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Some new bounds on eigenvalues of the Hadamard product and the Fan product of matrices ☆

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Let A and B be nonsingular M-matrices. A lower bound on the minimum eigenvalue $q(B \circ A^{-1})$ for the Hadamard product of A^{-1} and B, and a lower bound on the minimum eigenvalue $q(A \star B)$ for the Fan product of A and B are given. In addition, an upper bound on the spectral radius $\rho(A \circ B)$ of nonnegative matrices A and B is also obtained. These bounds improve several existing results in some cases and the estimating formulas are easier to calculate for they are only depending on the entries of matrices A and B.

1. Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$ throughout. We write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ for all $i, j \in N$. We write $A \geqslant 0$ if all $a_{ij} \geqslant 0$. If $A \geqslant 0$, we say A is a nonnegative matrix, and if $A > 0$, we say

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that A is a positive matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ is the set of all eigenvalues of A. Denote by Z_n the set of $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An matrix $A = (a_{ij}) \in R^{n\times n}$ is called an L-matrix if $A = (a_{ij}) \in Z_n$, and $a_{ii} > 0$. An $n \times n$ matrix A is called an M-matrix, if there exists an $n \times n$ nonnegative matrix B and a nonnegative real number λ such that $A = \lambda I - B$ and $\lambda \geqslant \rho(B)$ (where $\rho(B)$ is the spectral radius of B); if $\lambda > \rho(B)$ we call A is a nonsingular M-matrix; if $\lambda = \rho(B)$, we call A is a singular M-matrix. Denote by M_n the set of nonsingular M-matrices (see [1]).

Let $A \in Z_n$ and denote $q(A) = \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$. If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $q(A) = (\rho(A^{-1}))^{-1}$ is a positive real eigenvalue of A (see [2]).

For two matrices $A = (a_{ij}) \in R^{n\times n}$, $B = (b_{ij}) \in R^{n\times n}$, the Hadamard product of A and B is the matrix $A \circ B = (a_{ij}b_{ij}) \in R^{n\times n}$.

The Fan product of A and B is defined by $A \star B = (c_{ij}) \in R^{n\times n}$, where

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

The Hadamard product of matrices and the Fan product of matrices arise in a wide variety of ways, such as trigonometric moments of convolutions of periodic functions, products of integral equation kernels, the weak minimum principle in partial differential equations, characteristic functions in probability theory, the study of association schemes in combinatorial theory, and so on (see [3]). Motivated by these problems, estimation of the lower bounds of $q(B \circ A^{-1})$ and the lower bounds of $q(A \star B)$ for two matrices $A, B \in M_n$ has been a focus of attention of many researchers and some important results are presented (see [2,4–7,10] and the references). In this paper, we present several new estimating formulas of the lower bounds of $q(B \circ A^{-1})$ and the lower bounds of $q(A \star B)$ for two matrices $A, B \in M_n$. These bounds improve several existing results in some cases and our estimating formulas are easier to calculate for they are only depending on the entries of matrices A and B.

This paper is organized as follows: firstly, we exhibit a lower bound of $q(B \circ A^{-1})$ for two matrices $A, B \in M_n$ in Section 2; secondly, we exhibit a lower bound of $q(A \star B)$ for two matrices $A, B \in M_n$ in Section 3; finally, for two nonnegative matrices $A, B \in R^{n\times n}$, we exhibit an upper bound of $\rho(A \circ B)$ in Section 4.

For any $j, k, l \in N = \{1, 2, \ldots, n\}$, denote

$$R_i = \sum_{k\neq i} |a_{ik}|, \qquad d_i = \frac{R_i}{|a_{ii}|}, \quad i \in N;$$

