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On the zero-divisor graph of a commutative ring

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Abstract

Let R be a commutative ring and $\Gamma(R)$ be its zero-divisor graph. In this paper it is shown that for any finite commutative ring R, the edge chromatic number of $\Gamma(R)$ is equal to the maximum degree of $\Gamma(R)$, unless $\Gamma(R)$ is a complete graph of odd order. In [D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, in: Lecture Notes in Pure and Appl. Math., Vol. 220, Marcel Dekker, New York, 2001, pp. 61–72] it has been proved that if R and S are finite reduced rings which are not fields, then $\Gamma(R) \simeq \Gamma(S)$ if and only if $R \simeq S$. Here we generalize this result and prove that if R is a finite reduced ring which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_6$ and S is a ring such that $\Gamma(R) \simeq \Gamma(S)$, then $R \simeq S$.

Keywords: Zero-divisor graph; Edge coloring; Hamiltonian

Introduction

The concept of zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [6]. He let all elements of the ring be vertices of the graph and was interested mainly in colorings. In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures

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of rings. The zero-divisor graph of a commutative ring has been studied extensively by Anderson, Frazier, Lauve, Levy, Livingston and Shapiro, see [2–4]. The zero-divisor graph concept has recently been extended to non-commutative rings, see [7].

Throughout the paper, all rings are assumed to be commutative with unity $1 \neq 0$. If R is a ring, $Z(R)$ denotes its set of zero-divisors. A ring R is said to be *reduced* if R has no non-zero nilpotent element. A ring R is said to be *decomposable* if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise R is said to be *indecomposable*. If X is either an element or a subset of R, then $\mathrm{Ann}(X)$ denotes the annihilator of X in R. For any subset X of R, we define $X^* = X \backslash \{0\}$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^*$ in which two vertices x and y are adjacent if and only if $x \neq y$ and $xy = 0$.

For a graph G, the degree $d(v)$ of a vertex v in G is the number of edges incident to v. We denote the minimum and maximum degree of vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is *regular* if the degrees of all vertices of G are the same. We denote the *complete graph* with n vertices and *complete bipartite graph* with two parts of sizes m and n, by K_n and $K_{m,n}$, respectively. The complete bipartite graph $K_{1,n}$, is called a *star*. A *Hamiltonian cycle* of G is a cycle that contains every vertex of G. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. A subset X of the vertices of G is called a *clique* if the induced subgraph on X is a complete graph. A *k-vertex coloring* of a graph G is an assignment of k colors $\{1, \ldots, k\}$ to the vertices of G such that no two adjacent vertices have the same color. The *vertex chromatic number* $\chi(G)$ of a graph G, is the minimum k for which G has a k-vertex coloring. A *k-edge coloring* of a graph G is an assignment of k colors $\{1, \ldots, k\}$ to the edges of G such that no two adjacent edges have the same color. The *edge chromatic number* $\chi'(G)$ of a graph G, is the minimum k for which G has a k-edge coloring. A graph G is said to be *critical* if G is connected and $\chi'(G) = \Delta(G) + 1$ and for any edge e of G, we have $\chi'(G \setminus \{e\}) < \chi'(G)$.

Beck in [6] proved several interesting theorems for the vertex chromatic number of a zero-divisor graph. For example, he showed that for any commutative ring R, if R is a direct product of finitely many reduced rings and principal ideal rings, then $\chi(\Gamma(R))$ equals to the size of maximum clique of $\Gamma(R)$. Although Beck used a different graph, his results apply to the current setting. There are many interesting questions about zero-divisor graphs. For instance, Anderson, Frazier, Lauve and Livingston asked in [2]: “For which finite commutative rings R, is $\Gamma(R)$ planar?” In [1] it was proved that if R is a finite local ring such that $\Gamma(R)$ has at least 33 vertices, then $\Gamma(R)$ is not a planar graph.

Results

The vertex chromatic number of zero-divisor graphs has been studied extensively by Beck in [6]. Here we will study the edge chromatic number of zero-divisor graphs and prove that if R is a finite commutative ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.

If G is a graph, clearly in any edge coloring of G, the edges incident with one vertex should be colored with different colors. This observation implies that $\chi'(G) \geqslant \Delta(G)$. An important theorem due to Vizing is the following.

Vizing's Theorem [8, p. 16]. *If G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

Also the following lemma is a key to our proof.

