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Asymptotic Behavior with Respect to Thickness of Boundary Stabilizing Feedback for the Kirchoff Plate

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1. INTRODUCTION

1.1. *Statement of the Problem*

Let Ω be an open bounded domain in R^2 with a sufficiently smooth, say C^∞, boundary Γ. It would suffice to assume that the boundary Γ is C^4. This, however, would require more involved estimates at the level of pseudodifferential calculus used later. In order to avoid unessential complications, we prefer to assume C^∞ regularity of the boundary. In Ω, we consider the following model of the Kirchoff plate with homogeneous Dirichlet boundary conditions and a control, u, acting through a second order boundary condition (as a moment):

$$w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w = 0 \quad \text{in} \quad Q_T = (0, T) \times \Omega \tag{1.1.a}$$

$$\left.\begin{aligned} w(0, \cdot) &= w_0 \\ w_t(0, \cdot) &= w_1 \end{aligned}\right\} \quad \text{in } \Omega \tag{1.1b}$$

$$w = 0 \quad \text{on} \quad \Sigma_T = (0, T) \times \Gamma \tag{1.1.c}$$

$$\Delta w = u \in L_2(\Sigma_T) \quad \text{on} \quad \Sigma_T = (0, T) \times \Gamma, \tag{1.1.d}$$

where we assume $0 < \gamma < M$. This parameter, γ, is proportional to the square of the thickness of the plate and is therefore assumed to be small.

Closely related to this problem is the limit problem as $\gamma \downarrow 0$, which is called the Euler–Bernoulli plate equation and is described by the following model (see [10]):

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$$w_{tt} + \Delta^2 w = 0 \qquad \text{in } Q_T \tag{1.2.a}$$

$$\left.\begin{aligned} w(0, \cdot) &= w_0 \\ w_t(0, \cdot) &= w_1 \end{aligned}\right\} \qquad \text{in } \Omega \tag{1.2.b}$$

$$w = 0 \qquad \text{on } \Sigma_T \tag{1.2.c}$$

$$\Delta w = u \in L_2(\Sigma_T) \qquad \text{on } \Sigma_T. \tag{1.2.d}$$

We are interested in the question of uniform stabilization for both these models, i.e., can we express u in terms of w_t (as a feedback) so that the resulting system is well-posed and the energy of the system defined in an appropriate topology decays exponentially with respect to the energy at $t = 0$? Moreover, we would ideally like the feedback control designed for model (1.1) to remain effective for its corresponding limit problem (when $\gamma \to 0$), i.e., the model (1.2). This is to say that the properties of stabilization are "robust" (insensitive) with respect to the parameter of the equation which, in our case, is $\gamma > 0$.

The interest in studying sensitivity of feedback controls with respect to the parameter $\gamma > 0$ is motivated by several reasons. Among them is the fact that the value of the parameter $\gamma > 0$ is usually very small. Thus it would be highly undesirable if the feedback control "loses" its properties in the limit process. We also note that inclusion of $\gamma > 0$ changes the character of the models. Model (1.1) ($\gamma > 0$) is of hyperbolic type (with finite speed of propagation), while model (1.2) is of Petrovsky type (with infinite speed of propagation).

1.2. *Literature*

The study of exact controllability and boundary stabilization for both Euler–Bernoulli and Kirchoff models has attracted attention in recent years. We shall focus on equations with boundary conditions imposed on $w|_\Gamma$ and $\Delta w|_\Gamma$, as in the present models (1.1) and (1.2). For other cases of boundary conditions, we refer the reader to [11, 19] (exact controllability) and [2, 10, 22] (stabilization). First results were obtained for the easier problem of exact controllability (versus stabilization) with L_2-controls. Since the optimal regularity results for (1.1) give

$$u \in L_2(\Sigma_T) \Rightarrow (w, w_t) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)) \tag{1.3}$$

(see [16]), while for the corresponding Euler–Bernoulli plate, [13] has found that

$$u \in L_2(\Sigma_T) \Rightarrow (w, w_t) \in C([0, T]; H_0^1(\Omega) \times H^{-1}(\Omega)), \tag{1.4}$$

the above spaces seem a natural choice of topologies suitable for controllability. The initial controllability results for Euler–Bernoulli model (1.2) (respectively, Kirchoff model (1.1)) on spaces of optimal regularity as in (1.4) (respectively, (1.3)) were proved in [19] and with more extensive treatment in [14] (respectively, [11]). In all these references, a suitable differential multiplier method was used which produced exact controllability under the assumption that *two controls* are active on the boundary Σ_T (i.e., $w|_\Gamma = u_1$, $\Delta w|_\Gamma = u_2$). The problem of controlling the plate equation with only *one* control active (say $w|_\Gamma = 0$, $\Delta w|_\Gamma = u$) is more delicate and its solution requires more refined analysis. This consists of removing one boundary term of higher order in the appropriate original inequality. In the case of the Kirchoff model, exact controllability with one control has been obtained in [16] and later, by different methods, in [9]. For the Euler–Bernoulli model, the corresponding result has been proved in [18]. Later, [6] extended this result to general boundary conditions which account for moments of inertia and where the techniques of [18], based on a certain symmetry of the boundary conditions and the associated biharmonic operator, are not applicable. (See also a more recent unified treatment by [25] where the problem with a "reduced number of controls" is treated.)

When it comes to the *stabilization* problem, the situation is far more complicated. Indeed, the multiplier method used in the context of controllability does not produce the right estimates even in the case of very simple geometries (e.g., a disk). To cope with this problem, more sophisticated techniques, involving some pseudodifferential calculus, are needed even in the simplest cases. Relevant stabilization results that have been obtained for these two plates with *only* one control acting through $\Delta w|_{\Sigma_T}$ are

(i) uniform stabilization results for (1.1) with $u = -(\partial/\partial\nu)\, w_t$ in the space $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ under the assumption that the domain must be convex (see [16]);

(ii) uniform stabilization of model (1.2) with $u = -(\partial/\partial\nu)\, w_t$ in the space $H^2(\Omega) \times L_2(\Omega)$ under the assumption that Ω is a sphere (or a small deformation of a sphere) [15];

(iii) uniform stabilization of model (1.2) with $u = -(\partial/\partial\nu)\, \Delta^{-1} w_t$ on the space of optimal regularity, $H_0^1(\Omega) \times H^{-1}(\Omega)$, without geometric conditions (see [12] and [7] where boundary conditions accounted for moments of inertia).

The main goal and contribution of this paper is the construction of a stabilizing feedback control, $u = \mathscr{F}(w_t) \in L_2(\Sigma_T)$, acting as a moment *only*, which would

(1) provide exponential decay rates for the Kirchoff model (1.1) which are *independent* of the parameter $\gamma > 0$,

(2) not require that any geometric assumptions (except for the usual regularity hypothesis) be imposed on the domain Ω.

These results would also allow us to pass with the limit as $\gamma \downarrow 0$ and to obtain results for the limit equation (1.2) with $\gamma = 0$ on the space $H^2(\Omega) \times L_2(\Omega)$. This general type of problem was treated in [10], however, different boundary conditions were considered. Indeed, in [10], feedback control was acting via *two* boundary conditions, *moments* and *forces*. Moreover, geometric conditions of "star-shaped" type are assumed in [10]. In that case, the desired asymptotic estimates followed from multiplier techniques applied directly to the original equation. In our case, the technical difficulties are much greater. It suffices to mention that the rather complicated techniques of [16] (going for beyond multiplier methods and dealing with stabilization of the Kirchoff plate via moments only), even in the case of convex domains, are still not adequate because when $\gamma \downarrow 0$ critical constants in the estimates that are needed to prove uniform stabilization become infinite. Thus, another approach is necessary. In fact, the crux of the matter is in proving rather refined estimates (see Lemmas 2.2 and 2.3) involving microlocal analysis. We shall see that as a result of our analysis, we will obtain not only the "desirable" limit behavior, but we will also be able to improve results (i) and (ii) for both model (1.1) and model (1.2) by entirely deleting (rather severe) geometric conditions on the domain (particularly ii).

We note that geometric conditions imposed on the domain Ω (usually "star-shaped" type) are typical for most of the results on stabilization and controllability of plate equations (see [10, 19]) with the exception of [21] and [18] where some controllability results are obtained for Euler–Bernoulli plates by methods of geometric optics.

1.3. *The Feedback System and Uniform Stabilization*

Because of the regularity and exact controllability results found in [16], the appropriate space for stabilization of (1.1) is $H^2(\Omega) \times H_0^1(\Omega)$, with a suitable topology. We define the energy corresponding to system (1.1) to be

$$\begin{aligned} E_w(t) &= \|w_t(t)\|^2_{L_2(\Omega)} + \gamma\, \|\nabla w_t(t)\|^2_{L_2(\Omega)} + \|\Delta w(t)\|^2_{L_2(\Omega)} \\ &= \|w_t(t)\|^2_{H^1_{0,\gamma}(\Omega)} + \|\Delta w(t)\|^2_{L_2(\Omega)}, \end{aligned} \tag{1.5}$$

where $H^1_{0,\gamma}(\Omega)$ will be our notation for the Hilbert spaœ $H_0^1(\Omega)$ with norm

$$\|g\|^2_{H^1_{0,\gamma}(\Omega)} = \|g\|^2_{L_2(\Omega)} + \gamma\, \|\nabla g\|^2_{L_2(\Omega)}. \tag{1.6}$$

Note that if we set $u \equiv 0$ in (1.1), the resulting system is conservative.

As seen in [16], $u = -(\partial/\partial\nu) w_t$ is a reasonable candidate for the uniform stabilization of (1.1). With u so defined we will consider the following feedback system.

$$\begin{aligned} w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w &= 0 \quad \text{in } Q_\infty = (0, \infty) \times \Omega \\ \left.\begin{aligned} w(0, \cdot) &= w_0 \\ w_t(0, \cdot) &= w_1 \end{aligned}\right\} & \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \Sigma_\infty = (0, \infty) \times \Gamma \\ \Delta w &= -\frac{\partial}{\partial \nu} w_t \quad \text{on } \Sigma_\infty = (0, \infty) \times \Gamma. \end{aligned} \tag{1.7}$$

The following well-posedness and regularity results hold for the Kirchoff model (1.7).

THEOREM 1.1. (i) (Well-posedness on $H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$). *Let* $(w_0, w_1) \in H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$. *Then there exists a unique solution (in the sense of distributions),* $(w(t), w_t(t)) \in C([0, T]; H^2(\Omega) \times H^1_0(\Omega))$, *satisfying system* (1.7).

(ii) (L_2-boundedness of the feedback operator). *For* $(w_0, w_1) \in H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$, *the solution,* $w(t)$, *of* (1.7) *satisfies*

$$\int_0^\infty \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Gamma)} dt \leqslant E_w(0). \tag{1.8}$$

(iii) (Regularity). *If, in addition,* $(w_0, w_1) \in H^4(\Omega) \times H^3(\Omega)$ *and satisfies the appropriate compatibility conditions at the origin, i.e.,*

$$\begin{aligned} w_0|_\Gamma &= w_1|_\Gamma = 0 \\ \Delta w_0|_\Gamma &= -(\partial/\partial\nu) w_1, \end{aligned} \tag{1.9}$$

then the corresponding solution, $w(t)$, *of* (1.7) *satisfies*

$$w \in C(0, \infty; H^4(\Omega)), \quad w_t \in C(0, \infty; H^3(\Omega)), \quad w_{tt} \in C(0, \infty; H^2(\Omega)). \tag{1.10}$$

Parts (i) and (ii) of Theorem 1.1 have been proved in [16]. Part (iii) has been proved in [8].

By using the regularity result of Theorem 1.1, for every $(w_0, w_1) \in H^4(\Omega) \times H^3(\Omega)$ satisfying (1.9) and every $a \geqslant 0$,

$$E_w(t) + 2 \int_a^t \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Gamma)} dt = E_w(a). \tag{1.11}$$

The above equality can be extended by density to all $(w_0, w_1) \in H^2(\Omega) \times H_0^1(\Omega)$.

The corresponding well-posedness results for the Euler–Bernoulli model associated with (1.7) have been proved in [13] and are stated in the following theorem.

THEOREM 1.2. (i) (Well-posedness on $H^2(\Omega) \times L_2(\Omega)$). *Let* $(w_0, w_1) \in H^2(\Omega) \times L_2(\Omega)$. *Then there exists a unique solution (in the sense of distributions)* $(w(t), w_t(t)) \in C([0, T]; H^2(\Omega) \times L_2(\Omega))$, *satisfying system* (1.2) *with* $u = -(\partial/\partial v)\, w_t$.