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k\neq l,i} |a_{lk}|}, \quad l \neq i; \qquad r_i = \max_{l\neq i}\{r_{li}\}, \quad i \in N;$$

$$c_{il} = \frac{|a_{il}|}{|a_{ll}| - \sum_{k\neq l,i} |a_{kl}|}, \quad l \neq i, \qquad c_i = \max_{l\neq i}\{c_{il}\}, \quad i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_k}{|a_{jj}|}, \quad j \neq i, \qquad s_i = \max_{j\neq i}\{s_{ij}\}, \quad i \in N,$$

$$m_{ji} = |a_{ji}|h_j, h_j = \begin{cases} d_j, & d_j \neq 0, \\ 1, & d_j = 0, \end{cases} \qquad m_i = \max_{j\neq i}\{m_{ji}\}; \quad i, j \in N.$$

2. A lower bound for $q(B \circ A^{-1})$

In this section, we present a new lower bound for $q(B \circ A^{-1})$. Firstly, we give some lemmas which are mainly involving about some inequalities for the entries of matrix A^{-1}. They will be useful in the following proofs.

Lemma 2.1 [5]. (a) *If $A=(a_{ij})$ is an $n\times n$ strictly diagonally dominant matrix by row, that is, $|a_{ii}|>\sum_{j\neq i}|a_{ij}|$ for any $i\in N$, then $A^{-1}=(\beta_{ij})$ exists, and*

$$|\beta_{ji}|\leqslant\frac{\sum_{k\neq j}|a_{jk}|}{|a_{jj}|}|\beta_{ii}|,\qquad \text{for all } i\neq j. \tag{2.1}$$

(b) *If $A=(a_{ij})$ is an $n\times n$ strictly diagonally dominant matrix by column, that is, $|a_{ii}|>\sum_{j\neq i}|a_{ji}|$ for any $i\in N$, then $A^{-1}=(\beta_{ij})$ exists, and*

$$|\beta_{ij}|\leqslant\frac{\sum_{k\neq j}|a_{kj}|}{|a_{jj}|}|\beta_{ii}|,\qquad \text{for all } i\neq j. \tag{2.2}$$

Lemma 2.2. (a) *Let $A=(a_{ij})\in R^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1}=(\beta_{ij})$, we have*

$$\beta_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_k}{a_{jj}}\beta_{ii},\qquad \text{for all } j\neq i. \tag{2.3}$$

(b) *Let $A=(a_{ij})\in R^{n\times n}$ be a strictly column diagonally dominant M-matrix. Then, for $A^{-1}=(\beta_{ij})$, we have*

$$\beta_{ij}\leqslant\frac{|a_{ij}|+\sum_{k\neq j,i}|a_{kj}|c_k}{a_{jj}}\beta_{ii},\qquad \text{for all } j\neq i. \tag{2.4}$$

Proof. (a) For $i\in N$, let $r_i(\varepsilon)=\max\limits_{l\neq i}\left\{\frac{|a_{li}|+\varepsilon}{a_{ll}-\sum_{k\neq l,j}|a_{lk}|}\right\}$. Since A is strictly diagonally dominant, then $\frac{|a_{li}|}{a_{ll}-\sum_{k\neq l,j}|a_{lk}|}<1$. Hence, there exists $\varepsilon>0$ such that $0<r_i(\varepsilon)<1$, for all $i\in N$. Let

$$R_i(\varepsilon)=\mathrm{diag}(r_1(\varepsilon),\ldots,r_{i-1}(\varepsilon),1,r_{i+1}(\varepsilon),\ldots,r_n(\varepsilon)).$$

For a given $i\in N$, it is easy to check that the matrix $AR_i(\varepsilon)$ is again a strictly row diagonally dominant M-matrix. By Lemma 2.1 (a) , we derive the following inequality

$$r_j^{-1}(\varepsilon)\beta_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_k(\varepsilon)}{r_j(\varepsilon)a_{jj}}\beta_{ii},\qquad j\neq i,\ j\in N.$$

i.e.,

$$\beta_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_k(\varepsilon)}{a_{jj}}\beta_{ii},\qquad j\neq i,\quad j\in N.$$

Let $\varepsilon\longrightarrow 0$ to obtain

$$\beta_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_k}{a_{jj}}\beta_{ii},\qquad j\neq i,\quad j\in N.$$