Vizing's Adjacency Lemma [8, p. 24]. *If G is a critical graph, then G has at least $\Delta(G) - \delta(G) + 2$ vertices of maximum degree.*

Remark 1. We note that if G is a graph and $\chi'(G) = \Delta(G) + 1$, then there exists a subgraph of G, say G_1, such that $\chi'(G_1) = \Delta(G) + 1$ and for any edge e of G_1 we have $\chi'(G_1 \backslash \{e\}) = \Delta(G)$. Clearly G_1 has a connected subgraph, say H, such that $\chi'(H) = \Delta(G) + 1$. The graph H is a critical graph with maximum degree $\Delta(G)$. If x is a vertex of H with degree $\Delta(G)$, then by Vizing's Adjacency Lemma, H has at least $\Delta(G) - d_H(v) + 2$ vertices of degree $\Delta(G)$, for any vertex v which is adjacent to x. Therefore if G is a graph such that for every vertex u of maximum degree there exists an edge uv such that $\Delta(G) - d(v) + 2$ is more than the number of vertices with maximum degree in G, then by the above argument and Vizing's Theorem, we have $\chi'(G) = \Delta(G)$.

It is not hard to see that if R is an Artinian local ring, then the Jacobson radical of R equals $Z(R)$. Thus $Z(R)$ is a nilpotent ideal and this implies that if R is not a field, then $\mathrm{Ann}(Z(R)) \neq \{0\}$. Moreover, each element of $\mathrm{Ann}(Z(R))^*$ is adjacent to each other vertex of $\Gamma(R)$.

Theorem 1. *If R is a finite local ring which is not a field, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.*

Proof. Since R is a finite local ring, $\mathrm{Ann}(Z(R)) \neq \{0\}$. If $\Gamma(R)$ is a complete graph, then by [8, Theorem 1.2, p. 12], we are done. Thus suppose that $\Gamma(R)$ is not a complete graph and so $\mathrm{Ann}(Z(R)) \neq Z(R)$. If $x \in Z(R) \backslash \mathrm{Ann}(Z(R))$, then there is an element $a \in Z(R)$ such that $ax \neq 0$. This implies that x is adjacent to no vertices of $a + \mathrm{Ann}(Z(R))$. Therefore $d(x) \leqslant |Z(R)^*| - |\mathrm{Ann}(Z(R))|$. Hence $\Delta(\Gamma(R)) - d(x) + 2 \geqslant |\mathrm{Ann}(Z(R))| + 1$. Clearly, $\mathrm{Ann}(Z(R))^*$ is the set of all vertices of maximum degree in $\Gamma(R)$. So, by Remark 1, we have $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. □

Now using König's Theorem, we show that the previous theorem is true for any finite commutative ring.

König's Theorem [8, p. 11]. *For any bipartite graph G, we have $\chi'(G) = \Delta(G)$.*

Remark 2. Assume that $R = R_1 \times \cdots \times R_n$ is a finite decomposable commutative ring. We note that if $x = (x_1, \ldots, x_n)$ has maximum degree in $\Gamma(R)$, then x has exactly one non-zero component, say x_1. Now suppose that R_1 is a local ring. We consider two cases: If R_1 is a field, then $\Delta(\Gamma(R)) = d(x) = |R_2| \cdots |R_n| - 1$; If R_1 is not a field, then we have $x_1 \in \mathrm{Ann}(Z(R_1))^*$ and $\Delta(\Gamma(R)) = d(x) = |Z(R_1)||R_2| \cdots |R_n| - 2$.

Theorem 2. *If R is a finite decomposable ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$.*

