among the all vertices which are not adjacent to $y$. Since the degree of each vertex in $B$ is $|S_2| \cdots |S_m| - 1$, we should have $|R_2| \cdots |R_n| - 1 = |S_2| \cdots |S_m| - 1$.

If $R_1$ is a field and $S_1$ is not a field, as we saw in the previous case, we have $d(x) = |R_2| \cdots |R_n| - 1$ and $d(y) = |Z(S_1)| |S_2| \cdots |S_m| - 2$, hence $|R_2| \cdots |R_n| |(Z(S_1)| - 1) = 1$, a contradiction. Thus both $R_1$ and $S_1$ are fields or none of them are fields. First suppose that $R_1$ and $S_1$ are fields. Now, we know that $|A| = |R_1| - 2$ and $|B| = |S_1| - 2$ are equal. This implies that $|R_1| = |S_1|$. Since in this case $\Gamma'(R_1)$ and $\Gamma'(S_1)$ are empty, there is nothing to prove.

So, suppose that $R_1$ and $S_1$ are not fields. Hence $d(x) = |Z(R_1)||R_2| \cdots |R_n| - 2$ and $d(y) = |Z(S_1)| |S_2| \cdots |S_m| - 2$. This implies that $|Z(R_1)||R_2| \cdots |R_n| = |Z(S_1)| |S_2| \cdots |S_m|$ and so we obtain $|Z(R_1)| = |Z(S_1)|$. Now, we know that $|A| = |R_1| - |Z(R_1)|$ and $|B| = |S_1| - |Z(S_1)|$ are equal, hence $|R_1| = |S_1|$. Clearly, the restriction of $f$ to $A$ is a one to one correspondence between $A$ and $B$. So we may assume that $f(u, 0, 0, \ldots, 0) = (u, S_1 Z(S_1))$ if $a \in Z(R_1)$ and $f(a, 0, 0, 0) = (b_1, \ldots, b_m)$, we show that $b_2 = \cdots = b_m = 0$. Since every vertex adjacent to $(1, 0, 0, \ldots, 0)$ in $\Gamma'(R_1 \times \cdots \times R_n)$ is adjacent to $(a, 0, 0, \ldots, 0)$, every vertex adjacent to $(a, 0, 0, \ldots, 0)$ is adjacent to $(b_1, \ldots, b_m)$. Since, for any $i$, $2 \leq i \leq m$, the vertices $e_i$ are adjacent to $(u, 0, 0, \ldots, 0)$, we have $b_2 = \cdots = b_m = 0$, where $e_i$ is the element whose $i$th component is one and other components are zero. Thus $b_1 \neq 0$. This implies that the function $f_1 : \Gamma'(R_1) \rightarrow \Gamma'(S_1)$ defined by $a \rightarrow f(a, 0, 0, 0) = (b, 0, 0, \ldots, 0)$ is a graph isomorphism, and thus $\Gamma(R_1) \cong \Gamma(S_1)$.

If $(a_0, a_2, \ldots, a_n)$ is non-zero, then $f(0, a_2, \ldots, a_n)$ is adjacent to $(u, 0, 0, \ldots, 0)$. So, we may write $f(a_0, a_2, \ldots, a_n) = (0, b_2, \ldots, b_m)$. Now, we show that the function $f'(R_2 \times \cdots \times R_n) \rightarrow \Gamma(S_2 \times \cdots \times S_m)$ defined by $(a_0, a_2, \ldots, a_n) \rightarrow f(0, a_2, \ldots, a_n) = (0, b_2, \ldots, b_m) \rightarrow (b_2, \ldots, b_m)$ is well-defined. Indeed, if $(a_0, a_2, \ldots, a_n)$ is a vertex in $\Gamma'(R_2 \times \cdots \times R_n)$, then there exists an index $i$, $2 \leq i \leq m$, such that $b_i$ is a zero-divisor.

The reason is that otherwise $d((0, b_2, \ldots, b_m)) = |S_1| - 1$ whereas $d((0, a_2, \ldots, a_n)) > |R_1| - 1$, because at least one of the $a_i$’s is zero-divisor. Clearly $f'$ is a graph isomorphism
and therefore $\Gamma(R_2 \times \cdots \times R_n) \cong \Gamma(S_2 \times \cdots \times S_m)$. If $n, m \geq 3$, we repeat this procedure.

Suppose that $n > m$. Thus, by rearrangement, we may assume that $\Gamma(R_m \times \cdots \times R_n) \cong \Gamma(S_m)$. By part (i), we have $R_m \times \cdots \times R_n \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $|S_m| = 8$ or $9$.