(ii) (L_2-boundedness of the feedback operator). *For* $(w_0, w_1) \in H^2(\Omega) \times L_2(\Omega)$, *the solution*, $w(t)$, *of* (1.2) *with* $u = -(\partial/\partial v)\, w_t$ *satisfies*

$$\int_0^T \left\| \frac{\partial}{\partial v} w_t \right\|^2_{L_2(\Gamma)} dt \leqslant \|\Delta w_0\|^2_{L_2(\Omega)} + \|w_1\|^2_{L_2(\Omega)}. \tag{1.12}$$

We now state our uniform stabilization result for (1.7).

THEOREM 1.3. *The feedback system* (1.1), *with* $u = -(\partial/\partial v)\, w_t$, *is uniformly (exponentially) stable on the space* $H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$; *i.e., there exist constants*, $C > 0$, $\omega > 0$, *such that*

$$\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{H^1_{0,\gamma}(\Omega)} \leqslant Ce^{-\omega t}[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{H^1_{0,\gamma}(\Omega)}]. \tag{1.13}$$

Moreover, the constants C and ω *are* independent *of* γ.

COROLLARY 1.1. *The feedback system for the Euler–Bernoulli plate corresponding to the limit problem of* (1.7), *again with* $u = -(\partial/\partial v)\, w_t$, *is uniformly (exponentially) stable on the space* $H^2(\Omega) \times L_2(\Omega)$; *i.e., there exist constants*, $C > 0$, $\omega > 0$, *such that*

$$\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} \leqslant Ce^{-\omega t}[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L_2(\Omega)}]. \tag{1.14}$$

Remark. The results of Theorem 1.3 and Corollary 1.1, as well as the well-posedness results, can be extended to the case when the boundary condition (1.1.d) is replaced by

$$\Delta w + (1 - \mu)\, Bw = u \qquad \text{on } \Sigma_T, \tag{1.15}$$

where $0 < \mu < \frac{1}{2}$ is Poisson's ratio and the boundary operator B is defined by

$$Bw \equiv -\frac{\partial^2}{\partial \tau^2} w - k \frac{\partial}{\partial v} w = -k \frac{\partial}{\partial v} w, \tag{1.16}$$

where $k = k(x)$ is the geodesic curvature of the boundary Γ, and the second equality follows from (1.1.c) (see [8]). In fact, the techniques presented in this paper, together with estimates for the operator B given in [6], [7], can be adapted to this more general situation.

Proof of Corollary 1.1. Let $w_\gamma(t)$ denote the solution to (1.7) with $\gamma > 0$. Then by Theorem 1.3, $w_\gamma(t)$ satisfies

$$\|w_\gamma(t)\|^2_{H^2(\Omega)} + \|w_{\gamma,t}(t)\|^2_{H^1_{0,\gamma}(\Omega)} \leqslant Ce^{-\omega t}[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{H^1_{0,\gamma}(\Omega)}], \tag{1.17}$$

and

$$\int_0^\infty \left\|\frac{\partial}{\partial \nu} w_{\gamma,t}\right\|^2_{L_2(\Gamma)} dt \leqslant C[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{H^1_{0,\gamma}(\Omega)}],$$

where the constants C and ω are independent of γ. Hence there exists a subsequence, also denoted by $w_\gamma(t)$, such that

$$w_\gamma \xrightarrow{w^*} w \qquad \text{in } L_\infty(0, T; H^2(\Omega)), \tag{1.18}$$

$$w_{\gamma,t} \xrightarrow{w^*} w_t \qquad \text{in } L_\infty(0, T; L_2(\Omega)) \tag{1.19}$$

$$\frac{\partial}{\partial \nu} w_{\gamma,t} \xrightarrow{w} \frac{\partial}{\partial \nu} w_t \qquad \text{in } L_2(\Sigma_T). \tag{1.20}$$

Proof of 1.20. Since

$$\left\|\frac{\partial}{\partial \nu} w_{\gamma,t}\right\|_{L_2(\Sigma_T)} \leqslant C \Rightarrow \frac{\partial}{\partial \nu} w_{\gamma,t} \xrightarrow{w} l \qquad \text{in } L_2(\Sigma_T), \tag{1.21}$$

is $l \equiv (\partial/\partial \nu)\, w_t$? We know, that

$$\begin{aligned} w_\gamma &\xrightarrow{w} w && \text{in } L_2(0, T; H^2(\Omega)) \\ \Rightarrow w_{\gamma,t} &\xrightarrow{w} w_t && \text{in } H^{-1}(0, T; H^2(\Omega)). \end{aligned} \tag{1.22}$$

Let $D\colon H^s(\Gamma) \to H^s(\Omega)$ be the "Dirichlet map" defined by

$$Dg = v \Leftrightarrow \begin{cases} \Delta v = 0 & \text{in } \Omega \\ v|_\Gamma = g. \end{cases} \tag{1.23}$$

Hence, taking $g \in H^1_0(0, T; H^s(\Gamma)) \Rightarrow Dg \in H^1_0(0, T; H^s(\Omega))$, we find

$$\begin{aligned} \left(\frac{\partial}{\partial \nu} w_{\gamma,t}, g\right)_{L_2(\Sigma_T)} &= (\Delta w_{\gamma,t}, Dg)_{L_2(Q_T)} \\ &= -(\Delta w_\gamma, Dg_t)_{L_2(Q_T)} \xrightarrow[\gamma \to 0]{} -(\Delta w, Dg_t)_{L_2(Q_T)} \\ &= \left(\frac{\partial}{\partial \nu} w_t, g\right)_{L_2(\Sigma_T)} \qquad \forall g \in H^1_0(0, T; H^s(\Gamma)), \end{aligned} \tag{1.24}$$

where the convergence of the inner product follows from (1.22). On the other hand,

$$\left(\frac{\partial}{\partial \nu} w_{\gamma, t}, g\right)_{L_2(\Sigma_T)} \to (l, g)_{L_2(\Sigma_T)} \tag{1.25}$$

and, since $H_0^1(0, T; H^s(\Gamma))$ is dense in $L_2(\Sigma_T)$, we reach our desired conclusion,

$$l \equiv \frac{\partial}{\partial \nu} w_t. \quad \blacksquare \tag{1.26}$$

Using (1.18)–(1.20), we may pass to the limit as $\gamma \to 0$ in (1.7) to obtain

$$\begin{aligned} w_{tt} + \Delta^2 w &= 0 && \text{in } Q_T \\ \left.\begin{aligned} w(0, \cdot) &= w_0 \\ w_t(0, \cdot) &= w_1 \end{aligned}\right\} & && \text{in } \Omega \\ w &= 0 && \text{on } \Sigma_T \\ \Delta w &= -\frac{\partial}{\partial \nu} w_t && \text{on } \Sigma_T, \end{aligned} \tag{1.27}$$

where Q_T and Σ_T are as in Eq. (1.1). From (1.18), (1.19), (1.20), (1.27), and lowersemicontinuity of the norms, with $w_0 \in H^2(\Omega)$ and $w_1 \in H^1(\Omega)$, we obtain

$$\begin{aligned} \|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} &\leq \liminf_{\gamma \to 0} [\|w_\gamma(t)\|^2_{H^2(\Omega)} + \|w_{\gamma, t}(t)\|^2_{L_2(\Omega)}] \\ &\leq \liminf_{\gamma \to 0} [\|w_\gamma(t)\|^2_{H^2(\Omega)} \\ &\quad + \gamma \|\nabla w_{\gamma, t}\|^2_{L_2(\Omega)} + \|w_{\gamma, t}(t)\|^2_{L_2(\Omega)}] \\ &\leq Ce^{-\omega t}[\|w_0\|^2_{H^2(\Omega)} + \lim_{\gamma \to 0} \|w_1\|^2_{H^1_{0, \gamma}(\Omega)}] \\ &= Ce^{-\omega t}[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L_2(\Omega)}]. \end{aligned} \tag{1.28}$$

Extending by density the above inequality to all $w_1 \in L_2(\Omega)$ leads to (1.14). $\blacksquare$

Note that these results have *no geometric constraints* unlike previous results, (i), (ii), available in recent literature.

Remark. The stabilization result of Corollary 1.1 is different from the results on the space of optimal regularity in [15]. Indeed, the feedback control is different (local versus nonlocal in [15]) and the topology of energy is also different.

2. Proof of Theorem 1.3

2.1. *Preliminary Material*

Since we wish to represent the solution to (1.7) in semigroup form, a technique motivated by [1], we need the following operator definitions.

Let $A: L_2(\Omega) \to L_2(\Omega)$ be the positive, self-adjoint operator defined by

$$Ah \equiv \Delta^2 h, \qquad D(A) = \{h \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta h|_\Gamma = 0\}. \tag{2.1}$$

Note that

$$A^{1/2}h = -\Delta h, \qquad D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

In addition, the interpolation result of Grisvard [4] tells us that

$$D(A^\theta) \subset H^{4\theta}(\Omega), \tag{2.3}$$

and, in particular,

$$D(A^\theta) \sim H_0^{4\theta}(\Omega) \qquad \forall \theta < \tfrac{1}{2}, \tag{2.4}$$

with equivalent norms. Therefore,

$$\|f\|_{H_0^{4\theta}(\Omega)} \sim \|f\|_{D(A^\theta)} = \|A^\theta f\|_{L_2(\Omega)} \qquad \forall f \in D(A^\theta). \tag{2.5}$$

Next we define the Green map, $G: L_2(\Gamma) \to L_2(\Omega)$, by

$$Gg = v \Leftrightarrow \begin{cases} \Delta^2 v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma \\ \Delta v = g & \text{on } \Gamma. \end{cases} \tag{2.6}$$

Using these operators, we find that the abstract first-order system which models (1.7) is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} w \\ w_t \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -(I+\gamma A^{1/2})^{-1} A & (I+\gamma A^{1/2})^{-1} AGG^*A \end{bmatrix} \begin{bmatrix} w \\ w_t \end{bmatrix} \\ &\equiv \mathscr{A}_{\mathscr{F}} \begin{bmatrix} w \\ w_t \end{bmatrix}, \end{aligned} \tag{2.7}$$

where G^* is the adjoint of G defined by

$$(Gg, v)_{L_2(\Omega)} = \langle g, G^*v\rangle_{L_2(\Gamma)} \qquad \forall g \in L_2(\Gamma), \quad v \in L_2(\Omega), \tag{2.8}$$

and by using Green's theorem we can find (see [6])

$$G^*Af = \frac{\partial}{\partial \nu} f|_\Gamma \qquad \forall f \in D(A). \tag{2.9}$$

From elliptic regularity [20], we have

$$\begin{matrix} G \in \mathscr{L}(H^s(\Gamma) \to H^{s+5/2}(\Omega)) \\ G^* \in \mathscr{L}(H^s(\Omega) \to H^{s+5/2}(\Gamma)) \end{matrix} \qquad \text{for} \quad s \in R. \tag{2.10}$$

It was shown in [16] that $\mathscr{A}_{\mathscr{F}}$ with $D(\mathscr{A}_{\mathscr{F}}) = \{z \in D(A^{1/2}) \times H^1_{0,\gamma}(\Omega) : \mathscr{A}_{\mathscr{F}} z \in D(A^{1/2}) \times H^1_{0,\gamma}(\Omega)\}$, generates a semigroup of contractions on $H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$.

Let $\mathscr{S}(t)$ denote the semigroup generated by $\mathscr{A}_{\mathscr{F}}$. To establish the fact that the system (1.7) is uniformly stable on $H^2(\Omega) \times H^1_{0,\gamma}(\Omega)$ (with the rates independent of γ), it is sufficient to prove that $\exists T > 0$ such that

$$|\mathscr{S}(T)|_{\mathscr{L}(H^2(\Omega) \times H^1_{0,\gamma}(\Omega))} < \rho < 1, \tag{2.11}$$

where ρ is independent of γ (see [23]).

2.2. *Sufficient Conditions for* (2.11) *to Hold*

The major technical estimate needed for the proof of Theorem 1.3 is given in the following lemma.