Proof. It is well known that every commutative Artinian ring is isomorphic to the direct product of finitely many local rings, see [5, p. 90]. Suppose that $R = R_1 \times \cdots \times R_n$, where $n \geqslant 2$ and each R_i is a local ring. By Remark 2, without loss of generality suppose that the non-zero components of the vertices with maximum degree in $\Gamma(R)$ occur in $R_1, \ldots, R_k$. First we claim that all of the rings $R_1, \ldots, R_k$ are fields or none of them are fields. Working towards a contradiction suppose that R_1 is a field and R_2 is not a field. Now, every vertex with maximum degree in $R_1 \times \{0\} \times \cdots \times \{0\}$ has degree $|R_2| \cdots |R_n| - 1$ and each vertex with maximum degree in $\{0\} \times R_2 \times \{0\} \times \cdots \times \{0\}$ has degree $|R_1||Z(R_2)||R_3| \cdots |R_n| - 2$. Thus we have $|Z(R_2)||R_3| \cdots |R_n|(|R_1| - |R_2/Z(R_2)|) = 1$, a contradiction. Therefore by Remark 2, for any i, $1 \leqslant i \leqslant k$, $\Delta(\Gamma(R)) = |R_1| \cdots |R_{i-1}||Z(R_i)||R_{i+1}| \cdots |R_n| - \varepsilon$, where $\varepsilon = 1$ or 2. Hence, we have $|R_1/Z(R_1)| = \cdots = |R_k/Z(R_k)|$. Moreover, since for each j, $k+1 \leqslant j \leqslant n$, the degree of any vertex in $\{0\} \times \cdots \times \{0\} \times R_j \times \{0\} \times \cdots \times \{0\}$ is less than $\Delta(\Gamma(R))$, we have

$$\left|R_j/Z(R_j)\right| \geqslant \left|R_1/Z(R_1)\right|. \qquad (*)$$

For any t, $1 \leqslant t \leqslant n$, suppose that e_t is the element whose tth component is one and other components are zero. First, suppose that the rings $R_1, \ldots, R_k$ are not fields. Then $\Gamma(R)$ has $\sum_{t=1}^{k} |\mathrm{Ann}(Z(R_t))^*|$ vertices of maximum degree. Clearly, every vertex of maximum degree in $\Gamma(R)$ is adjacent to at least one of the e_t's. Now for any i, $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}
\Delta\bigl(\Gamma(R)\bigr) - d(e_i) + 2 &\geqslant \bigl(|R_1| \cdots |R_{i-1}|\bigl|Z(R_i)\bigr||R_{i+1}| \cdots |R_n| - 2\bigr) \\
&\quad - \bigl(|R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1\bigr) + 2 \\
&= |R_1| \cdots |R_{i-1}|\bigl(\bigl|Z(R_i)\bigr| - 1\bigr)|R_{i+1}| \cdots |R_n| + 1 \\
&> \sum_{t=1}^{k} \bigl|\mathrm{Ann}\bigl(Z(R_t)\bigr)^*\bigr|.
\end{aligned}$$

Hence by Remark 1, we conclude that $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. Next, suppose that the rings $R_1, \ldots, R_k$ are fields. Then $\Gamma(R)$ has $\sum_{t=1}^{k} |R_t^*|$ vertices of maximum degree. If $n > 2$, then every vertex of maximum degree in $\Gamma(R)$ is adjacent to $1 - e_t$, for some t, $1 \leqslant t \leqslant k$. Note that in this case $|R_1| = \cdots = |R_k|$ and if we set $|R_1| = a$, then by $(*)$ we have $|R_j| \geqslant a$, for any j, $j > k$. Now since $a^{n-1} - a + 2 > n(a-1)$, for any i, $1 \leqslant i \leqslant k$, we have

$$\begin{aligned}
\Delta\bigl(\Gamma(R)\bigr) - d(1 - e_i) + 2 &= \bigl(|R_1| \cdots |R_{i-1}||R_{i+1}| \cdots |R_n| - 1\bigr) - \bigl(|R_i| - 1\bigr) + 2 \\
&\geqslant a^{n-1} - a + 2 > \sum_{t=1}^{k} \bigl|R_t^*\bigr|.
\end{aligned}$$

Thus by Remark 1, we conclude that $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. So assume that $n = 2$. If $k = 1$ and R_2 is not a field, then by $(*)$ we have $|R_2| \geqslant 2|R_1|$. Since in this case any vertex of maximum degree in $\Gamma(R)$ is adjacent to e_2 and $\Delta(\Gamma(R)) - d(e_2) + 2 = (|R_2| - 1) - (|R_1| - 1) + 2 > |R_1^*|$, by Remark 1, we obtain $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$. If either $k = 1$ and R_2 is a field or $k = 2$, then $\Gamma(R)$ is a complete bipartite graph. Hence, by König's Theorem, we have $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$ and the proof is complete. □

Now we are in a position to assert our main theorem.

Theorem 3. *If R is a finite ring, then $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$, unless $\Gamma(R)$ is a complete graph of odd order.*

The question of when $\Gamma(R) \simeq \Gamma(S)$ implies that $R \simeq S$ is very interesting and this question has been investigated in [2] and [3]. In [3] it is shown that for any commutative ring R, $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic, where $T(R)$ is the ring of fractions of R with respect to the multiplicatively closed subset $R \setminus Z(R)$ of R.

