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On a One-Dimensional Steady-State Hydrodynamic Model for Semiconductors

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Abstract. We present a hydrodynamic model for semiconductors, where the energy equation is replaced by a pressure-density relationship. We prove existence of smooth solutions and a uniqueness result in the subsonic case, which is characterized by a smallness assumption on the current flowing through the device.

1. The Model

The hydrodynamic model for semiconductors [1,2] has recently attracted a lot of attention because of its capability of modelling hot electron effects which are not accounted for in the classical drift-diffusion model [3,4]. A mathematical analysis of the hydrodynamic model has not been presented yet; only preliminary results exist by now [5]. In this paper, we investigate a simplified hydrodynamic model in which the energy equation is replaced by the assumption that the pressure is a given function of the density only. This assumption is commonly used in gas dynamics for isentropic or isothermal flows and gives Bernoulli's law [6].

After appropriate scaling the one-dimensional time-dependent system in the case of one carrier type (e.g. electrons) reads:

$$\rho_t + (\rho u)_x = 0 \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \rho \Phi_x = -\frac{\rho u}{\tau} \tag{1.2}$$

$$\Phi_{xx} = \rho - C(x), \tag{1.3}$$

where $\rho(x,t)$, $u(x,t)$, $\Phi(x,t)$ denote the electron density, velocity, and the electrostatistic potential respectively. $p = p(\rho)$ is the pressure-density relation which satisfies:

$$\rho^2 p'(\rho) \text{ is strictly monotonically increasing from } [0,\infty[\text{ onto } [0,\infty[. \tag{1.4}$$

A commonly used hypothesis is $p(\rho) = \kappa \rho^\gamma$ were $\gamma \geq 1$ and $\kappa > 0$. $\tau = \tau(\rho, \rho u)$ is the momentum relaxation time of which we assume:

$$\tau(\rho, \rho u) \geq \tau_0 > 0 \quad \forall (\rho, \rho u) \in [0, \infty[\times \Re. \tag{1.5}$$

The device domain is the x-interval $(0,1)$; $C = C(x) \in L^\infty(0,1)$ is the doping profile (given background density), which satisfies $C(x) > 0$.

The system (1.1) to (1.3) is supplemented by the following boundary conditions:

$$\rho(0,t) = \rho_0 \ , \ \rho(1,t) = \rho_1 \tag{1.6}$$

$$\Phi(0,t) = 0\ ,\ \Phi(1,t) = \Phi_1 \tag{1.7}$$

The conditions (1.6) represent Ohmic contacts, and Φ_1 stands for the applied bias.

In this paper, we investigate the steady state case $\rho_t = (\rho u)_t = 0$. Then, introducing the current density $j = \rho u$, the system (1.1) to (1.3) reduces to

$$j(x) = \text{const.} \tag{1.8}$$

$$\left(\frac{j^2}{\rho} + p(\rho)\right)_x - \rho\Phi_x = -\frac{j}{\tau} \tag{1.9}$$

$$\Phi_{xx} = \rho - C(x). \tag{1.10}$$

Up to now, the system (1.8) to (1.10) is valid for the cases of discontinuous as well as regular solutions. For the case of regular solutions we recast (1.9) by differentiating, dividing by ρ and using (1.8). We obtain

$$\left(\frac{j^2}{2\rho^2} + h(\rho) - \Phi\right)_x + \frac{j}{\tau\rho} = 0\ ,\ h'(\rho) = \frac{1}{\rho}p'(\rho). \tag{1.11}$$

Integrating this equation over $(0,1)$ and using (1.6), (1.7) leads to the following current-voltage characteristic:

$$\Phi_1 = F(\rho_1, j) - F(\rho_0, j) + j\int_0^1 \frac{dx}{\tau(\rho(x), j)\rho(x)} \tag{1.12}$$

where we denoted

$$F(\rho, j) = \frac{j^2}{2\rho^2} + h(\rho). \tag{1.13}$$

The relation (1.12) shows that we can prescribe j instead of Φ_1. Then by differentiating (1.11) and using the Poisson equation (1.10), we obtain the following second order boundary value problem, parametrized by $j \in \Re$:

$$F(\rho, j)_{xx} + j\left(\frac{1}{\tau\rho}\right)_x - \rho = -C(x)\ ,\ 0 < x < 1, \tag{1.14}$$

$$\rho(0) = \rho_0\ ,\ \rho(1) = \rho_1. \tag{1.15}$$

Once ρ is known from (1.14), (1.15), Φ can be computed by solving the Poisson equation (1.10) using the boundary condition (1.12) and $\Phi(0) = 0$. Thus, finding regular steady-state solutions of the hydrodynamic model (1.1) to (1.7) amounts to solving (1.14)-(1.15).