LEMMA 2.1. *Let T be sufficiently large. Assume $w(t)$ is the solution to* (1.7). *Then there exists a constant $C_T > 0$ (independent of γ) such that*

$$E_w(T) \leqslant C_T(1+\gamma) \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)}. \tag{2.12}$$

Assuming the validity of the above lemma, the proof of Theorem 1.3 is now routine. Indeed, by the semigroup property, we have

$$E_w(T) = \|\mathscr{S}(T)(w_0, w_1)\|^2_{H^2(\Omega) \times H^1_{0,\gamma}(\Omega)}. \tag{2.13}$$

Then, our energy identity, (1.11), with $t = T$ and $a = 0$, implies that

$$\|\mathscr{S}(T)(w_0, w_1)\|^2_{H^2(\Omega) \times H^1_{0,\gamma}(\Omega)} + \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} = \|(w_0, w_1)\|^2_{H^2(\Omega) \times H^1_{0,\gamma}(\Omega)}. \tag{2.14}$$

Combining (2.12) and (2.14) yields

$$\|\mathcal{S}(T)(w_0, w_1)\|^2_{H^2(\Omega)\times H^1_{0,\gamma}(\Omega)} + \frac{1}{C_T(1+\gamma)} E_w(T) \leqslant \|(w_0, w_1)\|^2_{H^2(\Omega)\times H^1_{0,\gamma}(\Omega)}. \tag{2.15}$$

From (2.13) and (2.15),

$$\|\mathcal{S}(T)(w_0, w_1)\|^2_{H^2(\Omega)\times H^1_{0,\gamma}(\Omega)} \leqslant \frac{1}{1+(1/(C_T(1+\gamma))} \|(w_0, w_1)\|^2_{H^2(\Omega)\times H^1_{0,\gamma}(\Omega)}$$

$$\Rightarrow |\mathcal{S}(T)|_{\mathcal{L}(H^2(\Omega)\times H^1_{0,\gamma}(\Omega))} < \rho < 1, \tag{2.16}$$

where ρ is independent of γ, which proves inequality (2.11).

2.3. *Proof of Lemma* 2.1

Step 1. The first step is fairly standard by now as it uses multiplier methods (as in [10]). We begin by using the multipliers $h \cdot \nabla w$ and w with Eqs. (1.1), where we assume $h = x - x_0$, for some $x_0 \in R^2$. (Note that in [16] different, higher order multipliers were used.) Combining the resulting identities and noting the boundary conditions, we find

$$\begin{aligned}
2\int_{Q_T} (\Delta w)^2\, d\Omega\, dt \leqslant C \int_{\Sigma_T} \left|\frac{\partial}{\partial \nu} w_t\right|^2 d\Gamma\, dt + \int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu}(h\cdot\nabla w)\, d\Gamma\, dt \\
- \int_{\Sigma_T} \frac{\partial}{\partial \nu}(\Delta w)\, h\cdot\nabla w\, d\Gamma\, dt + \int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu} w\, d\Gamma\, dt \\
+ \gamma \int_{Q_T} |\nabla w_t|^2\, d\Omega\, dt - \gamma \int_{Q_T} \Delta w_t h\cdot\nabla w_t\, d\Omega\, dt \\
- [(w_t, h\cdot\nabla w)_{L_2(\Omega)}]_0^T + \gamma[(\Delta w_t, h\cdot\nabla w)_{L_2(\Omega)}]_0^T \\
- [(w_t, w)_{L_2(\Omega)}]_0^T + \gamma[(\Delta w_t, w)_{L_2(\Omega)}]_0^T.
\end{aligned} \tag{2.17}$$

Two identities that will be needed to simplify (2.17) are the following:

$$\begin{aligned}
\int_{Q_T} |\Delta w|^2\, d\Omega\, dt - \int_{Q_T} |w_t|^2\, d\Omega\, dt - \gamma \int_{Q_T} |\nabla w_t|^2\, d\Omega\, dt \\
= -[(w_t + \gamma \nabla w_t, w)_{L_2(\Omega)}]_0^T - 2\int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu} w\, d\Gamma\, dt
\end{aligned} \tag{2.18}$$

and

$$\int_{Q_T} \Delta w_t h\cdot\nabla w_t\, d\Omega\, dt = \int_{\Sigma_T} \frac{\partial}{\partial \nu} w_t h\cdot\nabla w_t\, d\Gamma\, dt - \frac{1}{2}\int_{\Sigma_T} \left|\frac{\partial}{\partial \nu} w_t\right|^2 h\cdot\nu\, d\Gamma\, dt. \tag{2.19}$$

With these identities, we find

$$\begin{aligned}
&\int_{Q_T} |\Delta w|^2 \, d\Omega \, dt + \gamma \int_{Q_T} |\nabla w_t|^2 \, d\Omega \, dt + \int_{Q_T} |w_t|^2 \, d\Omega \, dt \\
&\quad \leqslant C \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w_t \right|^2 d\Gamma \, dt \\
&\qquad + \int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu} (h \cdot \nabla w) \, d\Gamma \, dt - \int_{\Sigma_T} \frac{\partial}{\partial \nu} (\Delta w) \, h \cdot \nabla w \, d\Gamma \, dt \\
&\qquad + 4 \left\{ \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w_t \right|^2 d\Gamma \, dt + \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w \right|^2 d\Gamma \, dt \right\} + \gamma C \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w_t \right|^2 d\Gamma \, dt \\
&\qquad + \frac{3}{2} [(w_t, h \cdot \nabla w)_{L_2(\Omega)}]_0^T + \frac{3\gamma}{2} [(\Delta w_t, h \cdot \nabla w)_{L_2(\Omega)}]_0^T - [(w_t, w)_{L_2(\Omega)}]_0^T \\
&\qquad + \gamma [(\Delta w_t, w)_{L_2(\Omega)}]_0^T + (w_t + \gamma \nabla w_t, w)_{L_2(\Omega)}]_0^T. \qquad (2.20)
\end{aligned}$$

Note that in the right-hand side of the above inequality, the only two terms that do not involve the L_2-norms of our feedback control on the boundary or lower order terms are the two boundary integrals in the second line. To deal with these terms, we need *sharp* regularity results for these boundary traces. (Note that standard trace theory does not suffice to bound these terms by the energy of our system.) Sharp estimates for these traces are the main technical contribution of this paper, which may also be of independent interest in partial differential equations.

Step 2: Estimates for the traces on the boundary. At this point, it is expedient to state the main two estimates for the traces of the solution on the boundary. Derivation of these estimates is the most technical part of the proof, requiring both microlocal analysis and special regularity properties of a pseudodifferential (abstract) Schrödinger equation.

LEMMA 2.2. *Let w be the solution to* (1.7) *and let $0 < \alpha < T/2$. Then w satisfies the following inequality.*

$$\left\| \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau} w \right\|^2_{L_2(\alpha, T-\alpha; \Gamma)} \leqslant C \left\{ \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} + \|w\|^2_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))} \right\}, \qquad (2.21)$$

where $0 < \varepsilon < \frac{1}{2}$ and C is independent *of γ.*

Lemma 2.2 will be proved in the third section of this paper.

LEMMA 2.3. *Let w be the solution to* (1.7) *and α and ε be as above. Then w satisfies the following inequality.*

$$\left\| \frac{\partial}{\partial \nu} (\Delta w) \right\|^2_{H^{-1}(\alpha,\, T-\alpha;\, L_2(\Gamma))}$$

$$\leqslant C_T \left\{ (1+\gamma) \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} + (1+\gamma) \, \|w\|^2_{L_2(0,\, T;\, H^{3/2+\varepsilon}(\Omega))} \right\}, \tag{2.22}$$

where C_T *is* independent *of* γ *and* $H^{-1}(\alpha, T-\alpha; L_2(\Gamma))$ *is the dual (pivotal) to the space* $H^1(\alpha, T-\alpha; L_2(\Gamma))$.

The proof of Lemma 2.3 is in Section 4.

Remark. Note that the results of (2.21) and (2.22) do not follow from trace theory.

Step 3. Returning to the integral terms in (2.20), the first boundary integral may be split directly:

$$\left| \int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu} (h \cdot \nabla w) \, d\Gamma \, dt \right| \leqslant \int_{\Sigma_T} |\Delta w|^2 \, d\Gamma \, dt + \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} (h \cdot \nabla w) \right|^2 d\Gamma \, dt. \tag{2.23}$$

Noting that

$$\frac{\partial}{\partial \nu} (h \cdot \nabla w) = h \cdot \mathbf{v} \frac{\partial^2}{\partial \nu^2} w + h \cdot \mathbf{\tau} \frac{\partial^2}{\partial \nu \, \partial \tau} w + \frac{\partial}{\partial \nu} (h \cdot \mathbf{v}) \frac{\partial}{\partial \nu} w, \tag{2.24}$$

we find

$$\left| \frac{\partial^2}{\partial \nu^2} w \right| \leqslant |\Delta w| + \left| k \frac{\partial}{\partial \nu} w \right| \qquad \text{on } \Gamma \tag{2.25}$$

since (as $w|_\Gamma = 0$),

$$\Delta w|_\Gamma = \frac{\partial^2}{\partial \nu^2} w|_\Gamma + k \frac{\partial}{\partial \nu} w|_\Gamma, \tag{2.26}$$

where k is the Gaussian curvature. Therefore,

$$\left| \int_{\Sigma_T} \Delta w \frac{\partial}{\partial \nu} (h \cdot \nabla w) \, d\Gamma \, dt \right| \leqslant C \left\{ \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w_t \right|^2 d\Gamma \, dt + \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} w \right|^2 d\Gamma \, dt + \int_{\Sigma_T} \left| \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau} w \right|^2 d\Gamma \, dt \right\}. \tag{2.27}$$

Step 4. The second boundary integral is estimated as follows.

$$\left|\int_{\Sigma_T} \frac{\partial}{\partial \nu}(\Delta w)\, h\cdot\nabla w\, d\Gamma\, dt\right| \leqslant C \left\|\frac{\partial}{\partial \nu}(\Delta w)\right\|_{H^{-1}(0,T;L_2(\Gamma))} \left\|\frac{\partial}{\partial \nu} w\right\|_{H^1(0,T;L_2(\Gamma))}$$
$$\leqslant C \left\|\frac{\partial}{\partial \nu}(\Delta w)\right\|^2_{H^{-1}(0,T;L_2(\Gamma))} + C_1 \left\|\frac{\partial}{\partial \nu} w_t\right\|^2_{L_2(\Sigma_T)}$$
$$+ C_2 \|w\|^2_{L_2(0,T;H^{3/2+\varepsilon}(\Omega))}, \tag{2.28}$$

where we have used the duality pairing between $H^1(0, T)$ and $H^{-1}(0, T)$.

Step 5. Consider (2.20) over the time interval $(\alpha, T-\alpha)$ instead of $(0, T)$. Combining the result with (2.21), (2.22), (2.27), and (2.28), and recalling the energy relation (1.11), we find

$$(T-2\alpha)\, E_w(T) \leqslant \int_\alpha^{T-\alpha} \int_\Omega \{|\Delta w|^2 + \gamma\, |\nabla w_t|^2 + |w_t|^2\}\, d\Omega\, dt$$
$$\leqslant C_T \left\{(1+\gamma) \left\|\frac{\partial}{\partial \nu} w_t\right\|^2_{L_2(\Sigma_T)} + (1+\gamma)\, \|w\|^2_{L_2(0,T;H^{3/2+\varepsilon}(\Omega))}\right\}$$
$$+ CE_w(0), \tag{2.29}$$

where C and C_T are both independent of γ. Let T be sufficiently large so that $T-2\alpha > 2C$. Then

$$E_w(T) \leqslant C_T \left\{(1+\gamma) \left\|\frac{\partial}{\partial \nu} w_t\right\|^2_{L_2(\Sigma_T)} + (1+\gamma)\, \|w\|^2_{L_2(0,T;H^{3/2+\varepsilon}(\Omega))}\right\}. \tag{2.30}$$

Step 6: Compactness/uniqueness argument. Our final step is to use a compactness/uniqueness argument (in the style of [19]) to remove the lower order terms in the above expression.

LEMMA 2.4. *Assume $w(t)$ is the solution to* (1.7). *Then the following inequality holds.*

$$\|w\|_{C(0,T;H^{2-\varepsilon}(\Omega))} \leqslant C_T \left\|\frac{\partial}{\partial \nu} w_t\right\|_{L_2(\Sigma_T)}, \tag{2.31}$$

where C_T is independent *of $\gamma > 0$.*

Note that estimate (2.31), with C_T depending on γ, can be obtained by rather routine procedures (see Proposition 2.1). What is more interesting here is the fact that the constant C_T *does not* depend on $\gamma > 0$.

Proof of Lemma 2.4. We shall start with a preliminary estimate which follows essentially from [16]. To make this paper self-contained, we provide a brief proof.