Theorem 4. *If $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ are finite local rings, then the following hold*:

(i) *For $n \geqslant 2$, $\Gamma(R_1 \times \cdots \times R_n) \simeq \Gamma(S_1)$ if and only if $n = 2$ and either $R_1 \times R_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R_1 \times R_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$. In the first case either $S_1 \simeq \mathbb{Z}_9$ or $S_1 \simeq \mathbb{Z}_3[x]/(x^2)$ and in the later case S_1 is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$.*

(ii) *For $n, m \geqslant 2$, $\Gamma(R_1 \times \cdots \times R_n) \simeq \Gamma(S_1 \times \cdots \times S_m)$ if and only if $n = m$ and there exists a permutation π over $\{1, \ldots, n\}$ such that for any i, $1 \leqslant i \leqslant n$, $|R_i| = |S_{\pi(i)}|$ and $\Gamma(R_i) \simeq \Gamma(S_{\pi(i)})$.*

Proof. (i) Since $n \geqslant 2$, we have $\Gamma(R_1 \times \cdots \times R_n) \simeq \Gamma(S_1)$ is not empty and thus S_1 is not a field. Since $\Gamma(S_1)$ has a vertex which is adjacent to every other vertex in $\Gamma(S_1)$, by [4, Theorem 2.5], we have $R_1 \times \cdots \times R_n \simeq \mathbb{Z}_2 \times F$, where F is a finite field. Thus $n = 2$. On the other hand, since $\Gamma(S_1) \simeq \Gamma(\mathbb{Z}_2 \times F)$ is a star, by [4, Theorem 2.13], we conclude that $\Gamma(\mathbb{Z}_2 \times F)$ has fewer than four vertices. Hence $|F| \leqslant 3$, and $F \simeq \mathbb{Z}_2$ or $\mathbb{Z}_3$. Now, by [2, Example 2.1(a)], the proof is complete. The other direction of the theorem is proved by direct verification.

(ii) First suppose that $n = m$ and $|R_i| = |S_i|$ and $\Gamma(R_i) \simeq \Gamma(S_i)$ for any i, $1 \leqslant i \leqslant n$. Define the function $f_i : R_i \to S_i$, by $f_i(0) = 0$, and f_i is a one to one correspondence between $R_i \backslash Z(R_i)$ and $S_i \backslash Z(S_i)$ and the restriction of f_i to $Z(R_i)^*$ is a graph isomorphism between $\Gamma(R_i)$ and $\Gamma(S_i)$. Now, it is easy to see that the function $f : \Gamma(R_1 \times \cdots \times R_n) \to \Gamma(S_1 \times \cdots \times S_n)$ defined by $f(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n))$ is a graph isomorphism.

Conversely suppose that $f : \Gamma(R_1 \times \cdots \times R_n) \to \Gamma(S_1 \times \cdots \times S_m)$ is a graph isomorphism. By Remark 2, without loss of generality we may assume that $x = (r, 0, \ldots, 0)$ is a vertex with maximum degree in $\Gamma(R_1 \times \cdots \times R_n)$. Thus $f(x)$ in $\Gamma(S_1 \times \cdots \times S_m)$ has maximum degree. By applying a permutation, we may assume that $y = f(x) = (s, 0, \ldots, 0)$. Now, we show that $|R_1| = |S_1|$ and $\Gamma(R_1) \simeq \Gamma(S_1)$. First assume that $R_1 \simeq \mathbb{Z}_2$. Toward a contradiction, suppose that S_1 is not isomorphic to $\mathbb{Z}_2$. If $\mathcal{B} = (S_1 \backslash (Z(S_1) \cup \{s\})) \times \{0\} \times \cdots \times \{0\}$, then every vertex in $\mathcal{B}$ has maximum degree among all vertices in $\Gamma(S_1 \times \cdots \times S_m)$ which are not adjacent to y. But among all vertices of $\Gamma(R_1 \times \cdots \times R_n)$ which are not adjacent to x, those vertices having maximum degree are those whose first components are one and have just one non-zero component other than their first components. For instance, assume that $(1, t, 0, \ldots, 0)$ is one of these vertices. We know that $d((1, t, 0, \ldots, 0)) = |Z(R_2)||R_3| \cdots |R_n| - 1$ and the degree of each vertex in $\mathcal{B}$ is $|S_2| \cdots |S_m| - 1$. This implies that $|Z(R_2)||R_3| \cdots |R_n| = |S_2| \cdots |S_m|$. Also, we have $d(x) = |R_2| \cdots |R_n| - 1$. If S_1 is a field, then we have $d(y) = |S_2| \cdots |S_m| - 1$. It follows that $|R_2| \cdots |R_n| - 1 = |S_2| \cdots |S_m| - 1$. Therefore $|R_2| = |Z(R_2)|$, which is a contradiction. Thus we conclude that S_1 is not a field. Hence we find that $d(y) = |Z(S_1)||S_2| \cdots |S_m| - 2$. This yields $|R_2| \cdots |R_n| - 1 = |Z(S_1)||S_2| \cdots |S_m| - 2$, hence $|Z(R_2)||R_3| \cdots |R_n|(|Z(S_1)| - |R_2/Z(R_2)|) = 1$. Therefore $n = 2$ and $|Z(R_2)| = 1$. It follows that R_2 is a field. Thus x is adjacent to the all vertices of $\Gamma(R_1 \times \cdots \times R_n)$ and since $\mathcal{B} \neq \varnothing$, it is a contradiction. So $S_1 \simeq \mathbb{Z}_2$ and in this case the assertion is proved.