Hence $n = m + 1$. Since $[0] \times \cdots \times [0] \times R_m \times [0]$ contains a vertex of maximum degree in $\Gamma(R_1 \times \cdots \times R_n)$, by Remark 2, we have $R_1 \cong \cdots \cong R_{m-1} \cong \mathbb{Z}_2$. This implies that $S_1 \cong \cdots \cong S_{m-1} \cong \mathbb{Z}_2$. Now, we have $\Delta(\Gamma(R_1 \times \cdots \times R_n)) = 2^{n-1} - 1$ or $3 \cdot 2^{n-2} - 1$ and $\Delta(\Gamma(S_1 \times \cdots \times S_m)) = 2^{m-2}|S_m| - 1$. Thus $|S_m| = 4$ or $6$, a contradiction. Hence $n = m$. So, by repeating the above proof and rearrangement, we have $\Gamma(R_i) \cong \Gamma(S_i)$ for any $i, 1 \leq i \leq n$, and $|R_i| = |S_i|$ for any $i, 1 \leq i \leq n$. Now, since $\Gamma(R_1 \times \cdots \times R_n)$ and $\Gamma(S_1 \times \cdots \times S_m)$ have the same maximum degree we conclude that $|R_n| = |S_n|$ and the proof is complete. □

Recently Anderson, Frazier, Lauve, and Livingston in [2] have proved that if $R$ and $S$ are finite reduced rings which are not fields, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$. In what follows we generalize this result. Indeed we show that if one of the two rings is reduced the assertion remains true.

**Theorem 5.** Let $R$ be a finite reduced ring and $S$ be a ring such that $S$ is not an integral domain. If $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$, unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$ and $S$ is a local ring.

**Proof.** Since $\Gamma(S)$ is finite, by [4, Theorem 2.2], we have $S$ is finite. Since $\Gamma(R)$ is not empty, $R$ is not a field. Thus by [5, Theorem 8.7, p. 90] we may write $R \cong F_1 \times \cdots \times F_n$ and $S \cong S_1 \times \cdots \times S_m$, where $n \geq 2$ and $F_i$’s are finite fields and $S_i$’s are finite local rings. If $m = 1$, by part (i) of the previous theorem, $n = 2$ and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$. So, suppose that $n, m \geq 2$. Now, by part (ii) of the previous theorem, we have $n = m$ and there exists a permutation $\pi$ over $[1, \ldots, n]$ such that $\Gamma(S_i) \cong \Gamma(F_{\pi(i)})$ and $|S_i| = |F_{\pi(i)}|$. Since the $F_i$’s are finite fields, $S_i \cong F_{\pi(i)}$ for any $i, 1 \leq i \leq n$. Thus $R \cong S$ and the proof is complete. □

Now we want to characterize all regular graphs which can be the zero-divisor graph of a commutative ring. The following theorem shows that any infinite zero-divisor graph has a vertex with infinite degree.

**Theorem 6.** If $R$ is a ring such that $R$ is not an integral domain and every vertex of $\Gamma(R)$ has finite degree, then $R$ is a finite ring.

**Proof.** Suppose $R$ is an infinite ring. Let $x$ and $y$ be non-zero elements of $R$ such that $xy = 0$. Then $yR^* \subseteq \text{Ann}(x)$. If $yR^*$ is infinite, then $x$ has infinite degree in $\Gamma(R)$. If $yR^*$ is finite, there exists an infinite subset $A$ of $R^*$ such that if $a_1, a_2 \in A$, then $y(a_1 \cdots a_2)$. If $a_0$ is a fixed element of $A$, then $\{a_0 - a \mid a \in A\}$ is an infinite subset of $\text{Ann}(y)$ and so $y$ has infinite degree in $\Gamma(R)$, a contradiction. □

**Theorem 7.** Let $R$ be a finite ring. If $\Gamma(R)$ is a regular graph, then it is either a complete graph or a complete bipartite graph.
Similarly, $R$ includes all vertices of $\Gamma(R)$ to see that a Hamiltonian cycle in $x/\Gamma(R)$ then $R$. Hence suppose $\{Z(R)\}$ is a bipartite graph. Consequently, the proof is complete. 

Theorem 8. Let $R$ be a finite decomposable ring. If $\Gamma(R)$ is a Hamiltonian graph, then $\Gamma(R) \simeq K_{n,n}$ for some natural number $n$.

Proof. Since $R$ is a decomposable ring, we may write $R = R_1 \times R_2$. Clearly, it suffices to show that $R_1$ and $R_2$ are fields. Suppose that $Z(R_1) \neq \{0\}$. Put $A = Z(R_1)^* \times (R_2 \setminus Z(R_2))$ and $B = Z(R_1)^* \times \{0\}$. We note that $B$ is the set of all vertices adjacent to at least one vertex of $A$, and that there are no edges between the vertices of $A$. Now, it is easy to see that a Hamiltonian cycle in $\Gamma(R)$ contains a matching between $A$ and $B$ which includes all vertices of $A$. Hence $|A| \leq |B|$ and this implies that $|R_2 \setminus Z(R_2)| \leq 1$. Because a commutative Artinian ring is a finite direct product of local rings, and since the only non-zero-divisor element of $R_2$ is the identity, $R_2$ must be a finite direct product of $\mathbb{Z}_2$'s. Let $x$ be that element of $R_2$ whose first component is zero and other components are one. So $(1, x)$ is a vertex of degree 1 in $\Gamma(R)$, which is impossible. Thus $R_1$ and similarly $R_2$ are fields and the proof is complete. 

