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# On doubling inequalities for elliptic systems ☆

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ABSTRACT

We prove doubling inequalities for solutions of elliptic systems with an iterated Laplacian as diagonal principal part and for solutions of the Lamé system of isotropic linearized elasticity. These inequalities depend on global properties of the solutions.


## 1. Introduction

The doubling property is a basic measure theoretic concept [12,16]. Its connection with the strong unique continuation principle for elliptic partial differential equations became evident in the geometrical approach to unique continuation developed by N. Garofalo and F.-H. Lin [13,14]. Subsequently, it turned out to be an important tool for obtaining quantitative estimates suitable for stability estimates in inverse boundary value problems [2,20,21,23] and, also in connection with inverse boundary value problems, for volume bounds of unknown inclusions in terms of boundary measurements [5,7,8]. Let us illustrate the underlying idea with an example. Consider an elliptic equation

$$\operatorname{div}(\sigma\nabla u)=0, \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega$ is a bounded open set with sufficiently smooth boundary and $\sigma(x)=\{\sigma_{ij}(x)\}$ is a symmetric matrix of coefficients, satisfying a uniform ellipticity condition and such that $\sigma_{ij}\in C^{0,1}$, and consider the solution $u$ to (1.1) satisfying the Dirichlet condition

$$u=g, \quad \text{on } \partial\Omega. \tag{1.2}$$

The doubling property then says that for any compact subset $G$ of $\Omega$ and for any concentric balls $B_r, B_{2r}\subset G$ we have

$$\int_{B_{2r}} u^2 \leqslant K\int_{B_r} u^2, \tag{1.3}$$

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where $K$ is a constant which depends on $\Omega, G$, the ellipticity and regularity bounds on the coefficients, but also *necessarily*, on $u$. It is in fact evident, just by looking at homogeneous harmonic polynomials in the unit ball, that the above constant must diverge with the degree of the polynomials.

For the purposes of inverse boundary value problems, it is often important that such a constant $K$ is estimated in terms of the known boundary data $g$ and not on interior values of the solution $u$ which may be unknown. Typically one expects that $C$ can be bounded in terms of a ratio of the form

$$F(g) = \frac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}}. \tag{1.4}$$

This ratio is usually called a *frequency function*, and Garofalo and Lin [13] attributed this concept to Almgren [9]. The specific choice of the norms in the ratio may vary depending on the boundary value problem, and on the functional framework. But the general idea is that the norm on the numerator is of higher order than the one on the denominator so that $F(g)$ resembles a Rayleigh quotient.

This theory can be considered well-settled within the area of scalar elliptic equations [8]. In the case of systems, since the same issue of unique continuation maintains unanswered questions, the study of doubling inequalities is still in progress.

For the Lamé system of isotropic linearized elasticity, the strong unique continuation is known when the coefficients $\mu, \lambda \in C^{1,1}$ [3]. In fact, in [3] a doubling inequality of the following form was proved

$$\int_{B_{2r}} |u|^2 + |\operatorname{div} u|^2 \leqslant C \int_{B_r} |u|^2 + |\operatorname{div} u|^2, \tag{1.5}$$

from which the strong unique continuation can be easily derived. However, it is not clear whether, from such an inequality, one can derive a doubling inequality for $\int_{B_r} |u|^2$ only. In fact such form of the doubling inequality was claimed in [4, Theorem 3.9], see also [6], but, unfortunately, the proof given there contained a gap.

More recently, doubling inequalities have been studied for systems with diagonal principal part given by the iterated Laplacian $\Delta^l$ [17], and in fact the coefficients in the lower order terms are also allowed to be singular. See also for related previous results [18]. In these papers, local forms of doubling inequalities were obtained.

In this note, our aim is twofold. First, we show that for elliptic systems *with diagonal principal part given by* $\Delta^l$ *and bounded lower order terms*, a global form of doubling inequality holds, see Theorem 3.4. Second, we apply this result to the Lamé system, by observing that, assuming in addition $\mu, \lambda \in C^{2,1}$, such a system can be reduced to a 4th order system with $\Delta^2$ as its diagonal principal part. Thus by such means, we restore the validity of the claimed Theorem 3.9 in [4] and consequently of Proposition 4.3 in [4] and Theorem 4.8 in [5], at least under the regularity assumptions $\mu, \lambda \in C^{2,1}$.

It remains open the issue whether the doubling inequality holds under the assumption $\mu, \lambda \in C^{1,1}$, whereas it is well known that a challenging open question is whether unique continuation in general holds true when $\mu, \lambda \in C^{0,1}$. Incidentally, we recall that the situation in the 2-dimensional case is different, we refer to Lin and Wang [19] and Escauriaza [11], for the state of the art.