Thus we may assume that neither R_1 nor S_1 is isomorphic to $\mathbb{Z}_2$. If $\mathcal{A} = (R_1 \backslash (Z(R_1) \cup \{r\})) \times \{0\} \times \cdots \times \{0\}$, then every vertex in $\mathcal{A}$ has maximum degree among all vertices in $\Gamma(R_1 \times \cdots \times R_n)$ which are not adjacent to x. The degree of any vertex in $\mathcal{A}$ is equal to $|R_2| \cdots |R_n| - 1$. Also, since $S_1 \not\simeq \mathbb{Z}_2$, $\mathcal{B}$ is the set of all vertices in $\Gamma(S_1 \times \cdots \times S_m)$ with maximum degree among the all vertices which are not adjacent to y. Since the degree of each vertex in $\mathcal{B}$ is $|S_2| \cdots |S_m| - 1$, we should have $|R_2| \cdots |R_n| - 1 = |S_2| \cdots |S_m| - 1$.

If R_1 is a field and S_1 is not a field, as we saw in the previous case, we have $d(x) = |R_2| \cdots |R_n| - 1$ and $d(y) = |Z(S_1)||S_2| \cdots |S_m| - 2$, hence $|R_2| \cdots |R_n|(|Z(S_1)| - 1) = 1$, a contradiction. Thus both R_1 and S_1 are fields or none of them are fields. First suppose that R_1 and S_1 are fields. Now, we know that $|\mathcal{A}| = |R_1| - 2$ and $|\mathcal{B}| = |S_1| - 2$ are equal. This implies that $|R_1| = |S_1|$. Since in this case $\Gamma(R_1)$ and $\Gamma(S_1)$ are empty, there is nothing to prove.

So, suppose that R_1 and S_1 are not fields. Hence $d(x) = |Z(R_1)||R_2| \cdots |R_n| - 2$ and $d(y) = |Z(S_1)||S_2| \cdots |S_m| - 2$. This implies that $|Z(R_1)||R_2| \cdots |R_n| = |Z(S_1)||S_2| \cdots |S_m|$ and so we obtain $|Z(R_1)| = |Z(S_1)|$. Now, we know that $|\mathcal{A}| = |R_1| - |Z(R_1)|$ and $|\mathcal{B}| = |S_1| - |Z(S_1)|$ are equal, hence $|R_1| = |S_1|$. Clearly, the restriction of f to $\mathcal{A}$ is a one to one correspondence between $\mathcal{A}$ and $\mathcal{B}$. So we may assume that $f(1, 0, \ldots, 0) = (u, 0, \ldots, 0)$, where $u \in S_1 \backslash Z(S_1)$. If $a \in Z(R_1)$ and $f(a, 0, \ldots, 0) = (b_1, \ldots, b_m)$, we show that $b_2 = \cdots = b_m = 0$. Since every vertex adjacent to $(1, 0, \ldots, 0)$ in $\Gamma(R_1 \times \cdots \times R_n)$ is adjacent to $(a, 0, \ldots, 0)$, every vertex adjacent to $(u, 0, \ldots, 0)$ is adjacent to $(b_1, \ldots, b_m)$. Since, for any i, $2 \leqslant i \leqslant m$, the vertices e_i are adjacent to $(u, 0, \ldots, 0)$, we have $b_2 = \cdots = b_m = 0$, where e_i is the element whose ith component is one and other components are zero. Thus $b_1 \neq 0$. This implies that the function $f_1 : \Gamma(R_1) \to \Gamma(S_1)$ defined by $a \to f(a, 0, \ldots, 0) = (b, 0, \ldots, 0) \to b$ is a graph isomorphism, and thus $\Gamma(R_1) \simeq \Gamma(S_1)$.