(b) For matrix $C_i(\varepsilon)A$, where $C_i(\varepsilon)=\mathrm{diag}(c_1(\varepsilon),\ldots,c_{i-1}(\varepsilon),1,c_{i+1}(\varepsilon),\ldots,c_n(\varepsilon))$, $i\in N$ and

$$c_i(\varepsilon)=\max_{l\neq i}\left\{\frac{|a_{il}|+\varepsilon}{a_{ll}-\sum_{k\neq l,i}|a_{kl}|}\right\},\qquad i\in N,$$

by Lemma 2.1 (b) and the same technique as in the above proof (a), Lemma 2.2 (b) is obtained. $\square$

Lemma 2.3. *Let $A=(a_{ij})\in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1}=(\beta_{ij})$, we have*

$$\beta_{ii}\geqslant\frac{1}{a_{ii}}. \tag{2.5}$$

Proof. Since A is an M-matrix, then $A^{-1} \geqslant 0$. From $AA^{-1} = I$, we have

$$1 = \sum_{j=1}^{n} a_{ij}\beta_{ji} = a_{ii}\beta_{ii} - \sum_{j \neq i} |a_{ij}|\beta_{ji}, \qquad \text{for all } i \in N.$$

Hence

$$a_{ii}\beta_{ii} \geqslant 1, \text{ that is, } \frac{1}{a_{ii}} \leqslant \beta_{ii}, \qquad \text{for all } i \in N. \qquad \square$$

Lemma 2.4 [7]. *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = (1, 1, \ldots, 1)^T$.*

Lemma 2.5 [6]. *If $A = (a_{ij}) \in M_n$, and $A^{-1} = (\beta_{ij})$ is a doubly stochastic matrix, then*

$$\beta_{ii} \geqslant \frac{1}{1 + \sum_{j \neq i} s_{ji}}, \qquad i \in N. \tag{2.6}$$

Lemma 2.6 [8]. *Let $A = (a_{ij})$ be an arbitrary complex matrix and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region:*

$$\bigcup \left\{ z \in C : |z - a_{ii}| \leqslant x_i \sum_{j \neq i} \frac{1}{x_j} |a_{ji}|, \ i \in N \right\}.$$

Lemma 2.7 [9]. *If $A = (a_{ij}) \in R^{n \times n}$ is an M-matrix, then there exists diagonal matrix D with positive diagonal entries, such that $D^{-1}AD$ is strictly row diagonally dominant matrix.*

Lemma 2.8 [9]. *Let $A, B \in R^{n \times n}$, and suppose $D \in R^{n \times n}$, $E \in R^{n \times n}$ are diagonal matrices. Then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

Lemma 2.9 [9]. *If $A \in M_n$, and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, $d_i > 0$ $(i = 1, 2, \ldots, n)$, then $D^{-1}AD$ is also an M-matrix.*

Now, we consider the lower bound of $q(B \circ A^{-1})$. In 1991, Horn et al. [3, p. 375] showed the classical result: If $A = (a_{ij}), B = (b_{ij}) \in M_n$, and $A^{-1} = (\beta_{ij})$, then

$$q(B \circ A^{-1}) \geqslant q(B) \min_i \beta_{ii}.$$

Subsequently, Huang in [4] improved the bounds in some cases, and obtained the following results:

$$q(B \circ A^{-1}) \geqslant \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_A)} \min_{1 \leqslant i \leqslant n} \frac{b_{ii}}{a_{ii}},$$

where $\rho(J_A), \rho(J_B)$ are the spectral radius of the Jacobi iterative matrices J_A and J_B.

The two bounds are theoretical formulas and it is difficult to calculate the lower bound of $q(B \circ A^{-1})$ by using the formulas because of the difficulty of calculating $q(B), \beta_{ii}, \rho(J_A), \rho(J_B)$. Now, we present a new estimating formula of the lower bounds of $q(B \circ A^{-1})$ which is easier to calculate.