Proposition 2.1. *Assume $w(t)$ is the solution to (1.7). Then the following inequality holds for all $0<\varepsilon<\frac{1}{2}$.*

$$\|w\|_{C(0,T;H^{2-\varepsilon}(\Omega))} \leqslant C_{T,\gamma}\left\|\frac{\partial}{\partial\nu}w_t\right\|_{L_2(\Sigma_T)}. \tag{2.32}$$

Proof. Assume (2.32) does not hold. Then there exists a sequence, $\{w_{\gamma,n}(t)\}_{n=1}^{\infty}$, with

$$w_{\gamma,n}(0)=w_{\gamma,n,0}, \qquad w'_{\gamma,n}(0)=w_{\gamma,n,1}, \tag{2.33}$$

where w' denotes differentiation with respect to time, such that

$$\begin{aligned}\left\|\frac{\partial}{\partial\nu}w'_{\gamma,n}\right\|_{L_2(\Sigma_T)} &\to 0 \qquad \text{as} \quad n\to\infty,\\ \|w_{\gamma,n}\|_{C(0,T;H^{2-\varepsilon}(\Omega))} &= 1 \qquad \forall n.\end{aligned} \tag{2.34}$$

Then each $w_{\gamma,n}$ satisfies (2.30). Therefore,

$$\{w_{\gamma,n,0}, w_{\gamma,n,1}\} \text{ is uniformly bounded in } H^2(\Omega)\times H^1_{0,\gamma}(\Omega). \tag{2.35}$$

Therefore, there exists a subsequence, also denoted by $\{w_{\gamma,n}(t)\}_{n=1}^{\infty}$, with

$$w_{\gamma,n,0}\to\tilde{w}_{\gamma,0} \qquad \text{weakly in } H^2(\Omega), \tag{2.36}$$

and

$$w_{\gamma,n,1}\to\tilde{w}_{\gamma,1} \qquad \text{weakly in } H^1_{0,\gamma}(\Omega).$$

Since we have assumed $w_{\gamma,n}(t)$ satisfies (1.7) for all n, we can write $w_{\gamma,n}(t)$ and $w'_{\gamma,n}(t)$ implicitly as

$$\begin{aligned} w_{\gamma,n}(t) &= C(t)\,w_{\gamma,n,0}+S(t)\,w_{\gamma,n,1}-\int_0^t S(t-\tau)\,\mathscr{A}G\left(\frac{\partial}{\partial\nu}w_t\right)d\tau,\\ w'_{\gamma,n}(t) &= -\mathscr{A}S(t)\,w_{\gamma,n,0}+C(t)\,w_{\gamma,n,1}\\ &\quad -\int_0^t\frac{\partial}{\partial t}\left\{S(t-\tau)\,\mathscr{A}G\left(\frac{\partial}{\partial\nu}w_t\right)\right\}d\tau,\end{aligned} \tag{2.37}$$

where $\mathscr{A}\equiv(I+\gamma A^{1/2})^{-1}A$ and $S(t)$ and $C(t)$ respectively denote the sine and cosine operators corresponding to $\mathscr{A}$.

Consider the solution to the following problem:

$$\begin{aligned} &\tilde{w}_{tt} - \gamma\, \Delta \tilde{w}_{tt} + \Delta^2 \tilde{w} = 0 \\ &\tilde{w}(0, \cdot) = \tilde{w}_0 \\ &\tilde{w}_t(0, \cdot) = \tilde{w}_1 \\ &\tilde{w}|_\Gamma = \Delta \tilde{w}|_\Gamma = 0. \end{aligned} \tag{2.38}$$

We can write $\tilde{w}(t)$ as

$$\tilde{w}(t) = C(t)\, \tilde{w}_0 + S(t)\, \tilde{w}_1. \tag{2.39}$$

Then, using the Lebesque dominated convergence theorem and the properties of sine and cosine operators, we find

$$\{w_{\gamma,n}, w'_{\gamma,n}\} \xrightarrow{w^*} \{\tilde{w}, \tilde{w}'\} \quad \text{in} \quad L_\infty(0, T; H^2(\Omega) \times H^1_{0,\gamma}(\Omega)). \tag{2.40}$$

Therefore, $\{w_{\gamma,n}, w'_{\gamma,n}\}$ are uniformly bounded in $L_\infty(0, T; H^2(\Omega) \times H^1_{0,\gamma}(\Omega))$. Since $H^{2-\varepsilon}(\Omega) \underset{\text{compact}}{\subset} H^2(\Omega)$, a result of Simon [24] gives us

$$w_{\gamma,n}(t) \to \tilde{w}(t) \qquad \text{strongly in } C(0, T; H^{2-\varepsilon}(\Omega)). \tag{2.41}$$

Thus, by (2.34) and (2.41), we obtain

$$\|\tilde{w}\|_{C(0,T;H^{2-\varepsilon}(\Omega))} = 1. \tag{2.42}$$

We now introduce the change of variable

$$\psi \equiv \tilde{w}_t. \tag{2.43}$$

Then ψ satisfies the equation

$$\begin{aligned} &\psi_{tt} - \gamma\, \Delta \psi_{tt} + \Delta^2 \psi = 0 \\ &\psi|_\Gamma = \Delta \psi|_\Gamma = 0 \\ &\frac{\partial}{\partial \nu} \psi|_\Gamma = 0. \end{aligned} \tag{2.44}$$

From [11, page 150], we now know that $\psi \equiv 0$. Thus, $\tilde{w} \equiv 0$, which contradicts (2.34). Hence, (2.32) holds. ∎

Proof of Lemma 2.4 (continued). By Proposition 2.1, we know $w(t)$ satisfies (2.32) with the constant $C_{T,\gamma}$ possibly dependent on γ. Assume

(2.31) does not hold. Then there exists a sequence, $\{w_\gamma(t)\}$, $\gamma \to 0^+$, such that $w_\gamma(t)$ satisfies (1.7) and

$$\left\| \frac{\partial}{\partial \nu} w'_\gamma \right\|_{L_2(\Sigma_T)} \to 0 \qquad \text{as} \quad \gamma \to 0^+, \tag{2.45}$$
$$\|w_\gamma\|_{C(0,\, T;\, H^{2-\varepsilon}(\Omega))} = 1 \qquad \text{for each } \gamma.$$

Then each w_γ satisfies (2.30). Therefore, from (1.11) and (2.30), as $\gamma \to 0^+$,

$$\{w_\gamma, w'_\gamma\} \text{ remains in a bounded set of } C(0, T; H^2(\Omega) \times L_2(\Omega)). \tag{2.46}$$

Hence, there exists a subsequence, also denoted by $\{w_\gamma(t)\}$, with

$$w_\gamma \xrightarrow{w^*} w \qquad \text{in } L_\infty(0, T; H^2(\Omega))$$
$$w'_\gamma \xrightarrow{w^*} w' \qquad \text{in } L_\infty(0, T; L_2(\Omega)). \tag{2.47}$$

Passing the limit on equation (1.7), we find that $w(t)$ satisfies (see [10]):

$$w_{tt} + \Delta^2 w = 0$$
$$w|_\Gamma = \Delta w|_\Gamma = 0 \tag{2.48}$$
$$\frac{\partial}{\partial \nu} w_t|_\Gamma = 0.$$

Since $H^{2-\varepsilon}(\Omega) \underset{\text{compact}}{\subset} H^2(\Omega)$, a result of Simon [24] gives us

$$w_\gamma(t) \to w(t) \qquad \text{strongly in } C(0, T; H^{2-\varepsilon}(\Omega)). \tag{2.49}$$

Thus, by (2.45) and (2.49), we obtain

$$\|w\|_{C(0,\, T;\, H^{2-\varepsilon}(\Omega))} = 1. \tag{2.50}$$

Introducing the change of variable

$$\psi \equiv w_t, \tag{2.51}$$

we find that ψ satisfies the equation

$$\psi_{tt} + \Delta^2 \psi = 0$$
$$\psi|_\Gamma = \Delta\psi|_\Gamma = 0 \tag{2.52}$$
$$\frac{\partial}{\partial \nu} \psi|_\Gamma = 0.$$

However, from a result of Lions (see [19, Corollary 3.2, p. 256]), we know that $\psi \equiv 0$, which in turn implies $w \equiv 0$. Since this contradicts (2.45), (2.31) holds. ∎

Step 7. Combining the results from our compactness/uniqueness argument with (2.30), we find

$$E_w(T) \leqslant C_T(1+\gamma) \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)}, \tag{2.53}$$

as desired for (2.12).

3. Proof of Lemma 2.2

3.1. *Microlocal Analysis*

We consider the following, more general, problem related to our system.

$$\begin{aligned} Pw &= f \quad \text{in } Q_T \\ w &= g_1 \quad \text{on } \Sigma_T \\ Bw &= g_2 \quad \text{on } \Sigma_T \end{aligned} \tag{3.1}$$

where the operator P (modulo lower order terms) is defined by

$$\begin{aligned} P(x, y; D_t, D_x, D_y) = & -a(x, y)\, D_t^2 \\ & + \gamma a(x, y)\, D_t^2 (a_1(x, y)\, D_y^2 + 2a_2(x, y)\, D_y D_x + D_x^2) \\ & + (a_1(x, y)\, D_y^2 + 2a_2(x, y)\, D_y D_x + D_x^2)^2, \end{aligned} \tag{3.2}$$

and the boundary operator B (modulo lower order terms) is defined by

$$B(x, y; D_t, D_x, D_y) \equiv a_1(x, y)\, D_y^2 + 2a_2(x, y)\, D_y D_x + D_x^2, \tag{3.3}$$

where $a(x, y) > 0$, $a_j(x, y)$ depends smoothly on (x, y) for $j = 1, 2$, $a_j(x, y)$ satisfy the appropriate ellipticity conditions, i.e.,

$$\begin{aligned} a_1(x, y) > \rho > 0 \qquad \forall x, y \in \Omega \\ a_1(x, y) - a_2^2(x, y) > \rho > 0, \end{aligned} \tag{3.4}$$

hence B is a regular boundary operator consistent with a Dirichlet system. In the definitions of P and B, we have adopted the notation

$$D_x = \frac{1}{i} \frac{\partial}{\partial x}, \quad D_y = \frac{1}{i} \frac{\partial}{\partial y}, \quad D_t = \frac{1}{i} \frac{\partial}{\partial t}. \tag{3.5}$$

We shall prove the following.

PROPOSITION 3.1. *Let $w \in H^2(Q_T)$ be the solution to (3.1). Then w satisfies the following estimate.*

$$\left\| \frac{\partial}{\partial \tau} \frac{\partial}{\partial \nu} w \right\|^2_{L_2(\alpha, T-\alpha; \Gamma)} \leqslant C \left\{ \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} + \|g_1\|^2_{H^1(0, T; L_2(\Gamma))} + \|g_1\|^2_{L_2(0, T; H^2(\Gamma))} + \|g_2\|^2_{L_2(\Sigma_T)} + \|f\|^2_{H^{-3/2+\varepsilon}(Q_T)} + (1+\gamma) \|w\|^2_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))} \right\}, \tag{3.6}$$

where $0 < \varepsilon < \frac{1}{2}$ and the constant C does not depend on $\gamma > 0$.

It is well known that, via a partition of unity and a flattening of the boundary procedure, it is enough to establish the estimate of Lemma 2.2 for P with $\Omega = \{x, y; x \geqslant 0\}$, $\Gamma = \{x = 0\}$, and $a(x, y)$, $a_j(x, y)$ constant outside a compact set.

Remark. In [17], a somewhat similar result as that of Proposition 3.1 was stated and proved. The main difference, however, is that [17] treats different (higher order) boundary conditions and, moreover, the analysis of [17] does not provide estimates which are independent of $\gamma > 0$.