If $(0, a_2, \ldots, a_n)$ is non-zero, then $f(0, a_2, \ldots, a_n)$ is adjacent to $(u, 0, \ldots, 0)$. So, we may write $f(0, a_2, \ldots, a_n) = (0, b_2, \ldots, b_m)$. Now, we show that the function $f' : \Gamma(R_2 \times \cdots \times R_n) \to \Gamma(S_2 \times \cdots \times S_m)$ defined by $(a_2, \ldots, a_n) \to f(0, a_2, \ldots, a_n) = (0, b_2, \ldots, b_m) \to (b_2, \ldots, b_m)$ is well-defined. Indeed, if $(a_2, \ldots, a_n)$ is a vertex in $\Gamma(R_2 \times \cdots \times R_n)$, then there exists an index i, $2 \leqslant i \leqslant m$, such that b_i is a zero-divisor. The reason is that otherwise $d((0, b_2, \ldots, b_m)) = |S_1| - 1$ whereas $d((0, a_2, \ldots, a_n)) > |R_1| - 1$, because at least one of the a_i's is zero-divisor. Clearly f' is a graph isomorphism and therefore $\Gamma(R_2 \times \cdots \times R_n) \simeq \Gamma(S_2 \times \cdots \times S_m)$. If $n, m \geqslant 3$, we repeat this procedure. Suppose that $n > m$. Thus, by rearrangement, we may assume that $\Gamma(R_m \times \cdots \times R_n) \simeq \Gamma(S_m)$. By part (i), we have $R_m \times \cdots \times R_n \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $|S_m| = 8$ or 9. Hence $n = m + 1$. Since $\{0\} \times \cdots \times \{0\} \times R_m \times \{0\}$ contains a vertex of maximum degree in $\Gamma(R_1 \times \cdots \times R_n)$, by Remark 2, we have $R_1 \simeq \cdots \simeq R_{m-1} \simeq \mathbb{Z}_2$. This implies that $S_1 \simeq \cdots \simeq S_{m-1} \simeq \mathbb{Z}_2$. Now, we have $\Delta(\Gamma(R_1 \times \cdots \times R_n)) = 2^{n-1} - 1$ or $3 \cdot 2^{n-2} - 1$ and $\Delta(\Gamma(S_1 \times \cdots \times S_m)) = 2^{m-2}|S_m| - 1$. Thus $|S_m| = 4$ or 6, a contradiction. Hence $n = m$. So, by repeating the above proof and rearrangement, we have $\Gamma(R_i) \simeq \Gamma(S_i)$ for any i, $1 \leqslant i \leqslant n$, and $|R_i| = |S_i|$ for any i, $1 \leqslant i \leqslant n-1$. Now, since $\Gamma(R_1 \times \cdots \times R_n)$ and $\Gamma(S_1 \times \cdots \times S_n)$ have the same maximum degree we conclude that $|R_n| = |S_n|$ and the proof is complete. □

Recently Anderson, Frazier, Lauve, and Livingston in [2] have proved that if R and S are finite reduced rings which are not fields, then $\Gamma(R) \simeq \Gamma(S)$ if and only if $R \simeq S$. In what follows we generalize this result. Indeed we show that if one of the two rings is reduced the assertion remains true.