The plan of the paper is as follows. In Section 2 we set up our notation and formulate the pure traction boundary value problem for the Lamé system of linearized elasticity. In Section 3 we first formulate a three-spheres inequality for solutions of systems with $\Delta^l$ as diagonal principal part, Theorem 3.1, which is an immediate consequence of a result of C.-L. Lin et al. [17]. Next we apply such a three-spheres inequality to derive a so-called estimate of *propagation of smallness*, Theorem 3.2. Then, in Theorem 3.3, we recall the local version of the doubling inequality proved by C.-L. Lin et al. [17, Theorem 1.3], and we arrive at our global version, Theorem 3.4. The doubling inequality we obtain has a constant $K$ which, among other quantities, depends on a frequency function given by the ratio $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$. Depending on which is the appropriate boundary condition that may be prescribed, such ratio could be dominated by a suitable ratio of norms which only involve the boundary data. This process is exemplified in the following Theorem 3.7 where the doubling inequality for the Lamé system is obtained and the doubling constant $K$ is controlled in terms of a ratio of norms of the boundary traction field $\varphi$. The bridge between the two Theorems 3.4, 3.7 is provided by Proposition 3.5 which enables to reduce the Lamé system to a system with $\Delta^2$ as diagonal principal part.

## 2. Notation

Throughout this paper we shall consider a bounded domain $\Omega$ in $\mathbb{R}^n$, $n \geqslant 2$, having Lipschitz boundary with constants $r_0$, $M_0$ according to the following definition.

**Definition 2.1** *(Lipschitz regularity).* Given a domain $\Omega$, we shall say that $\partial\Omega$ is of *Lipschitz class with constants* $r_0$, $M_0$, if, for any $x_0 \in \partial\Omega$, there exists a rigid transformation of coordinates under which we have $x_0 = 0$ and

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \mid x_n > \psi(x')\},$$

where for $x \in \mathbb{R}^n$, we set $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and where $\psi$ is a Lipschitz continuous function on $B_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\psi(0) = 0$$

and

$$\|\psi\|_{C^{0,1}(B_{r_0}(0))} \leqslant M_0 r_0.$$

Given a bounded domain $\Omega \subset \mathbb{R}^n$, for any $d > 0$ we shall denote

$$\Omega_d = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > d\}. \tag{2.1}$$

Moreover, when no ambiguity occurs, we shall denote for brevity by $B_R$ any ball in $\mathbb{R}^n$ of radius $R$.

Let us consider weak solutions $u \in H^1(\Omega, \mathbb{R}^n)$ to the Lamé system

$$\mathrm{div}\big(\mu\big(\nabla u + (\nabla u)^T\big)\big) + \nabla(\lambda\, \mathrm{div}\, u) = 0 \quad \text{in } \Omega, \tag{2.2}$$

which describes the equilibrium of a body $\Omega$ made by linear elastic isotropic material when body forces are absent. Here, $(\nabla u)^T$ denotes the transpose of the matrix $\nabla u$. In Eq. (2.2), $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are the Lamé moduli of the material.

In this paper we shall assume $\mu \in C^{2,1}(\overline{\Omega})$, $\lambda \in C^{2,1}(\overline{\Omega})$ with

$$\|\mu\|_{C^{2,1}(\overline{\Omega})} + \|\lambda\|_{C^{2,1}(\overline{\Omega})} \leqslant M, \tag{2.3}$$

for some positive constant $M$.

We shall say that $\mu$ and $\lambda$ satisfy the *strong ellipticity* condition if

$$\mu(x) \geqslant \alpha_0 > 0, \qquad 2\mu(x) + \lambda(x) \geqslant \beta_0 > 0 \quad \text{in } \overline{\Omega}, \tag{2.4}$$

where $\alpha_0$, $\beta_0$, $\gamma_0$ are positive constants.

We shall prescribe a boundary traction field $\varphi \in L^2(\partial\Omega, \mathbb{R}^n)$ satisfying the compatibility condition

$$\int_{\partial\Omega} \varphi \cdot r = 0 \tag{2.5}$$

for every *infinitesimal rigid displacement* $r$, that is $r(x) = c + Wx$, where $c$ any constant $n$-vector and $W$ is any constant skew $n \times n$ matrix. Namely, we shall consider weak solutions $u \in H^1(\Omega, \mathbb{R}^n)$ of the following problem:

$$\mathrm{div}\big(\mu\big(\nabla u + (\nabla u)^T\big)\big) + \nabla(\lambda\, \mathrm{div}\, u) = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$\big(\mu\big(\nabla u + (\nabla u)^T\big) + \lambda(\mathrm{div}\, u) I_n\big)\nu = \varphi \quad \text{on } \partial\Omega, \tag{2.7}$$

where $I_n$ is the $n \times n$ identity matrix and $\nu$ is the unit exterior normal to $\partial\Omega$.