Remark 1. The equation (1.9) can be regarded as a modified steady-state drift-diffusion model with a nonlinear current dependent diffusion term.

2. Existence in the subsonic case

Since we have

$$\frac{\partial F}{\partial\rho}(\rho, j) = -\frac{j^2}{\rho^3} + \frac{1}{\rho}p'(\rho) > 0 \Leftrightarrow \rho^2 p'(\rho) > j^2 \tag{2.1}$$

We conclude from (1.4) that there exists a unique $\rho_m = \rho_m(j) \geq 0$ such that $\frac{\partial F}{\partial\rho}(\rho, j) > 0$ for $\rho > \rho_m$. A typical plot of $\rho \to F(\rho, j)$ for fixed $|j| > 0$ is shown in Figure 1. Note that, by (2.1), the minimal point ρ_m of $\rho \to F(\rho, j)$ is a strictly increasing function of j, and $\rho_m(j = 0) = 0$.

The preceding considerations show that the equation (1.14) is uniformly elliptic for $\rho \geq \rho_* > \rho_m$. By (2.1), this condition implies $|u| < c$ where $c = (p'(\rho))^{1/2}$ denotes the speed of sound. Thus, the following theory applies to a fully subsonic flow. We prove an existence result in this case:

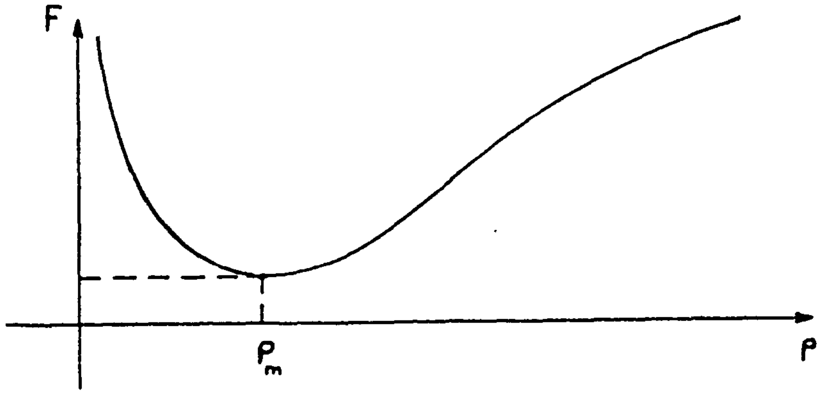

Figure 1: Plot of $\rho \to F(\rho, j)$.

THEOREM 1.

Let the assumptions (1.4), (1.5), $C \in L^\infty(0,1)$ hold and additionally assume that p', τ and $\partial\tau/\partial\rho$ are continuous with respect to ρ. Let j be such that

$$\rho_0, \rho_1, inf_{x \in (0,1)} C(x) > \rho_m(j). \tag{2.2}$$

Then, the problem (1.14), (1.15) has a solution which satisfies

$$Min(\rho_0, \rho_1, \underline{C}) \leq \rho(x) \leq Max(\rho_0, \rho_1, \overline{C}),\ x \in [0,1] \tag{2.3}$$

where

$$\underline{C} := inf_{x \in [0,1]} C(x)\ ,\ \overline{C} := sup_{x \in [0,1]} C(x).$$

Remark 2. By the above discussion of ρ_m, there exists a unique current j_M such that (2.2) is equivalent to $|j| < j_M . j_M$ is given by

$$\rho_m(j_M) = \mathrm{Min}(\rho_0, \rho_1, \underline{C}).$$

Thus, (2.2) represents a smallness assumption on the current flowing through the device, and gives a condition for fully subsonic flow.
Proof: We consider the operator $T : \Psi \to \rho$ defined by solving

$$\left(\frac{\partial F}{\partial \rho}(\Psi, j)\rho_x\right)_x - jG(\Psi, j)\rho_x - \rho = -C,\ 0 < x < 1, \tag{2.4}$$

$$\rho(0) = \rho_0\ ,\ \rho(1) = \rho_1, \tag{2.5}$$

where we denoted

$$G(\rho, j) = \frac{\frac{\partial \tau}{\partial \rho}(\rho, j)\rho + \tau(\rho, j)}{\tau(\rho, j)^2 \rho^2}. \tag{2.6}$$