Theorem 5. *Let R be a finite reduced ring and S be a ring such that S is not an integral domain. If $\Gamma(R) \simeq \Gamma(S)$, then $R \simeq S$, unless $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6$ and S is a local ring.*

Proof. Since $\Gamma(S)$ is finite, by [4, Theorem 2.2], we have S is finite. Since $\Gamma(R)$ is not empty, R is not a field. Thus by [5, Theorem 8.7, p. 90] we may write $R \simeq F_1 \times \cdots \times F_n$ and $S \simeq S_1 \times \cdots \times S_m$, where $n \geqslant 2$ and F_i's are finite fields and S_i's are finite local rings. If $m = 1$, by part (i) of the previous theorem, $n = 2$ and $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$. So, suppose that $n, m \geqslant 2$. Now, by part (ii) of the previous theorem, we have $n = m$ and there exists a permutation π over $\{1, \ldots, n\}$ such that $\Gamma(S_i) \simeq \Gamma(F_{\pi(i)})$ and $|S_i| = |F_{\pi(i)}|$. Since the F_i's are finite fields, $S_i \simeq F_{\pi(i)}$ for any i, $1 \leqslant i \leqslant n$. Thus $R \simeq S$ and the proof is complete. □

Now we want to characterize all regular graphs which can be the zero-divisor graph of a commutative ring. The following theorem shows that any infinite zero-divisor graph has a vertex with infinite degree.

Theorem 6. *If R is a ring such that R is not an integral domain and every vertex of $\Gamma(R)$ has finite degree, then R is a finite ring.*

Proof. Suppose R is an infinite ring. Let x and y be non-zero elements of R such that $xy = 0$. Then $yR^* \subseteq \mathrm{Ann}(x)$. If yR^* is infinite, then x has infinite degree in $\Gamma(R)$. If yR^* is finite, there exists an infinite subset A of R^* such that if $a_1, a_2 \in A$, then $ya_1 = ya_2$. If a_0 is a fixed element of A, then $\{a_0 - a \mid a \in A\}$ is an infinite subset of $\mathrm{Ann}(y)$ and so y has infinite degree in $\Gamma(R)$, a contradiction. □

Theorem 7. *Let R be a finite ring. If $\Gamma(R)$ is a regular graph, then it is either a complete graph or a complete bipartite graph.*

Proof. Assume that $\Gamma(R)$ is a regular graph of degree r. First we assume that $R = R_1 \times R_2$ is a decomposable ring. Since the degree of $(1,0)$ is $|R_2| - 1$ and the degree of $(0,1)$ is $|R_1| - 1$, we have $|R_1| = |R_2| = r + 1$. We show that R_1 is a field. If not, then there exist two non-zero elements a and b in R_1 such that $ab = 0$. But $(\{0\} \times R_2) \cup \{(b,1)\} \subseteq \mathrm{Ann}((a,0))$ and it follows that $d((a,0)) \geqslant r + 1$, a contradiction. Similarly, R_2 must be a field. So in this case, $\Gamma(R) \simeq K_{r,r}$. Now, suppose that R is an indecomposable ring. By [5, Theorem 8.7, p. 90], R is a local ring and $Z(R)$ is a nilpotent ideal. Thus $\mathrm{Ann}(Z(R)) \neq \{0\}$ and since $\Gamma(R)$ is a regular graph, we conclude that $\Gamma(R)$ is a complete graph. □

In the sequel we determine a family of commutative rings whose zero-divisor graphs are Hamiltonian.

Theorem 8. *Let R be a finite decomposable ring. If $\Gamma(R)$ is a Hamiltonian graph, then $\Gamma(R) \simeq K_{n,n}$, for some natural number n.*

Proof. Since R is a decomposable ring, we may write $R = R_1 \times R_2$. Clearly, it suffices to show that R_1 and R_2 are fields. Suppose that $Z(R_1) \neq \{0\}$. Put $\mathcal{A} = Z(R_1)^* \times (R_2 \backslash Z(R_2))$ and $\mathcal{B} = Z(R_1)^* \times \{0\}$. We note that $\mathcal{B}$ is the set of all vertices adjacent to at least one vertex of $\mathcal{A}$, and that there are no edges between the vertices of $\mathcal{A}$. Now, it is easy to see that a Hamiltonian cycle in $\Gamma(R)$ contains a matching between $\mathcal{A}$ and $\mathcal{B}$ which includes all vertices of $\mathcal{A}$. Hence $|\mathcal{A}| \leqslant |\mathcal{B}|$ and this implies that $|R_2 \backslash Z(R_2)| \leqslant 1$. Because a commutative Artinian ring is a finite direct product of local rings, and since the only non-zero-divisor element of R_2 is the identity, R_2 must be a finite direct product of $\mathbb{Z}_2$'s. Let x be that element of R_2 whose first component is zero and other components are one. So $(1,x)$ is a vertex of degree 1 in $\Gamma(R)$, which is impossible. Thus R_1 and similarly R_2 are fields and the proof is complete. □