Regarding existence, we recall that, provided the compatibility condition (2.5) is satisfied, a solution of the traction problem (2.6), (2.7) exists as long as the Lamé moduli $\mu$ and $\lambda$ are continuous and satisfy the strong ellipticity condition, see for instance Valent [22, §III].

With respect to uniqueness, it is well known that the solution $u$ to the above problem is uniquely determined up to an infinitesimal rigid displacement. In order to uniquely identify such solution, we shall assume from now on that $u$ satisfies the following normalization conditions

$$\int_{\Omega} u = 0, \qquad \int_{\Omega} \big(\nabla u - (\nabla u)^T\big) = 0. \tag{2.8}$$

## 3. Results

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of Lipschitz class with constants $r_0$ and $M_0$. Let $u = (u^1, \ldots, u^n) \in H^{2l}(\Omega, \mathbb{R}^n)$ be a solution to the system of differential inequalities

$$\big|\Delta^l u^i\big| \leqslant K_0 \sum_{|\alpha| \leqslant [\frac{3l}{2}]} \big|D^\alpha u\big|, \quad i = 1, \ldots, n. \tag{3.1}$$

**Theorem 3.1** *(Three-spheres inequality). Let $B_R \subset \Omega$. There exists a positive number $\vartheta < e^{-1/2}$, only depending on n, l, $K_0$, such that for every $r_1, r_2, r_3$, $0 < r_1 < r_2 < \vartheta r_3$, $r_3 \leqslant R$, we have*

$$\int_{B_{r_2}} |u|^2\, dx \leqslant C\bigg(\int_{B_{r_1}} |u|^2\, dx\bigg)^{\delta}\bigg(\int_{B_{r_3}} |u|^2\, dx\bigg)^{1-\delta}, \tag{3.2}$$

*for every $u \in H^{2l}(\Omega, \mathbb{R}^n)$ satisfying (3.1), where the constants C and $\delta$, $C > 0$, $0 < \delta < 1$, only depend on n, l, $K_0$, $r_1/r_3$, $r_2/r_3$, and where the balls $B_{r_i}$, $i = 1, 2, 3$, have the same center as $B_R$.*

**Proof.** This is in fact a special case of Theorem 1.1 in [17], where, in the inequalities (3.1), suitable singularities at the center of the balls $B_{r_i}$, $B_R$ are also allowed. □

**Theorem 3.2** *(Lipschitz propagation of smallness). Under the previous assumptions, for every $\rho > 0$ and for every $x \in \Omega_{\frac{4\rho}{\vartheta}}$, we have*

$$\int_{B_\rho(x)} |u|^2\,dx \geqslant C_\rho \int_\Omega |u|^2\,dx, \tag{3.3}$$

*where $\vartheta$ has been defined in Theorem 3.1 and $C_\rho$ only depends on $n$, $l$, $K_0$, $r_0$, $M_0$, $|\Omega|$, $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$ and $\rho$.*

**Proof.** By an iterative application of the three-spheres inequality (3.2) over balls having fixed values of the ratio $r_1/r_3$, $r_2/r_3$, and by repeating the arguments in [4, Proposition 4.1] we have

$$\frac{\|u\|_{L^2(\Omega_{\frac{5\rho}{\theta}})}}{\|u\|_{L^2(\Omega)}} \leqslant \frac{C}{\rho^{n/2}}\left(\frac{\|u\|_{L^2(B_\rho(x))}}{\|u\|_{L^2(\Omega)}}\right)^{\delta^L}, \tag{3.4}$$

with $L \leqslant \frac{|\Omega|}{\omega_n \rho^n}$. Here, the constants $C > 0$ and $\delta$, $0 < \delta < 1$, only depend on $n, l, K_0$.

We can rewrite the square of the left-hand side of (3.4) as

$$\frac{\|u\|^2_{L^2(\Omega_{\frac{5\rho}{\theta}})}}{\|u\|^2_{L^2(\Omega)}} = 1 - \frac{\|u\|^2_{L^2(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})}}{\|u\|^2_{L^2(\Omega)}}. \tag{3.5}$$

By Hölder's inequality

$$\|u\|^2_{L^2(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})} \leqslant |\Omega \setminus \Omega_{\frac{5\rho}{\theta}}|^{1/n} \|u\|^2_{L^{2n/(n-1)}(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})} \tag{3.6}$$

and by Sobolev inequality (see, for instance, [1])

$$\|u\|^2_{L^{2n/(n-1)}(\Omega)} \leqslant C\|u\|^2_{H^{1/2}(\Omega)}, \tag{3.7}$$

we have

$$\|u\|^2_{L^2(\Omega\setminus\Omega_{\frac{5\rho}{\theta}})} \leqslant C|\Omega \setminus \Omega_{\frac{5\rho}{\theta}}|^{1/n} \|u\|^2_{H^{1/2}(\Omega)}, \tag{3.8}$$

where $C > 0$ only depends on $r_0$, $M_0$ and $|\Omega|$.