Theorem 2.1. *Let $A = (a_{ij}), B = (b_{ij}) \in M_n$, and $A^{-1} = (\beta_{ij})$. Then*

$$q(B \circ A^{-1}) \geqslant \min_i \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\}. \tag{2.7}$$

Proof. Let A be an M-matrix. By Lemmas 2.7–2.9, there exists diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is strictly row diagonally dominant matrix. The matrix $D^{-1}AD$ is again an M-matrix and satisfies $q(B \circ A^{-1}) = q(D^{-1}(B \circ A^{-1})D) = q(B \circ (D^{-1}AD)^{-1})$.

So, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix. Therefore, $r_k < 1$ from the definition of r_k.

First, we assume that A and B are irreducible. For convenience, we denote

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_k, \qquad j \in N.$$

Then for any $j \in N$, we have

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_k \leqslant |a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k \leqslant R_j = \sum_{k \neq j} |a_{jk}| \leqslant a_{jj}. \tag{2.8}$$

Therefore, there exists a real number $\alpha_{ji} (0 \leqslant \alpha_{ji} \leqslant 1)$, such that

$$|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k = \alpha_{ji} R_j + (1 - \alpha_{ji}) R_j^r. \tag{2.9}$$

Hence

$$s_{ji} = \frac{\alpha_{ji} R_j + (1 - \alpha_{ji}) R_j^r}{a_{jj}} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{a_{jj}}. \tag{2.10}$$

Let $\alpha_j = \max\limits_{i \neq j} \{\alpha_{ji}\}$. Then $0 < \alpha_j \leqslant 1$ (if $\alpha_j = 0$, then A is reducible, which is a contradiction) and

$$s_j = \max_{i \neq j} \{s_{ji}\} = \frac{\alpha_j R_j + (1 - \alpha_j) R_j^r}{a_{jj}}, \qquad j \in N.$$

From $0 < \alpha_j \leqslant 1$, (2.8), (2.9) and (2.10), we have

$$0 < s_j \leqslant 1.$$

Now let λ be an eigenvalue of $B \circ A^{-1}$ and satisfy $q(B \circ A^{-1}) = \lambda$. Thus, by Lemma 2.6, there exists $i_0 (1 \leqslant i_0 \leqslant n)$, such that

$$|\lambda - b_{i_0 i_0} \beta_{i_0 i_0}| \leqslant s_{i_0} \sum_{j \neq i_0} \frac{1}{s_j} |b_{j i_0} \beta_{j i_0}|.$$

Then,

$$\begin{aligned}
\lambda &\geqslant b_{i_0 i_0} \beta_{i_0 i_0} - s_{i_0} \sum_{j \neq i_0} \frac{1}{s_j} |b_{j i_0} \beta_{j i_o}| \\
&\geqslant b_{i_0 i_0} \beta_{i_0 i_0} - s_{i_o} \sum_{j \neq i_0} \frac{1}{s_j} |b_{j i_0}| \frac{|a_{j i_0}| + \sum_{k \neq j, i_0} |a_{jk}| r_k}{a_{jj}} \beta_{i_0 i_0} \qquad \text{(by Lemma 2.2)} \\
&\geqslant b_{i_0 i_0} \beta_{i_0 i_0} - s_{i_0} \sum_{j \neq i_0} |b_{j i_0}| \beta_{i_0 i_0} \\
&= \left(b_{i_0 i_0} - s_{i_0} \sum_{j \neq i_0} |b_{j i_0}| \right) \beta_{i_0 i_0} \\
&\geqslant \left(b_{i_0 i_0} - s_{i_0} \sum_{j \neq i_0} |b_{j i_0}| \right) \frac{1}{a_{i_0 i_0}} \qquad \text{(by Lemma 2.3)} \\
&\geqslant \min_i \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\}.
\end{aligned}$$

Now assume that one of A and B is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $T = (t_{ij})$ the $n \times n$ permutation matrix with

$$t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1,$$

the remaining $t_{ij} = 0$, then both $A + \varepsilon T$and $B + \varepsilon T$ are irreducible nonsingular M-matrices for any chosen positive real number ε, sufficiently small such that all the leading principal minors of both $A + \varepsilon T$ and $B + \varepsilon T$ are positive. Now we substitute $A + \varepsilon T$ and $B + \varepsilon T$ for A and B, respectively in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity, that is, the result holds. $\square$

By using Lemma 2.5 and Theorem 2.1, we can get the following corollary.