Proof of Proposition 3.1. Step 1. Let $\psi(t) \in C_0^\infty(R)$ be a cutoff function defined such that $0 \leqslant \psi(t) \leqslant 1$ $\forall t$ and

$$\psi(t) = \begin{cases} 1 & \text{in } (\alpha, T-\alpha) \\ 0 & \text{outside } (\alpha/2, T-\alpha/2). \end{cases} \tag{3.7}$$

Define $w_c(t, \cdot) \equiv \psi(t)\, w(t, \cdot)$. Then $w_c(t, \cdot)$ satisfies

$$\begin{aligned} Pw_c &= [P, \psi]\, w + \psi f && \text{in} \quad Q_\infty \equiv \Omega \times (-\infty, \infty) \\ w_c &= \psi g_1 && \text{on} \quad \Sigma_\infty \equiv \Gamma \times (-\infty, \infty) \\ Bw_c &= \psi g_2 && \text{on} \quad \Sigma_\infty, \end{aligned} \tag{3.8}$$

where $[A, B]$ denotes the commutator of two operators A and B and, in our case,

$$[P, \psi]\, w = 2i\gamma a(\tilde{D}_x^2 + \tilde{D}_y^2)\, D_t(\psi_t w) + \text{lower order terms}. \tag{3.9}$$

Step 2. If P is defined as in (3.2), then the symbol of P is

$$p(x, y; s, D_x, \eta) = -as^2 + \gamma a s^2(a_1 \eta^2 + 2a_2 \eta D_x + D_x^2) + (a_1 \eta^2 + 2a_2 \eta D_x + D_x^2)^2. \tag{3.10}$$

In this equation, we denote the dual variables corresponding to time, t, and the tangential direction, y, by s and η, respectively; i.e.,

$$\frac{1}{i} D_t \to s = \kappa + i\sigma$$
$$D_y \to \eta. \tag{3.11}$$

For convenience, we now make the following definitions. Let

$$\tilde{\xi} \equiv D_x + a_2 \eta \tag{3.12}$$

with corresponding operator

$$\tilde{D}_x = D_x + a_2 D_y, \tag{3.13}$$

and let $\tilde{\eta}^2$ be the symbol of the operator $\tilde{D}_y$ defined by

$$\tilde{D}_y^2 \equiv (a_1 - a_2^2)\, D_y^2. \tag{3.14}$$

The fact that $a_1 - a_2^2 > \rho > 0$ justifies the notation $\tilde{D}_y^2$ which has a positive symbol, $(a_1 - a_2^2)\,\eta^2 > \rho\eta^2$. Thus, P can be written as

$$P = -aD_t^2 + \gamma a D_t^2(\tilde{D}_x^2 + \tilde{D}_y^2) + (\tilde{D}_x^2 + \tilde{D}_y^2)^2, \tag{3.15}$$

with corresponding symbol

$$p = -as^2 + \gamma a s^2(\tilde{\xi}^2 + \tilde{\eta}^2) + (\tilde{\xi}^2 + \tilde{\eta}^2)^2. \tag{3.16}$$

Our next step is to "microlocalize" problem (3.8). It suffices to consider only the quarter space $R^2(+)$ where $\sigma > 0$ and $\eta > 0$ (analysis in the remaining quarters is the same). We define mutually disjoint regions (cones) as follows:

$$\mathscr{R}_1 \equiv \{(x, y, \sigma, \eta) \in \Omega \times R^n : \sigma < c_0\, |\eta|\}$$
$$\mathscr{R}_{tr} \equiv \{(x, y, \sigma, \eta) \in \Omega \times R^n : c_0\, |\eta| \leqslant \sigma < 2c_0\, |\eta|\} \tag{3.17}$$
$$\mathscr{R}_2 \equiv \{(x, y, \sigma, \eta) \in \Omega \times R^n : 2c_0\, |\eta| \leqslant \sigma\}.$$

Consider Fig. 3.1. In $\mathscr{R}_1 \cup \mathscr{R}_{tr}$, provided c_0 is sufficiently small, from (3.14) and (3.16) it can be readily seen that the symbol $p(x, y; s, \xi, \eta)$ is elliptic of order four in $\mathscr{R}_1 \cup \mathscr{R}_{tr}$; i.e.,

$$p(x, y; s, \xi, \eta) \geqslant \delta(\tilde{\xi}^4 + |\eta|^4 + \sigma^4) \qquad \text{in} \quad \mathscr{R}_1 \cup \mathscr{R}_{tr} \equiv \mathscr{E}_1 \text{ for some } \delta > 0. \tag{3.18}$$

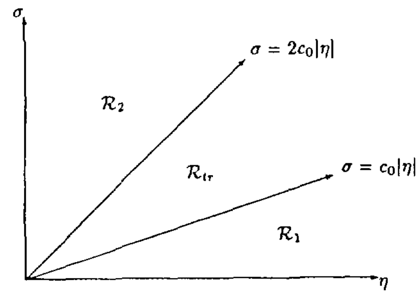

FIG. 3.1. Elliptic and non-elliptic regions.

Therefore the problem consisting of the operator p and boundary conditions $w|_\Gamma$, $Bw|_\Gamma$, is microelliptic in $\mathscr{R}_1$. To take advantage of this, we define a new function (pseudodifferential operator), λ, which would localize the problem to the elliptic region.

Step 3. Let $\lambda(x, y; \sigma, \eta) \in C^\infty$ be a cutoff function defined such that $0 \leqslant \lambda(x, y; \sigma, \eta) \leqslant 1$, λ is homogeneous of order zero in σ and η, and

$$\lambda(x, y; \sigma, \eta) = \begin{cases} 1 & \text{in } \mathscr{R}_1 \\ 0 & \text{in } \mathscr{R}_2, \end{cases} \tag{3.19}$$

i.e., $\operatorname{supp} \lambda \subset \mathscr{R}_1 \cup \mathscr{R}_{\mathrm{tr}} \equiv \mathscr{E}_1$. $\lambda(x) \in \mathscr{S}^0(R^2_{t,y})$ is a zero-order pseudodifferential operator in the variables t, y for a fixed $x \in \Omega$. This is to say that the corresponding pseudodifferential operator (still, denoted by λ) $\lambda \in C^\infty(R^+; OPS^0(R^2))$. If we write $w_c = \lambda w_c + (1-\lambda) w_c$, then we can see that λw_c satisfies

$$\begin{aligned} P(\lambda w_c) &= [P, \lambda]\, w_c + \lambda[P, \psi]\, w + \lambda\psi f && \text{in } Q_\infty \\ \lambda w_c &= \lambda\psi g_1 && \text{on } \Sigma_\infty \\ B(\lambda w_c) &= \lambda\psi g_2 - [B, \lambda]\, w_c && \text{on } \Sigma_\infty. \end{aligned} \tag{3.20}$$

Since p is elliptic of order four on $\operatorname{supp} \lambda$, this problem satisfies elliptic estimates in all variables and, in particular, λw_c satisfies the following inequality (see [3]):

$$\begin{aligned} &\|\lambda w_c\|_{L_2(0, \infty; H^{5/2}(\Omega))} + \left\| \frac{\partial}{\partial v} \frac{\partial}{\partial \tau} (\lambda w_c) \right\|_{L_2(\Sigma_\infty)} \\ &\quad \leqslant C \{ \|\lambda w_c\|_{H^2(\Sigma_\infty)} + \|B(\lambda w_c)\|_{L_2(\Sigma_\infty)} + \|[P, \lambda]\, w_c\|_{H^{-3/2+\varepsilon}(Q_\infty)} \\ &\qquad + \|\lambda[P, \psi]\, w\|_{H^{-3/2+\varepsilon}(Q_\infty)} + \|\lambda\psi f\|_{H^{-3/2+\varepsilon}(Q_\infty)} \}. \end{aligned} \tag{3.21}$$

Step 4. In the non-elliptic region, $\mathscr{R}_2 \cup \mathscr{R}_{\mathrm{tr}}$, we can estimate more directly. From Fig. 3.1, it is clear that

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\frac{\partial}{\partial \tau}(1-\lambda)\,w_c\right\|^2_{L_2(\Sigma_\infty)} &\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\eta|^2\left|\frac{\partial}{\partial x}(1-\lambda)\,\hat{w}_c(\sigma,\eta,x)|_{x=0}\right|^2 d\sigma\, d\eta \\ &\leqslant \frac{1}{c_0^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial x}\sigma(1-\lambda)\,\hat{w}_c(\sigma,\eta,x)|_{x=0}\right|^2 d\sigma\, d\eta \\ &= \frac{1}{c_0^2}\left\|\frac{\partial}{\partial x}[(1-\lambda)\,w_c]_t\right\|^2_{L_2(\Sigma_\infty)},\end{aligned} \tag{3.22}$$

where $\hat{w}_c$ is the Fourier–Laplace transform of w_c, i.e., the Fourier transform in the tangential direction and the Laplace transform in time.

By combining (3.21) and (3.22), we arrive at

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\frac{\partial}{\partial \tau}w_c\right\|_{L_2(\Sigma_\infty)} \leqslant \frac{1}{c_0}\left\|\frac{\partial}{\partial x}[(1-\lambda)\,w_c]_t\right\|_{L_2(\Sigma_\infty)} &+ C\{\|\lambda w_c\|_{L_2(0,\infty;H^2(\Gamma))} \\ &+ \|B(\lambda w_c)\|_{L_2(\Sigma_\infty)} + \|[P,\lambda]\,w_c\|_{H^{-3/2+\varepsilon}(Q_\infty)} \\ &+ \|\lambda[P,\psi]\,w\|_{H^{-3/2+\varepsilon}(Q_\infty)} + \|\lambda\psi f\|_{H^{-3/2+\varepsilon}(Q_\infty)}\}.\end{aligned} \tag{3.23}$$

To obtain the desired estimate in (3.6), we need to estimate all the terms on the right-hand side of (3.23), in particular the ones involving commutators.

Step 5. By using formulas for an asymptotic expansion of the symbols corresponding to the appropriate commutators (see [5, p. 70]) and noting that supp $\lambda \subset \mathscr{E}_1$, we obtain

$$\begin{cases}\operatorname{symb}\{[P,\lambda]\} = (1+\gamma)\,\mathcal{O}(\tilde{\xi}^3 + |\eta|\,\tilde{\xi}^2 + |\eta|^2\,\tilde{\xi} + |\eta|^3) & \text{in } \mathscr{E}_1 \\ \operatorname{supp}\operatorname{symb}\{[P,\lambda]\} \subset \mathscr{E}_1, & \end{cases} \tag{3.24}$$

$$\begin{cases}\operatorname{symb}\{\lambda[P,\psi]\} = \gamma\mathcal{O}(|\eta|\,\tilde{\xi}^2 + |\eta|^3) & \text{in } \mathscr{E}_1 \\ \operatorname{supp}\operatorname{symb}\{\lambda[P,\psi]\} \subset \mathscr{E}_1, & \end{cases} \tag{3.25}$$

$$\begin{cases}\operatorname{symb}\{[B,\lambda]\} = \mathcal{O}(\tilde{\xi} + |\eta|) & \text{in } \mathscr{E}_1 \\ \operatorname{supp}\operatorname{symb}\{[B,\lambda]\} \subset \mathscr{E}_1. & \end{cases} \tag{3.26}$$

Hence, in particular for any $0 < \varepsilon < \frac{1}{2}$,

$$[P,\lambda] \in \mathscr{L}(L_2(-\infty,\infty;H^{3/2-\varepsilon}(\Omega)) \to H^{-3/2+\varepsilon}(Q_\infty)), \tag{3.27}$$

$$\lambda[P,\psi] \in \mathscr{L}(L_2(-\infty,\infty;H^{3/2-\varepsilon}(\Omega)) \to H^{-3/2+\varepsilon}(Q_\infty)), \tag{3.28}$$

$$[B,\lambda] \in \mathscr{L}(L_2(-\infty,\infty;H^{3/2-\varepsilon}(\Omega)) \to H^{1/2+\varepsilon}(Q_\infty)), \tag{3.29}$$

$$[D_x, 1-\lambda]|_{x=0} \in \mathscr{L}(L_2(\Sigma_\infty) \to L_2(\Sigma_\infty)). \tag{3.30}$$

Step 6. From (3.27) and (3.28),

$$\begin{aligned} \|[P, \lambda]\, w_c\|_{H^{-3/2+\varepsilon}(Q_\infty)} &\leqslant C(1+\gamma)\, \|w\|_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))}, \\ \|\lambda[P, \psi]\, w\|_{H^{-3/2+\varepsilon}(Q_\infty)} &\leqslant C(1+\gamma)\, \|w\|_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))}, \end{aligned} \tag{3.31}$$

hence the last three terms in (3.23) are bounded by

$$C(1+\gamma)\, \|w\|_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))} + \|f\|_{H^{-3/2+\varepsilon}(Q_T)}. \tag{3.32}$$

The second term on the right-hand side of (3.23) is plainly estimated by

$$\|\lambda w_c\|_{L_2(0, \infty; H^2(\Gamma))} \leqslant C\, \|g_1\|_{L_2(0, T; H^2(\Gamma))}, \tag{3.33}$$

where we have used the fact that λ is a local operator in the normal direction, x, and $\lambda \in C^\infty(R^+; OPS^0(R^2))$.

To estimate the first term on the right-hand side of (3.23), we first note that

$$\begin{aligned} D_x D_t(1-\lambda)\, w_c &= D_x(1-\lambda)\, D_t w_c + D_x[D_t, 1-\lambda]\, w_c \\ &= (1-\lambda)\, D_x D_t w_c + [D_x, 1-\lambda]\, D_t w_c + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}, 1-\lambda\right] w_c. \end{aligned} \tag{3.34}$$

By (3.30) and homogeneity properties of λ,

$$\|(1-\lambda)\, D_x D_t w_c\|_{L_2(\Sigma_\infty)} \leqslant C \left\|\frac{\partial}{\partial \nu} w_t\right\|_{L_2(\Sigma_T)}, \tag{3.35}$$

$$\|[D_x, 1-\lambda]\, D_t w_c\|_{L_2(\Sigma_\infty)} \leqslant C\, \|w_t\|_{L_2(\Sigma_T)}, \tag{3.36}$$

$$\left\|\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}, 1-\lambda\right] w_c\right\|_{L_2(\Sigma_\infty)} \leqslant C\, \|w\|_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))}, \tag{3.37}$$

where in (3.37) we have used the fact that $[\partial/\partial t, 1-\lambda] \in C^\infty(R^+; OPS^0(R^2_{t,y}))$ and trace theory.