Theorem 9. *Let R be a finite principal ideal ring. If $\Gamma(R)$ is a Hamiltonian graph, then it is either a complete graph or a complete bipartite graph.*

Proof. If R is a decomposable ring, then by the previous theorem, $\Gamma(R)$ is a complete bipartite graph. Hence suppose R is an indecomposable ring. Now by [5, Theorem 8.7, p. 90], R is a local ring and $Z(R)$ is a principal ideal. Let $Z(R) = Rx$. If $\mathrm{Ann}(x) \neq Z(R)$, then $x \notin \mathrm{Ann}(x)$. Since $\mathrm{Ann}(x) = \mathrm{Ann}(Z(R))$, the vertices of $x + \mathrm{Ann}(x)$ are adjacent to all vertices of $\mathrm{Ann}(x)^*$ and not adjacent to any other vertex. Now, along a Hamiltonian cycle, when we leave a vertex of $x + \mathrm{Ann}(x)$ we reach a vertex of $\mathrm{Ann}(x)^*$, but this is impossible, since $|\mathrm{Ann}(x)^*| < |x + \mathrm{Ann}(x)|$. Thus $\mathrm{Ann}(x) = Z(R)$ and $\Gamma(R)$ is a complete graph. □

Corollary 1. *The graph $\Gamma(\mathbb{Z}_n)$ is a Hamiltonian graph if and only if $n = p^2$, where p is a prime more than 3 and in this case $\Gamma(\mathbb{Z}_n) \simeq K_{p-1}$.*

Proof. If $\mathbb{Z}_n$ is a decomposable ring as we saw in the proof of Theorem 8, then $\mathbb{Z}_n \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime number, a contradiction. Now if $\mathbb{Z}_n$ is an indecomposable ring, then $n = p^r$, where p is a prime number and r is a natural number. If $r \geqslant 3$, then two vertices p and $2p$ are not adjacent and according to the proof of Theorem 9, we get a contradiction. Therefore $r = 2$ and $\Gamma(\mathbb{Z}_n) \simeq K_{p-1}$. □

Remark 3. If $R = \mathbb{Z}_3[x, y]/(x^2 + xy, y^2 + 2xy)$, then R is a local ring with unique maximal ideal $Z(R)$ such that $Z(R)^3 = \{0\}$. Note that $\Gamma(R)$ is a Hamiltonian graph which is neither a complete graph nor a complete bipartite graph (since $\bar{x}$ and $\bar{y}$ are not adjacent and $\bar{x},\ \bar{x}\bar{y},\ 2\bar{x}\bar{y}$ are mutually adjacent). The following sequence shows a Hamiltonian cycle in $\Gamma(R)$:

$$\begin{aligned}
&\bar{x} \to \bar{x} + \bar{y} \to \bar{x} + \bar{x}\bar{y} \to \bar{x} + \bar{y} + \bar{x}\bar{y} \to \bar{x} + 2\bar{x}\bar{y} \to \bar{x} + \bar{y} + 2\bar{x}\bar{y} \to 2\bar{x} \\
&\quad \to 2\bar{x} + 2\bar{y} \to 2\bar{x} + \bar{x}\bar{y} \to 2\bar{x} + 2\bar{y} + \bar{x}\bar{y} \to 2\bar{x} + 2\bar{x}\bar{y} \to 2\bar{x} + 2\bar{y} + 2\bar{x}\bar{y} \to \bar{x}\bar{y} \\
&\quad \to \bar{y} \to \bar{x} + 2\bar{y} \to \bar{y} + \bar{x}\bar{y} \to \bar{x} + 2\bar{y} + \bar{x}\bar{y} \to \bar{y} + 2\bar{x}\bar{y} \to \bar{x} + 2\bar{y} + 2\bar{x}\bar{y} \to 2\bar{y} \\
&\quad \to 2\bar{x} + \bar{y} \to 2\bar{y} + \bar{x}\bar{y} \to 2\bar{x} + \bar{y} + \bar{x}\bar{y} \to 2\bar{y} + 2\bar{x}\bar{y} \to 2\bar{x} + \bar{y} + 2\bar{x}\bar{y} \to 2\bar{x}\bar{y} \to \bar{x}.
\end{aligned}$$

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