Corollary 2.1. *If $A, B \in M_n$ and $A^{-1} = (\beta_{ij})$ is a doubly stochastic matrix, then*

$$q(B \circ A^{-1}) \geqslant \min_i \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{1 + \sum_{j \neq i} s_{ji}} \right\}.$$

Example 2.1. Let

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}.$$

It is easy to check that $A, B \in M_4$.

If we apply Theorem 5.7.31 of [3], we have

$$q(B \circ A^{-1}) \geqslant q(B) \min_i \beta_{ii} = 0.07.$$

If we apply Theorem 9 of [4], we have

$$q(B \circ A^{-1}) \geqslant \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_i \frac{b_{ii}}{a_{ii}} = 0.0707.$$

Now, applying Theorem 2.1, we have

$$q(B \circ A^{-1}) \geqslant \min_i \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\} = 0.08.$$

Example 2.2. Let

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -5 \\ -3 & 8 \end{pmatrix}.$$

Then $2 = q(B \circ A^{-1}) = \min_i \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\} = 2.$

It is surprise to see that our bound is the minimum eigenvalue of $B \circ A^{-1}$.

Remark 2.1. The given numerical examples show that the bound in Theorem 2.1 is better than 5.7.31 of [3] and Theorem 9 of [4] in some cases. On the other hand, it is only depending on the entries of matrices A and B. So the bound (2.7) is more easily derived than others.

3. A lower bound for $q(A \star B)$

In this section, we present a lower bound of the minimum eigenvalue $q(A \star B)$ for the Fan product of M-matrices.

In 1991, Horn et al. [3, p. 359] proved the classical result: If $A, B \in M_n$, then

$$q(A \star B) \geqslant q(A)q(B). \tag{3.1}$$

Recently, in [4, p. 1554], Huang proved

$$q(A \star B) \geqslant (1 - \rho(J_A)\rho(J_B)) \min_{1 \leqslant i \leqslant n} (a_{ii}b_{ii}), \tag{3.2}$$

where $\rho(J_A), \rho(J_B)$ are the spectral radius of the Jacobi iterative matrices J_A and J_B.

In [10, p. 13], Fang improved (3.1), and obtained

$$q(A \star B) \geqslant \min_{1 \leqslant i \leqslant n} \{a_{ii}q(B) + b_{ii}q(A) - q(A)q(B)\}. \tag{3.3}$$

These bounds are theoretical formulas and it is difficult to calculate the lower bound of $q(A \star B)$ by using the formulas because of the difficulty of calculating $q(A), q(B), \rho(J_A), \rho(J_B)$. Now, we present a new estimating formula of the lower bounds of $q(A \star B)$ which is easier to calculate.

Theorem 3.1. *Let $A, B \in M_n$. Then*

$$q(A \star B) \geqslant \min_{1 \leqslant i \leqslant n} \left\{ a_{ii}b_{ii} - m_i \sum_{j \neq i} \frac{|b_{ji}|}{h_j} \right\}. \tag{3.4}$$

Proof. It is evident that (3.4) holds with equality for $n = 1$. Therefore, we assume that $n \geqslant 2$ and divide two cases to prove.

Case 1. Let $A \star B$ be irreducible. Then A and B are irreducible. Now let λ be an eigenvalue of $A \star B$ and satisfy

$$q(A \star B) = \lambda.$$

By Lemma 2.6, there exists $i_0 (1 \leqslant i_0 \leqslant n)$, such that

$$|\lambda - a_{i_0i_0}b_{i_0i_0}| \leqslant m_{i_0} \sum_{j \neq i_0} \frac{1}{m_j} |a_{ji_0}b_{ji_0}|,$$