Collecting (3.35)–(3.37) yields

$$\begin{aligned} &\left\|\frac{\partial}{\partial x}\, [(1-\lambda)\, w_c]_t\right\|_{L_2(\Sigma_\infty)} \\ &\quad \leqslant \left[\left\|\frac{\partial}{\partial \nu} w_t\right\|_{L_2(\Sigma_T)} + \|w_t\|_{L_2(\Sigma_T)} + \|w\|_{L_2(0, T; H^{3/2+\varepsilon}(\Omega))}\right]. \end{aligned} \tag{3.38}$$

Step 7: Estimates for the third term on the right-hand side of (3.23). By using the boundary conditions in (3.20), we find

$$\|B(\lambda w_c)\|_{L_2(\Sigma_\infty)} \leqslant \|\lambda\psi g_2\|_{L_2(\Sigma_\infty)} + \|[B, \lambda]\, w_c\|_{L_2(\Sigma_\infty)}$$
$$\leqslant C\, \|g_2\|_{L_2(\Sigma_T)} + \|w\|_{L_2(0,\, T;\, H^{3/2+\varepsilon}(\Omega))}, \tag{3.39}$$

by (3.29), trace theory, and the property $\lambda\psi \in C^\infty(R^+;\, OPS^0(R^2_{t,\, y}))$.

Step 8. Combining all of the above inequalities, (3.23), (3.32), (3.33), (3.38), and (3.39), we arrive at our desired estimate,

$$\left\| \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau} w \right\|^2_{L_2(\Sigma_{T\infty})} \leqslant \left\| \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau} w_c \right\|^2_{L_2(\Sigma_\infty)}$$
$$\leqslant C \left\{ \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} + \|g_1\|^2_{H^1(0,\, T;\, L_2(\Gamma))} \right.$$
$$+ \|g_1\|^2_{L_2(0,\, T;\, H^2(\Gamma))} + \|g_2\|^2_{L_2(\Sigma_T)}$$
$$\left. + (1+\gamma)\, \|w\|^2_{L_2(0,\, T;\, H^{3/2+\varepsilon}(\Omega))} + \|f\|^2_{H^{-3/2+\varepsilon}(Q_T)} \right\}. \tag{3.40}$$

4. Proof of Lemma 2.3

Step 1. We again consider the function $w_c(t, \cdot)$ which is defined as $w_c \equiv \psi w$, where ψ is as in Section 3.1 and w is our original solution to (1.1). Thus, w_c satisfies

$$\begin{aligned} w_{c,\,tt} - \gamma\, \Delta w_{c,\,tt} + \Delta^2 w_c &= [\tilde{P}, \psi]\, w && \text{in } Q_\infty \\ \left.\begin{aligned} w_c(0) &= 0 \\ w_{c,\,t}(0) &= 0 \end{aligned}\right\} & && \text{in } \Omega \\ w_c &= 0 && \text{on } \Sigma_\infty \\ \Delta w_c &= -\psi \frac{\partial}{\partial \nu} w_t && \text{on } \Sigma_\infty, \end{aligned} \tag{4.1}$$

where $\tilde{P} w = w_{tt} - \gamma\, \Delta w_{tt} + \Delta^2 w$. Rewriting this equation for w_c as

$$\begin{aligned} w_{c,\,tt} + (I-\gamma\Delta)^{-1}\, \Delta^2 w_c &= (I-\gamma\Delta)^{-1}\, [\tilde{P}, \psi]\, w && \text{in } Q_\infty \\ \left.\begin{aligned} w_c(0) &= 0 \\ w_{c,\,t}(0) &= 0 \end{aligned}\right\} & && \text{in } \Omega \\ w_c &= 0 && \text{on } \Sigma_\infty \\ \Delta w_c &= -\psi \frac{\partial}{\partial \nu} w_t && \text{on } \Sigma_\infty, \end{aligned} \tag{4.2}$$

we see that $w_c(t, \cdot)$ can be written implicitly using the variation of parameters formula. Therefore,

$$w_c(t) = \frac{1}{2i} \sqrt{A_\gamma}^{-1} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} - e^{-i\sqrt{A_\gamma}(t-\tau)}) F(\tau)\, d\tau, \tag{4.3}$$

where

$$\begin{aligned} A_\gamma &\equiv (I + \gamma A^{1/2})^{-1} A \\ F(t) &\equiv (I + \gamma A^{1/2})^{-1} \left\{ \psi A^{1/2} D \left(\frac{\partial}{\partial \nu} w_t(t) \right) + [\tilde{P}, \psi]\, w(t) \right\}, \end{aligned} \tag{4.4}$$

and Dg is defined to be the harmonic extension of the function g from the boundary into the interior; i.e.,

$$Dg = v \Leftrightarrow \begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = g & \text{on } \Gamma. \end{cases} \tag{4.5}$$

Define

$$\begin{aligned} w_2(t) &\equiv \frac{1}{2i} \sqrt{A_\gamma}^{-1} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} - e^{-i\sqrt{A_\gamma}(t-\tau)})(I + \gamma A^{1/2})^{-1} \\ &\quad \times \left\{ \psi A^{1/2} D \left(\frac{\partial}{\partial \nu} w_t(\tau) \right) \right\} d\tau \\ w_3(t) &\equiv \frac{1}{2i} \sqrt{A_\gamma}^{-1} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} - e^{-i\sqrt{A_\gamma}(t-\tau)}) \\ &\quad \times (I + \gamma A^{1/2})^{-1} [\tilde{P}, \psi]\, w(\tau)\, d\tau. \end{aligned} \tag{4.6}$$

Step 2. Expression for $w_2(t)$. Since w_c is compactly supported on $(0, T)$,

$$\|w_c\|_{H^{-1}(0,T)} = \left\| \int_0^t w_c(\tau)\, d\tau \right\|_{L_2(0,T)} \tag{4.7}$$

Hence,

$$\left\| \frac{\partial}{\partial \nu} \Delta w_c \right\|_{H^{-1}(0,\infty; L_2(\Gamma))} = \left\| \frac{\partial}{\partial \nu} A^{1/2} \int_0^t (w_2 + w_3)(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)}. \tag{4.8}$$

Thus, integrating $w_2(t)$ from 0 to t, we find

$$\frac{\partial}{\partial v} A^{1/2} \int_0^t w_2(\tau)\, d\tau = -\frac{1}{2} \frac{\partial}{\partial v} A^{1/2} A_\gamma^{-1} (I + \gamma A^{1/2})^{-1} A^{1/2} \times \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)} - 2)\, \psi(\tau)\, D\left(\frac{\partial}{\partial v} w_t(\tau)\right) d\tau. \tag{4.9}$$

Differentiating $w_2(t)$ with respect to t gives us

$$w_2'(t) = \frac{1}{2} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)})(I + \gamma A^{1/2})^{-1} \times \psi(\tau)\, A^{1/2} D\left(\frac{\partial}{\partial v} w_t(\tau)\right) d\tau. \tag{4.10}$$

By noting that

$$A^{1/2} A_\gamma^{-1} (I + \gamma A^{1/2})^{-1} A^{1/2} = I \tag{4.11}$$

and

$$\gamma A^{1/2} (I + \gamma A^{1/2})^{-1} - I = (I + \gamma A^{1/2})^{-1}, \tag{4.12}$$

we can compare (4.9) and (4.10) and find that

$$\frac{\partial}{\partial v} A^{1/2} \int_0^t w_2(\tau)\, d\tau = -\gamma \frac{\partial}{\partial v} w_2'(t) + \frac{\partial}{\partial v} \int_0^t \psi(\tau)\, D\left(\frac{\partial}{\partial v} w_t(\tau)\right) d\tau - \frac{1}{2} \frac{\partial}{\partial v} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)})(I + \gamma A^{1/2})^{-1} \times \psi(\tau)\, D\left(\frac{\partial}{\partial v} w_t(\tau)\right) d\tau. \tag{4.13}$$

Step 3. Expression for $w_3(t)$. We proceed exactly as we did for $w_2(t)$, which allows us to find

$$\frac{\partial}{\partial v} A^{1/2} \int_0^t w_3(\tau)\, d\tau = -\gamma \frac{\partial}{\partial v} w_3'(t) + \frac{\partial}{\partial v} \int_0^t A^{-1/2} [\tilde{P}, \psi]\, w(\tau)\, d\tau - \frac{1}{2} \frac{\partial}{\partial v} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)}) \times (I + \gamma A^{1/2})^{-1} A^{-1/2} [\tilde{P}, \psi]\, w(\tau)\, d\tau. \tag{4.14}$$

Combining (4.8), (4.13) and (4.14) gives us

$$\begin{aligned}
&\left\| \frac{\partial}{\partial \nu} A^{1/2} w_c \right\|_{H^{-1}(0,\infty;L_2(\Gamma))} \\
&\quad \leqslant \gamma \left\| \frac{\partial}{\partial \nu} (w_c)_t \right\|_{L_2(\Sigma_\infty)} + \left\| \frac{\partial}{\partial \nu} \int_0^t \psi(\tau)\, D\left(\frac{\partial}{\partial \nu} w_t(\tau) \right) d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\quad + \frac{1}{2} \left\| \frac{\partial}{\partial \nu} \int_0^t \left(e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)}\right)(I+\gamma A^{1/2})^{-1} \right. \\
&\quad \left. \times \psi(\tau)\, D\left(\frac{\partial}{\partial \nu} w_t(\tau) \right) d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\quad + \left\| \frac{\partial}{\partial \nu} \int_0^t A^{-1/2}[\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\quad + \frac{1}{2} \left\| \frac{\partial}{\partial \nu} \int_0^t \left(e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)}\right)(I+\gamma A^{1/2})^{-1} \right. \\
&\quad \left. \times A^{-1/2}[\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)}
\end{aligned} \tag{4.15}$$

Step 4. To bound the second term on the right-hand side of (4.15), we note that

$$\int_0^t \psi(\tau)\, D\left(\frac{\partial}{\partial \nu} w_t(\tau) \right) d\tau = \psi(t)\, D\left(\frac{\partial}{\partial \nu} w(t) \right) - \int_0^t \psi'(\tau)\, D\left(\frac{\partial}{\partial \nu} w(\tau) \right) d\tau. \tag{4.16}$$

Thus,

$$\left\| \frac{\partial}{\partial \nu} \int_0^t \psi(\tau)\, D\left(\frac{\partial}{\partial \nu} w_t(\tau) \right) d\tau \right\|_{L_2(\Sigma_T)} \leqslant M_\psi \left\| \frac{\partial}{\partial \nu} D\left(\frac{\partial}{\partial \nu} w \right) \right\|_{L_2(\Sigma_T)}. \tag{4.17}$$

But, we also know, by elliptic regularity, that

$$\begin{aligned}
\left\| \frac{\partial}{\partial \nu} D\left(\frac{\partial}{\partial \nu} w \right) \right\|_{L_2(\Sigma_T)} &\leqslant C \left\| D\left(\frac{\partial}{\partial \nu} w \right) \right\|_{L_2(0,T;H^{3/2}(\Omega))} \\
&\leqslant C \left\| \frac{\partial}{\partial \nu} w \right\|_{L_2(0,T;H^1(\Gamma))} \leqslant C \left\| \frac{\partial}{\partial \tau} \frac{\partial}{\partial \nu} w \right\|_{L_2(\Sigma_T)}
\end{aligned} \tag{4.18}$$

Step 5. By writing $[\tilde{P}, \psi] w$ explicitly as

$$[\tilde{P}, \psi]\, w \equiv 2\gamma A^{1/2}(\psi' w)_t + \psi'' w - 2(\psi' w)_t + \gamma \psi'' A^{1/2} w, \tag{4.19}$$

the fourth term on the right-hand side of (4.15) can be bounded directly by trace theory as follows:

$$\left\| \frac{\partial}{\partial \nu} \int_0^t A^{-1/2}[\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)}$$

$$= \left\| \frac{\partial}{\partial \nu} A^{-1/2}[\tilde{P}, \psi]\, w \right\|_{H^{-1}(0,\infty; L_2(\Gamma))}$$

$$\leqslant C \left\{ \left\| \frac{\partial}{\partial \nu} A^{-1/2} \gamma A^{1/2} (\psi' w)_t \right\|_{H^{-1}(0,\infty; L_2(\Gamma))} + \|A^{-1/2} \psi'' w\|_{H^{-1}(0,T; H^{3/2+\varepsilon}(\Omega))} \right.$$

$$\left. + \|A^{-1/2} \psi' w\|_{H^{-1}(0,T; H^{3/2+\varepsilon}(\Omega))} + \gamma \left\| \frac{\partial}{\partial \nu} w \right\|_{L_2(\Sigma_T)} \right\}$$

$$\leqslant C \left\{ \gamma \left\| \frac{\partial}{\partial \nu} (\psi' w)_t \right\|_{H^{-1}(0,\infty; L_2(\Gamma))} + \|w\|_{L_2(0,T; L_2(\Omega))} + \gamma \left\| \frac{\partial}{\partial \nu} w \right\|_{L_2(\Sigma_T)} \right\}$$

$$\leqslant C \left\{ \gamma \left\| \frac{\partial}{\partial \nu} w \right\|_{L_2(\Sigma_T)} + \|w\|_{L_2(0,T; L_2(\Omega))} \right\} \tag{4.20}$$

To bound the third and the last terms on the right-hand side of (4.15) we will need the following proposition.