Theorem 9. Let $R$ be a finite principal ideal ring. If $\Gamma(R)$ is a Hamiltonian graph, then it is either a complete graph or a complete bipartite graph.

Proof. If $R$ is a decomposable ring, then by the previous theorem, $\Gamma(R)$ is a complete bipartite graph. Hence suppose $R$ is an indecomposable ring. Now by [5, Theorem 8.7, p. 90], $R$ is a local ring and $Z(R)$ is a principal ideal. Let $Z(R) = Rx$. If $Ann(x) \neq Z(R)$, then $x \notin Ann(x)$. Since $Ann(x) = Ann(Z(R))$, the vertices of $x + Ann(x)$ are adjacent to all vertices of $Ann(x)^*$ and not adjacent to any other vertex. Now, along a Hamiltonian cycle, when we leave a vertex of $x + Ann(x)$ we reach a vertex of $Ann(x)^*$, but this is impossible, since $|Ann(x)^*| < |x + Ann(x)|$. Thus $Ann(x) = Z(R)$ and $\Gamma(R)$ is a complete graph. 

Corollary 1. The graph $\Gamma(\mathbb{Z}_n)$ is a Hamiltonian graph if and only if $n = p^2$, where $p$ is a prime more than 3 and in this case $\Gamma(\mathbb{Z}_n) \simeq K_{p-1}$.

Proof. If $\mathbb{Z}_n$ is a decomposable ring as we saw in the proof of Theorem 8, then $\mathbb{Z}_n \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime number, a contradiction. Now if $\mathbb{Z}_n$ is an indecomposable
then \( n = p^r \), where \( p \) is a prime number and \( r \) is a natural number. If \( r \geq 3 \), then two vertices \( p \) and \( 2p \) are not adjacent and according to the proof of Theorem 9, we get a contradiction. Therefore \( r = 2 \) and \( \Gamma(\mathbb{Z}_n) \cong K_{p-1} \). □

**Remark 3.** If \( R = \mathbb{Z}_3[x, y]/(x^2 + xy, y^2 + 2xy) \), then \( R \) is a local ring with unique maximal ideal \( Z(R) \) such that \( Z(R)^3 = \{0\} \). Note that \( \Gamma(R) \) is a Hamiltonian graph which is neither a complete graph nor a complete bipartite graph (since \( \bar{x} \) and \( \bar{y} \) are not adjacent and \( \bar{x}, \bar{x} \bar{y}, 2 \bar{x} \bar{y} \) are mutually adjacent). The following sequence shows a Hamiltonian cycle in \( \Gamma(R) \):

\[
\begin{align*}
\bar{x} &\rightarrow \bar{x} + \bar{y} \rightarrow \bar{x} + \bar{x} \bar{y} \rightarrow \bar{x} + \bar{y} + \bar{x} \bar{y} \rightarrow \bar{x} + 2\bar{x} \bar{y} \rightarrow \bar{x} + \bar{y} + 2\bar{x} \bar{y} \rightarrow 2\bar{x} \\
&\rightarrow 2\bar{x} + 2\bar{y} \rightarrow 2\bar{x} + \bar{x} \bar{y} \rightarrow 2\bar{x} + 2\bar{y} + \bar{x} \bar{y} \rightarrow 2\bar{x} + 2\bar{x} \bar{y} \rightarrow 2\bar{x} + 2\bar{y} + 2\bar{x} \bar{y} \rightarrow \bar{x} \bar{y} \\
&\rightarrow \bar{y} \rightarrow \bar{x} + 2\bar{y} \rightarrow \bar{y} + \bar{x} \bar{y} \rightarrow \bar{x} + 2\bar{y} + \bar{x} \bar{y} \rightarrow \bar{y} + 2\bar{x} \bar{y} \rightarrow \bar{x} + 2\bar{y} + 2\bar{x} \bar{y} \rightarrow 2\bar{y} \\
&\rightarrow 2\bar{y} + \bar{y} \rightarrow 2\bar{y} + \bar{x} \bar{y} \rightarrow 2\bar{y} + \bar{y} + \bar{x} \bar{y} \rightarrow 2\bar{y} + 2\bar{x} \bar{y} \rightarrow 2\bar{y} + \bar{y} + 2\bar{x} \bar{y} \rightarrow 2\bar{x} \bar{y} \rightarrow \bar{x}. 
\end{align*}
\]

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**References**