i.e.,

$$\begin{aligned} \lambda &\geqslant a_{i_0i_0}b_{i_0i_0} - m_{i_0} \sum_{j \neq i_0} \frac{1}{m_j} |a_{ji_0}||b_{ji_0}| \\ &\geqslant a_{i_0i_0}b_{i_0i_0} - m_{i_0} \sum_{j \neq i_0} \frac{1}{|a_{ji_0}|h_j} |a_{ji_0}||b_{ji_0}| \\ &= a_{i_0i_0}b_{i_0i_0} - m_{i_0} \sum_{j \neq i_0} \frac{|b_{ji_0}|}{h_j} \\ &\geqslant \min_i \left\{ a_{ii}b_{ii} - m_i \sum_{j \neq i} \frac{|b_{ji}|}{h_j} \right\}. \end{aligned}$$

Case 2. Let $A \star B$ be reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $T = (t_{ij})$ the $n \times n$ permutation matrix with

$$t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1,$$

the remaining $t_{ij} = 0$, then both $A - \varepsilon T$ and $B - \varepsilon T$ are irreducible nonsingular M-matrices for any chosen positive real numbers ε, sufficiently small such that all the leading principal minors of both $A - \varepsilon T$ and $B - \varepsilon T$ are positive. Now we substitute $A - \varepsilon T$ and $B - \varepsilon T$ for A and B, respectively in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. $\square$

Example 3.1. Now we again consider the numerical example in the Example 2.1.

Let

$$A=\begin{pmatrix}4&-1&-1&-1\\-2&5&-1&-1\\0&-2&4&-1\\-1&-1&-1&4\end{pmatrix},\quad B=\begin{pmatrix}1&-1/2&0&0\\-1/2&1&-1/2&0\\0&-1/2&1&-1/2\\0&0&-1/2&1\end{pmatrix}.$$

If we apply the classical result (3.1), we have

$$q(A\star B)\geqslant q(A)q(B)=0.191.$$

If we apply (3.2), that is, Theorem 9 of [4], we have

$$q(A\star B)\geqslant \min_{1\leqslant i\leqslant n}\{a_{ii}q(B)+b_{ii}q(A)-q(A)q(B)\}=1.573.$$

If we apply (3.3), that is, Theorem 4 of [10], we have

$$q(A\star B)\geqslant(1-\rho(J_A)\rho(J_B))\min_{1\leqslant i\leqslant n}(a_{ii}b_{ii})=0.1808.$$

If we apply Theorem 3.1, we have

$$q(A\star B)\geqslant\min_i\left\{a_{ii}b_{ii}-m_i\sum_{j\neq i}\frac{|b_{ji}|}{h_j}\right\}=2.4333.$$

In fact, $q(A\star B)=3.2296$.

Remark 3.1. The example shows that the bound (3.4) in Theorem 3.1 is better than (3.1), (3.2) and (3.3) in some cases. On the other hand, the bound (3.4) is only depending on the entries of matrices A and B. So, the bound is more easily derived than others.

4. An upper bound for the spectral radius of the Hadamard product of two nonnegtive matrices

In this section, we present an upper bound of $\rho(A\circ B)$ for nonnegtive matrices A,B.

In [3,4,10], the following bounds of $\rho(A\circ B)$ are given for $A,B\geqslant 0$ for nonnegtive matrices A,B, respectively.

$$\rho(A\circ B)\leqslant\rho(A)\rho(B), \tag{4.1}$$

$$\rho(A\circ B)\leqslant(1+\rho(J_A')\rho(J_B'))\max_{1\leqslant i\leqslant n}a_{ii}b_{ii}, \tag{4.2}$$

$$\rho(A\circ B)\leqslant\max_{1\leqslant i\leqslant n}\{2a_{ii}b_{ii}+\rho(A)\rho(B)-a_{ii}\rho(B)-b_{ii}\rho(A)\}. \tag{4.3}$$

These bounds are theoretical formulas and it is difficult to calculate the upper bound of $\rho(A\circ B)$ by using the formulas because of the difficulty of calculating $\rho(A),\rho(B)$ and $\rho(J_A'),\rho(J_B')$. Now, we present a new estimating formula of the upper bounds of $\rho(A\circ B)$ which is easier to calculate.