Proposition 4.1. *Consider the following three "abstract" Schrödinger problems:*

$$z_t = i\sqrt{A_\gamma}\, z + f \qquad z(0) = 0, \tag{4.21}$$

$$z_t = i\sqrt{A_\gamma}\, z + (I + \gamma A^{1/2})^{-1} f \qquad z(0) = 0, \tag{4.22}$$

$$z_t = i\sqrt{A_\gamma}\, z + (I + \gamma A^{1/2})^{-1} Df \qquad z(0) = 0. \tag{4.23}$$

Then

(i) *if $z(t)$ satisfies* (4.21), *then for every $0 < \alpha < \frac{1}{2}$,*

$$\|A^\alpha z(t)\|_{L_2(\Omega)} \leqslant C \|f\|_{L_1(0,T; D(A^\alpha))} \text{ uniformly in } \gamma. \tag{4.24}$$

(ii) *If $z(t)$ satisfies* (4.22), *then*

$$\left\| \frac{\partial}{\partial \nu} z \right\|_{L_2(\Sigma_T)} \leqslant C(T+1)\, \|A^{1/4}(I + \gamma A^{1/2})^{-1} f\|_{L_1(0,T; L_2(\Omega))}, \tag{4.25}$$

which in turn implies both

$$\left\| \frac{\partial}{\partial \nu} z \right\|_{L_2(\Sigma_T)} \leqslant C(T+1) \|f\|_{L_1(0,T;H^1(\Omega))}$$

and (4.26)

$$\left\| \frac{\partial}{\partial \nu} z \right\|_{L_2(\Sigma_T)} \leqslant \frac{C(T+1)}{\gamma} \|f\|_{L_1(0,T;H^{-1}(\Omega))}.$$

(iii) *If* $z(t)$ *satisfies* (4.23), *then*

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C_T \|f\|^2_{L_2(\Sigma_T)}, \tag{4.27}$$

and the constant C *is* independent *of* $\gamma > 0$.

Remark. The result of part (i) is classical. However, note that the results of parts (ii) and (iii) are not the standard ones. By using standard regularity theory, we would only expect at most

$$\frac{\partial}{\partial \nu} z(t) \in H^{-1}(\Gamma). \tag{4.28}$$

Therefore, these results give us a bonus of "one derivative."

Remark. Notice that results in the same spirit as those in (4.22) and (4.23) were proved in [7]. The main differences, however, are that in the present context,

(i) we deal with an "abstract" Schrödinger equation ($A_\gamma^{1/2}$ is a pseudodifferential operator) rather than a standard Schrödinger operator, $i\Delta$, as in [7];

(ii) we need to keep track of the dependence on $\gamma > 0$ so that the final constant C does not depend on $\gamma > 0$.

Since the proof of Proposition 4.1 is rather technical, the proof is relegated to Appendix A.

We continue with the proof of Lemma 2.3. To bound the last term on the right-hand side of (4.15), it is enough to estimate

$$\left\| \frac{\partial}{\partial \nu} \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I + \gamma A^{1/2})^{-1} A^{-1/2} [\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)}. \tag{4.29}$$

(The argument with the $e^{-i\sqrt{A_\gamma}}$ term is the same.) Recalling (4.19) and applying part (ii) of Proposition 4.1 to the second and fourth terms in (4.19) with $f \equiv A^{-1/2} w\psi''$ and $f \equiv w\psi''$, respectively, we find

$$\begin{aligned}
&\left\| \frac{\partial}{\partial \nu} \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I+\gamma A^{1/2})^{-1} A^{-1/2} [\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\quad \leqslant 2 \left\| \frac{\partial}{\partial \nu} \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} \gamma A^{1/2} (I+\gamma A^{1/2})^{-1} A^{-1/2} (\psi' w)_t\, d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\qquad + 2 \left\| \frac{\partial}{\partial \nu} \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I+\gamma A^{1/2})^{-1} A^{-1/2} (\psi' w)_t\, d\tau \right\|_{L_2(\Sigma_\infty)} \\
&\qquad + C(T+1)\, \gamma \|w\|_{L_2(0,T;H^1(\Omega))} \\
&\quad \equiv p_1 + p_2 + C(T+1)\, \gamma \|w\|_{L_2(0,T;H^1(\Omega))}.
\end{aligned} \tag{4.30}$$

We will be frequently using the estimate

$$\|\gamma A^{1/2} (I+\gamma A^{1/2})^{-1} x\|_{\mathscr{H}} \leqslant C \|x\|_{\mathscr{H}} \qquad \text{uniformly in } \gamma > 0, \tag{4.31}$$

and the result by Grisvard [4, (2.4)].

Bound for p_1. We integrate by parts

$$\begin{aligned}
&\int_0^t e^{i\sqrt{A_\gamma}}(t-\tau)\, \gamma A^{1/2} (I+\gamma A^{1/2})^{-1} A^{-1/2} (\psi' w)_t\, d\tau \\
&\quad = (I+\gamma A^{1/2})^{-1} \gamma (\psi' w)(t) \\
&\qquad - \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I+\gamma A^{1/2})^{-1/2} A^{1/2} \gamma (I+\gamma A^{1/2})^{-1} (\psi' w)(\tau)\, d\tau \\
&\quad \equiv p_{11} + p_{12}.
\end{aligned} \tag{4.32}$$

We can bound p_{11} by trace theory, (2.4), and (4.31):

$$\begin{aligned}
\left\| \frac{\partial}{\partial \nu} p_{11} \right\|_{L_2(\Sigma_\infty)} &\leqslant C \|A^{3/8+\varepsilon} p_{11}\|_{L_2(Q_\infty)} \\
&\quad C \|A^{3/8+\varepsilon} \gamma (I+\gamma A^{1/2})^{-1} (\psi' w)\|_{L_2(Q_\infty)} \\
&\leqslant C \|\gamma A^{1/2} (I+\gamma A^{1/2})^{-1} (\psi' w)\|_{L_2(Q_\infty)} \leqslant C \|w\|_{L_2(Q_T)}.
\end{aligned} \tag{4.33}$$

As for p_{12}, we shall use part (ii) of Proposition 4.1 with $f \equiv \sqrt{A_\gamma}\, \gamma \psi' w$. Then

$$\left\| \frac{\partial}{\partial \nu} p_{12} \right\|_{L_2(\Sigma_\infty)}$$
$$\leqslant C(T+1) \| A^{1/4} (I + \gamma A^{1/2})^{-1} f \|_{L_2(0, \infty; L_2(\Omega))}$$
$$= C(T+1) \left\| A^{1/4} (I + \gamma A^{1/2})^{-1} \gamma A^{1/2} (I + \gamma A^{1/2})^{-1/2} w \right\|_{L_2(0, T; L_2(\Omega))}$$
$$\leqslant C(T+1) \| A^{1/4} w \|_{L_2(0, T; L_2(\Omega))} \leqslant C(T+1) \| w \|_{L_2(0, T; H^1(\Omega))}. \tag{4.34}$$

Collecting (4.32)–(4.34), we obtain

$$p_1 \leqslant C(T+1) \| w \|_{L_2(0, T; H^1(\Omega))}. \tag{4.35}$$

Bound for p_2: We integrate by parts

$$\int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I + \gamma A^{1/2})^{-1} A^{-1/2} (\psi' w)_t \, d\tau$$
$$= (I + \gamma A^{1/2})^{-1} A^{-1/2} (\psi' w)(t)$$
$$- \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)} (I + \gamma A^{1/2})^{-1} (I + \gamma A^{1/2})^{-1/2} (\psi' w)(\tau) \, d\tau$$
$$\equiv p_{21} + p_{22}. \tag{4.36}$$

As in the case of p_{11}, to bound p_{21} we apply trace theory and (2.4) in a straightforward manner.

$$\left\| \frac{\partial}{\partial \nu} p_{21} \right\|_{L_2(\Sigma_\infty)} \leqslant \| w \|_{L_2(Q_T)}. \tag{4.37}$$

For p_{22}, we apply part (ii) of Proposition 4.1 with $f \equiv (I + \gamma A^{1/2})^{-1/2} \psi' w$. Since

$$\| f \|_{L_2(0, \infty; H^1(\Omega))} \leqslant C \| w \|_{L_2(0, T; H^1(\Omega))}, \tag{4.38}$$

we obtain

$$\left\| \frac{\partial}{\partial \nu} p_{22} \right\|_{L_2(\Sigma_\infty)} \leqslant C(T+1) \| w \|_{L_2(0, T; H^1(\Omega))}. \tag{4.39}$$

Combining (4.38) and (4.39), we find

$$p_2 \leqslant C(T+1) \| w \|_{L_2(0, T; H^1(\Omega))}. \tag{4.40}$$

Therefore, from (4.30), (4.35), and (4.40), we obtain

$$\left\| \frac{\partial}{\partial v} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)})(I+\gamma A^{1/2})^{-1} A^{-1/2}[\tilde{P}, \psi]\, w(\tau)\, d\tau \right\|_{L_2(\Sigma_\alpha)}$$
$$\leqslant C(T+1)(1+\gamma)\|w\|_{L_2(0,T;H^1(\Omega))}. \tag{4.41}$$

Step 6. Our last step is to bound the third term on the right-hand side of (4.15). We apply part (iii) of Proposition 4.1 with $f \equiv \psi(\partial/\partial v) w_t$ to find

$$\left\| \frac{\partial}{\partial v} \int_0^t (e^{i\sqrt{A_\gamma}(t-\tau)} + e^{-i\sqrt{A_\gamma}(t-\tau)})(I+\gamma A^{1/2})^{-1} D\left(\psi(\tau)\frac{\partial}{\partial v} w_t(\tau)\right) d\tau \right\|^2_{L_2(\Sigma_\alpha)}$$
$$\leqslant C_T \left\| \frac{\partial}{\partial v} w_t \right\|^2_{L_2(\Sigma_T)}. \tag{4.42}$$

Therefore, by combining all of the above estimates, (4.15), (4.18), (4.20), (4.41), (4.42), and using trace theory, we find

$$\left\| \frac{\partial}{\partial v}(\Delta w_c) \right\|_{H^{-1}(0,T;L_2(\Gamma))}$$
$$\leqslant C_T(1+\gamma)\left\| \frac{\partial}{\partial v} w_t \right\|^2_{L_2(\Sigma_T)} + C\left\{ \left\| \frac{\partial}{\partial v}\frac{\partial}{\partial \tau} w \right\|^2_{L_2(\Sigma_T)} \right.$$
$$\left. + (T+1)(1+\gamma)\|w\|^2_{L_2(0,T;H^{3/2+\varepsilon}(\Omega))} \right\}. \tag{4.43}$$

By taking the above estimate from α to $T-\alpha$ and noting Lemma 2.2, we arrive at our desired result,

$$\left\| \frac{\partial}{\partial v}(\Delta w) \right\|_{H^{-1}(\alpha,T-\alpha;L_2(\Gamma))}$$
$$\leqslant C_T(1+\gamma)\left\| \frac{\partial}{\partial v} w_t \right\|^2_{L_2(\Sigma_T)} + C(T+1)(1+\gamma)\|w\|^2_{L_2(0,T,H^{3/2+\varepsilon}(\Omega))}. \tag{4.44}$$

APPENDIX A: Proof of Proposition 4.1

A.1. *Proof of Part* (i)