Now, suppose that Ψ satisfies

$$\rho_s := \mathrm{Min}(\rho_0, \rho_1, \underline{C}) \leq \Psi \leq \rho_u := \mathrm{Max}(\rho_0, \rho_1, \overline{C}). \tag{2.7}$$

By the assumption (2.2), the linear equation (2.4) is uniformly elliptic. Thus, the smoothness assumptions on p and τ guarantee the existence of a unique $H^1(0,1)$-solution ρ. The maximum principle [7] implies that $\rho = T(\Psi)$ also satisfies (2.7). Then, by multiplying (2.4) by $(\rho - \rho_B)$ where ρ_B is any function assuming the boundary conditions an $H^1(0,1)$ bound on ρ independent of Ψ is easily obtained. By the compact imbedding of $H^1(0,1)$ into $C^0([0,1])$, and by a standard continuity argument we conclude the existence of a fixed point of T from Schauder's theorem. ∎
Remark 3. If $|j|$ is so large that the condition (2.2) is violated, then the flow may at least be partly supersonic and the occurrence of shocks cannot be excluded.

3. UNIQUENESS FOR SMALL $|j|$

We prove the following uniqueness result for small currents.

THEOREM 2.

Let the assumptions of Theorem 1 hold, and additionally assume that τ and $\partial\tau/\partial\rho$ are jointly continuous with respect to (ρ, j). Then, there exists a constant $j_0 > 0$ such that for any current which satisfies $|j| < j_0$, there exists at most one regular solution ρ of the problem (1.14), (1.15) in the set of functions which satisfy $\rho(x) \geq \rho^r > \rho_m(j)$ for all x in $[0,1]$.

Proof: Let ρ_1 and ρ_2 be two solutions of the problem (1.14), (1.15) which satisfy $\rho_i(x) \geq \rho_* > \rho_m(j)$ for $i = 1, 2$. Then by subtracting the equations we obtain:

$$(a(x)e)_{xx} + j(b(x)e)_x - e = 0 \tag{3.1}$$

where $e(x) = (\rho_2 - \rho_1)(x)$ and

$$a(x) = \int_0^1 \frac{\partial F}{\partial \rho}(\rho_1(x) + v(\rho_2(x) - \rho_1(x)), j)dv \tag{3.2}$$

$$b(x) = \frac{1}{\tau(\rho_1, j)\rho_1\tau(\rho_2, j)\rho_2} \int_0^1 \frac{\partial(\rho\tau(\rho, j))}{\partial \rho}(\rho_1(x) + v(\rho_2(x) - \rho_1(x)))dv. \tag{3.3}$$

Multiplying (3.1) by ae and integration by parts give:

$$\int_0^1 |(ae)_x|^2 dx + j\int_0^1 b(x)e(ae)_x dx + \int_0^1 a|e|^2 dx = 0. \tag{3.4}$$

We estimate, using the positivity of a:

$$\begin{aligned}\left|\int_0^1 b(x)e(ae)_x dx\right| \leq & \frac{1}{2}\int_0^1 |b(x)|\sqrt{a(x)}|e(x)|^2 dx \\ & + \frac{1}{2}\int_0^1 \frac{|b(x)|}{\sqrt{a(x)}}|(ae)_x|^2 dx.\end{aligned} \tag{3.5}$$

Thus we obtain:

$$\begin{aligned}&\int_0^1 \left(1 - \frac{|j|}{2}\frac{|b(x)|}{\sqrt{a(x)}}\right)|(ae)_x|^2 dx \\ &+ \int_0^1 \left(a(x) - \frac{|j|}{2}|b(x)|\sqrt{a(x)}\right)|e|^2 dx \leq 0.\end{aligned} \tag{3.6}$$

Both terms on the left hand side are positive if

$$|j||b(x)| \leq \sqrt{a(x)}. \tag{3.7}$$

By the assumptions of Theorem 2, there exists a constant $j_0 > 0$ such that (3.7) holds for any $|j| \leq j_0$.

Remark 4. The condition (3.7) can also be regarded as a condition on τ: for a given j, uniqueness holds if τ is sufficiently large and constant. In particular, for $\tau = \infty$ (infinitely fast momentum relaxation), we have uniqueness for all j in the subsonic regime.

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