Lemma 4.1 [1]. *Let $A\in R^{n\times n}$ be a given nonnegative matrix. Then either A is irreducible or there exists a permutation P such that*

$$P^TAP=\begin{pmatrix}A_1&A_{12}&\cdots&A_{1k}\\&A_2&\cdots&A_{2k}\\&\mathbf{0}&\ddots&\vdots\\&&&A_k\end{pmatrix} \tag{4.4}$$

in which each A_i is irreducible, $i=1,\ldots,k$.

Lemma 4.2 [1]. *Eq. (4.4) is called the irreducible normal form. Note that* $\sigma(A) = \bigcup_{i=1}^{k} \sigma(A_i)$ *and* $\rho(A) = \max\{\rho(A_i) : i = 1, \ldots, k\}$.

Theorem 4.1. *Let* $A, B \in R^{n\times n}$ *be two nonnegative matrices. Then*

$$\rho(A \circ B) \leqslant \max_i \left\{ a_{ii}b_{ii} + m_i \sum_{j\neq i} \frac{|b_{ji}|}{h_j} \right\}. \tag{4.5}$$

Proof. Let $C = A \circ B$. First assume that C is irreducible, obviously A and B are irreducible.

Let λ be an eigenvalue of C and satisfy $\rho(C) = \lambda$. Thus, by Lemma 2.6, there exists $i_0 (1 \leqslant i_0 \leqslant n)$, such that

$$|\lambda - a_{i_0i_0}b_{i_0i_0}| \leqslant m_{i_0} \sum_{j\neq i_0} \frac{1}{m_j} a_{ji_0}b_{ji_0}$$

i.e.,

$$\begin{aligned}
\lambda &\leqslant a_{i_0i_0}b_{i_0i_0} + m_{i_0} \sum_{j\neq i_0} \frac{1}{m_j} a_{ji_0}b_{ji_0} \\
&\leqslant a_{i_0i_0}b_{i_0i_0} + m_{i_0} \sum_{j\neq i_0} \frac{1}{a_{ji_0}h_j} a_{ji_0}b_{ji_0} \\
&= a_{i_0i_0}b_{i_0i_0} + m_{i_0} \sum_{j\neq i_0} \frac{b_{ji_0}}{h_j} \\
&\leqslant \max_i \left\{ a_{ii}b_{ii} + m_i \sum_{j\neq i} \frac{b_{ji}}{h_j} \right\}.
\end{aligned}$$

Now, let C be reducible. We may assume that C has a block upper triangular form with irreducible diagonal blocks $C_i = A_i \circ B_i$ for $i = 1, \ldots, s$. This means that A_i and B_i are also irreducible. By Lemma 4.2, we have

$$\rho(A \circ B) = \max_i \rho(A_i \circ B_i).$$

From the equality, we easily get that the conclusion holds for $A \geqslant 0$ and $B \geqslant 0$. $\square$

Example 4.1. Let

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 5 & 1 & 1 \\ 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

If we apply (4.1), we have

$$\rho(A \circ B) \leqslant \rho(A)\rho(B) = 50.1274.$$

If we apply (4.2), we have

$$\rho(A \circ B) \leqslant (1 + \rho(J_A)\rho(J_B)) \max_{1\leqslant i\leqslant n} a_{ii}b_{ii} = 39.7468.$$

If we apply (4.3), we have

$$\max_{1\leqslant i\leqslant n} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 25.5364.$$

But, if we apply Theorem 4.1, we have

$$\max_i \left\{ a_{ii}b_{ii} + m_i \sum_{j \neq i} \frac{|b_{ji}|}{h_j} \right\} = 23.2.$$

In fact, $\rho(A \circ B) = 20.7439$.

Remark 4.1. The example shows that the bound (4.5) in Theorem 4.1 is better than (4.1), (4.2) and (4.3) in some cases. On the other hand, the bound (4.5) is only depending on the entries of matrices A and B. So, the bound is more easily derived than others.

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