Let z satisfy

$$z_t = i\sqrt{A_\gamma}\, z + f$$
$$z(0) = 0. \tag{A.1}$$

Multiplying the equation by $A^{2\alpha}\bar{z}$ and integrating by parts, we find

$$\int_0^t (z_t, A^{2\alpha}\bar{z})_{L_2(\Omega)} = i(\sqrt{A_\gamma}z, A^{2\alpha}\bar{z})_{L_2(0,T;L_2(\Omega))} + (f, A^{2\alpha}\bar{z})_{L_2(0,T;L_2(\Omega))}$$

$$\Rightarrow \|A^\alpha z(t)\|^2_{L_2(\Omega)} \leqslant \|A^\alpha f\|_{L_1(0,T;L_2(\Omega))} \|A^\alpha z\|_{L_\infty(0,T;L_2(\Omega))}, \tag{A.2}$$

since A and A_γ are both self-adjoint with respect to $L_2(\Omega)$ and $z(0)=0$. Because the above inequality holds for all $t \in [0, T]$, we find

$$\|A^\alpha z\|_{L_\infty(0,T;L_2(\Omega))} \leqslant \|A^\alpha f\|_{L_1(0,T;L_2(\Omega))}, \tag{A.3}$$

which in turn implies our desired result,

$$\|A^\alpha z(t)\|_{L_2(\Omega)} \leqslant \|A^\alpha f\|_{L_1(0,T;L_2(\Omega))} \qquad \forall t \in [0, T]. \tag{A.4}$$

A.2. *Proof of Part* (ii)

Let z satisfy

$$\begin{aligned} z_t &= i\sqrt{A_\gamma}z + (I+\gamma A^{1/2})^{-1} f \\ z(0) &= 0. \end{aligned} \tag{A.5}$$

Define

$$\hat{f} \equiv (I+\gamma A^{1/2})^{-1/2} f. \tag{A.6}$$

Then we can rewrite the equation for z as

$$(I+\gamma A^{1/2})^{1/2} z_t = iA^{1/2} z + \hat{f}. \tag{A.7}$$

Step 1. Multiplying the left-hand side of (A.7) by $h \cdot \nabla\bar{z}$ and integrating by parts, we find (recalling $A^{1/2}z = -\Delta z$)

$$\begin{aligned} 2\Im m((I+\gamma A^{1/2})^{1/2} z_t, h\cdot\nabla\bar{z})_{L_2(Q_T)} &= ((I+\gamma A^{1/2})^{1/2} z, h\cdot\nabla\bar{z})_{L_2(\Omega)}\big|_0^T \\ &\quad - ((I+\gamma A^{1/2})^{1/2} z, h\cdot\nabla\bar{z}_t)_{L_2(Q_T)} \\ &\quad - ((I+\gamma A^{1/2})^{1/2} \bar{z}_t, h\cdot\nabla z)_{L_2(Q_T)}. \end{aligned} \tag{A.8}$$

Let

$$F \equiv h\bar{z}_t(I+\gamma A^{1/2})^{1/2} z. \tag{A.9}$$

Then

$$\begin{aligned} \operatorname{div} F = h\cdot\nabla\bar{z}_t(I+\gamma A^{1/2})^{1/2} z + h\cdot\nabla((I+\gamma A^{1/2})^{1/2} z)\bar{z}_t \\ + \operatorname{div} h(\bar{z}_t(I+\gamma A^{1/2})^{1/2} z). \end{aligned} \tag{A.10}$$

Noting that

$$(I+\gamma A^{1/2})^{1/2}-I=\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}, \tag{A.11}$$

we find

$$\begin{aligned}\operatorname{div} F &= h\cdot\nabla\bar{z}_t(I+\gamma A^{1/2})^{1/2}z+h\bar{z}_t\nabla z\\ &\quad+h\cdot\nabla(\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z)\\ &\quad+\operatorname{div} h(\bar{z}_t z)+\operatorname{div} h(\bar{z}_t\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z).\end{aligned} \tag{A.12}$$

Thus we find,

$$\begin{aligned}&2\Im m((I+\gamma A^{1/2})^{1/2}z_t,\, h\cdot\nabla\bar{z})_{L_2(Q_T)}\\ &\quad=((I+\gamma A^{1/2})^{1/2}z,\, h\cdot\nabla\bar{z})_{L_2(\Omega)}\big|_0^T-\int_0^T\int_\Omega \operatorname{div} F\, d\Omega\, dt\\ &\qquad+(\bar{z}_t,\, h\cdot\nabla(\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z))_{L_2(Q_T)}\\ &\qquad+(\bar{z}_t,\, z\operatorname{div} h)_{L_2(Q_T)}\\ &\qquad+(\bar{z}_t,\, \gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z\operatorname{div} h)_{L_2(Q_T)}\\ &\qquad-(\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}\bar{z}_t,\, h\cdot\nabla z)_{L_2(Q_T)}.\end{aligned} \tag{A.13}$$

If we assume $h=x-x_0$ for some $x_0\in R^2$, then $\operatorname{div} h=2$. Substituting the expression for z_t from (A.7) into the above identity, then combining it with the results of the same multiplier with the right-hand side of (A.7), we find

$$\begin{aligned}&\left\|\frac{\partial}{\partial\nu}z\right\|^2_{L_2(\Sigma_T)}\\ &\quad\leqslant C\{(T+1)\|\nabla z\|^2_{L_\infty(0,T;L_2(\Omega))}+\|\hat{f}\|^2_{L_1(0,T;L_2(\Omega))}\}\\ &\qquad+2|(A^{1/2}(I+\gamma A^{1/2})^{-1/2}\bar{z},\, \gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z)_{L_2(Q_T)}|\\ &\qquad+2|((I+\gamma A^{1/2})^{-1}f,\, \gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z)_{L_2(Q_T)}|\\ &\qquad+|((I+\gamma A^{1/2})^{-1/2}\bar{z},\, h\cdot\nabla\{\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z\})_{L_2(Q_T)}|\\ &\qquad+|(I+\gamma A^{1/2})^{-1}f,\, h\cdot\nabla\{\gamma A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z\})_{L_2(Q_T)}|.\end{aligned} \tag{A.14}$$

Since

$$\begin{aligned}&|\gamma(A^{1/2}(I+\gamma A^{1/2})^{-1/2}\bar{z},\, A^{1/2}[I+(I+\gamma A^{1/2})^{1/2}]^{-1}z)_{L_2(Q_T)}|\\ &\quad\leqslant C\gamma\|A^{1/2}(I+\gamma A^{1/2})^{-1/2}z\|^2_{L_2(Q_T)},\end{aligned} \tag{A.15}$$

and from (2.4) we have

$$\gamma\|A^{1/2}(I+\gamma A^{1/2})^{-1/2}z\|_{L_2(\Omega)}\leqslant C\|z\|_{H^1(\Omega)} \quad \text{uniformly in } \gamma, \tag{A.16}$$

combining the above two inequalities with (A.14) gives the result

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C\{(T+1)\|\nabla z\|^2_{L_\infty(0,T;L_2(\Omega))} + \|A^{1/4}(I+\gamma A^{1/2})^{-1} f\|^2_{L_1(0,T;L_2(\Omega))} + \|(I+\gamma A^{1/2})^{-1/2} f\|^2_{L_1(0,T;L_2(\Omega))}\}. \tag{A.17}$$

Step 2. Estimates for the particular solution. Using the variations of parameters formula, we can write $z(t)$ as

$$z(t) = \int_0^t e^{i\sqrt{A_\gamma}(t-\tau)}(I+\gamma A^{1/2})^{-1} f(\tau)\, d\tau. \tag{A.18}$$

Using this formulation for $z(t)$, we find

$$\|A^{1/4} z(t)\|_{L_2(\Omega)} \leqslant \int_0^t \|A^{1/4}(I+\gamma A^{1/2})^{-1} f(\tau)\|_{L_2(\Omega)}\, d\tau$$

$$\leqslant \|A^{1/4}(I+\gamma A^{1/2})^{-1} f\|_{L_1(0,T;L_2(\Omega))} \qquad \forall t \in (0, T). \tag{A.19}$$

Therefore,

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C\{(T+1)\|A^{1/4}(I+\gamma A^{1/2})^{-1} f\|^2_{L_1(0,T;L_2(\Omega))} + \|(I+\gamma A^{1/2})^{-1/2} f\|^2_{L_1(0,T;L_2(\Omega))}. \tag{A.20}$$

Step 3. Completion of proof. Finally, by noting that

$$\|A^{-1/4}(I+\gamma A^{1/2})^{1/2} f\|_{L_2(\Omega)} \leqslant C\|f\|_{L_2(\Omega)}, \tag{A.21}$$

where C is independent of $0 < \gamma < M$, we arrive at our desired result

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C(T+1)\|A^{1/4}(I+\gamma A^{1/2})^{-1} f\|^2_{L_1(0,T;L_2(\Omega))}, \tag{A.22}$$

where C_T is independent of $\gamma > 0$.

A.3. *Proof of Part* (iii)

Step 1. We begin as in the proof of part (ii), but we replace f by Df in our equation. Therefore, by using the same techniques as in the first step of the proof of part (ii), we find

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C\{(T+1)\|\nabla z\|^2_{L_\infty(0,T;L_2(\Omega))} + \|(I+\gamma A^{1/2})^{-1/2} Df\|^2_{L_1(0,T;L_2(\Omega))}. \tag{A.23}$$

Step 2. Estimates for the particular solution. Using the variations of parameters formula, we can write $z(t)$ as

$$z(t)=\int_0^t e^{i\sqrt{A_\gamma}(t-\tau)}(I+\gamma A^{1/2})^{-1} Df(\tau)\,d\tau. \tag{A.24}$$

Define the closed, densely defined operator $L: L_2(\Sigma_T)\to L_2(Q_T)$ by

$$(Lf)(t)\equiv A^{1/2}\int_0^t e^{i\sqrt{A_\gamma}(t-\tau)}(I+\gamma A^{1/2})^{-1} Df(\tau)\,d\tau. \tag{A.25}$$

Then we can show that (see [6])

$$\begin{aligned}(L^*g)(t)&\equiv D^*A^{1/2}\int_t^T e^{-i\sqrt{A_\gamma}(t-\tau)}(I+\gamma A^{1/2})^{-1} g(\tau)\,d\tau\\ &=\frac{\partial}{\partial v}\int_t^T e^{-i\sqrt{A_\gamma}(t-\tau)}(I+\gamma A^{1/2})^{-1} g(\tau)\,d\tau,\end{aligned} \tag{A.26}$$

where the adjoints are taken with respect to the L_2-topology.

Let $\eta(t)$ be defined to be the integral in the above expression. Then η satisfies

$$\begin{aligned}\eta_t&=-i\sqrt{A_\gamma}\,\eta+(I+\gamma A^{1/2})^{-1}g\\ \eta(T)&=0.\end{aligned} \tag{A.27}$$

Therefore, from our previous calculations, η satisfies:

$$\begin{aligned}&\left\|\frac{\partial}{\partial v}\eta\right\|^2_{L_2(\Sigma_T)}\\ &\quad\leqslant C\{(T+1)\|\nabla\eta\|^2_{L_\infty(0,T;L_2(\Omega))}+\|(I+\gamma A^{1/2})^{-1/2}g\|^2_{L_1(0,T;L_2(\Omega))}\}\\ &\quad\leqslant C\{(T+1)\|\nabla\eta\|^2_{L_\infty(0,T;L_2(\Omega))}+\|g\|^2_{L_1(0,T;L_2(\Omega))}\}.\end{aligned} \tag{A.28}$$

By using the equation for η directly, we find

$$\begin{aligned}\|A^{1/4}\eta\|_{L_2(\Omega)}&\leqslant\int_t^T\|A^{1/4}(I+\gamma A^{1/2})^{-1}g(\tau)\|_{L_2(\Omega)}\,d\tau\\ &\leqslant C\|A^{1/4}g\|_{L_1(0,T;L_2(\Omega))}.\end{aligned} \tag{A.29}$$

Therefore,

$$\left\| \frac{\partial}{\partial \nu} \eta \right\|^2_{L_2(\Sigma_T)} \leqslant C(T+1) \| g \|^2_{L_1(0, T; H_0^1(\Omega))}$$

$$\Rightarrow L^* \in \mathscr{L}(L_1(0, T; H_0^1(\Omega)) \to L_2(\Sigma_T))$$

$$\Rightarrow L \in \mathscr{L}(L_2(\Sigma_T) \to L_\infty(0, T; H^{-1}(\Omega))$$

$$\Rightarrow K \equiv A^{-1/2} L \in \mathscr{L}(L_2(\Sigma_T) \to L_\infty(0, T; H_0^1(\Omega)). \tag{A.30}$$

Thus,

$$\| z \|_{L_\infty(0, T; H_0^1(\Omega))} \leqslant C_T \| f \|_{L_2(\Sigma_T)}. \tag{A.31}$$

Step 3. Completion of proof. Combining (A.14) and (A.31), we arrive at our desired result,

$$\left\| \frac{\partial}{\partial \nu} z \right\|^2_{L_2(\Sigma_T)} \leqslant C_T \| f \|^2_{L_2(\Sigma_T)}, \tag{A.32}$$

where C_T is independent of $\gamma > 0$.

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