Moreover,

$$|\Omega \setminus \Omega_{\frac{5\rho}{\theta}}| \leqslant C\rho, \tag{3.9}$$

where $C > 0$ only depends on $r_0$, $M_0$ and $|\Omega|$ (see estimate (A.3) in [7] for details).

By (3.5), (3.8) and (3.9) we have that there exists $\bar\rho > 0$, only depending on $r_0, M_0, |\Omega|$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$ such that

$$\frac{\|u\|^2_{L^2(\Omega_{\frac{5\rho}{\theta}})}}{\|u\|^2_{L^2(\Omega)}} \geqslant \frac{1}{2}, \tag{3.10}$$

for every $\rho$, $0 < \rho \leqslant \bar\rho$.

Therefore, from (3.4) and (3.10) the thesis follows when $0 < \rho \leqslant \bar\rho$.

For larger values of $\rho$, inequality (3.3) is trivial. □

**Theorem 3.3** *(Local doubling inequality). Let $u \in H^{2l}(B_1, \mathbb{R}^n)$ be a nontrivial solution to (3.1) in $B_1 \subset \mathbb{R}^n$. There exist constants $R_0 \in (0, 1)$, $\vartheta^* \in (0, \frac{1}{2})$, $K > 0$ such that*

$$\int_{B_{2r}} |u|^2\,dx \leqslant K \int_{B_r} |u|^2\,dx, \quad \textit{for every } r,\ 0 < r \leqslant \vartheta^*. \tag{3.11}$$

*Here $R_0$ only depends on $n, l, K_0$, whereas $\vartheta^*$, $K$ only depend on $n, l, K_0$ and on the ratio*

$$F_{\mathrm{loc}} = \frac{\|u\|_{L^2(B_{R_0^2})}}{\|u\|_{L^2(B_{R_0^4})}}. \tag{3.12}$$

**Proof.** We refer to Theorem 1.3 in [17]. The present statement is merely adapted in terms of notation and of a more explicit expression of the dependencies of the various constants $R_0$, $\vartheta^*$, $K$. $\square$

**Theorem 3.4** *(Doubling inequality). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with boundary of Lipschitz class with constants $r_0$, $M_0$ and let $u \in H^{2l}(\Omega, \mathbb{R}^n)$ be a nontrivial solution to* (3.1)*. There exists a constant $\vartheta$, $0 < \vartheta < 1$, only depending on $n, l, K_0$, such that for every $\bar{r} > 0$ and for every $x_0 \in \Omega_{\bar{r}}$, we have*

$$\int_{B_{2r}(x_0)} |u|^2\,dx \leqslant K \int_{B_r(x_0)} |u|^2\,dx, \quad \textit{for every } r,\ 0 < r \leqslant \frac{\vartheta}{2}\bar{r}, \tag{3.13}$$

*where $K > 0$ only depends on $n$, $l$, $K_0$, $r_0$, $M_0$, $|\Omega|$, $\bar{r}$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$.*

**Proof.** By the unique continuation property, $u$ is a nontrivial solution to (3.1) in $B_{\bar{r}}(x_0) \subset \Omega$.

Let

$$v(y) = u(\bar{r}y + x_0).$$

Then $v \in H^{2l}(B_1, \mathbb{R}^n)$ is a nontrivial solution in $B_1$ to

$$\left|\Delta^l v^i(y)\right| \leqslant \widetilde{K_0} \sum_{|\alpha| \leqslant [\frac{3l}{2}]} \left|D^\alpha v(y)\right|, \quad i = 1, \ldots, n, \tag{3.14}$$

with $\widetilde{K_0}$ only depending on $n$, $l$, $K_0$, $\bar{r}$.

By Theorem 3.3 and coming back to the old variables, we have

$$\int_{B_{2s}(x_0)} |u|^2\,dx \leqslant K \int_{B_s(x_0)} |u|^2\,dx, \quad \text{for every } s,\ 0 < s \leqslant \vartheta^* \bar{r}, \tag{3.15}$$

with $\vartheta^* \in (0, \frac{1}{2})$, $K > 0$ only depending on $n$, $l$, $K_0$, $\bar{r}$ and, increasingly, on

$$\widetilde{F}_{\mathrm{loc}} = \frac{\|u\|_{L^2(B_{R_0^2\bar{r}}(x_0))}}{\|u\|_{L^2(B_{R_0^4\bar{r}}(x_0))}}. \tag{3.16}$$

Let $\vartheta \in (0, 1)$ be the constant introduced in Theorem 3.1. If $\vartheta^* \geqslant \frac{\vartheta}{2}$, then (3.15) holds for $s \leqslant \frac{\vartheta}{2}\bar{r}$. Otherwise, given $s \in (\vartheta^*\bar{r}, \frac{\vartheta}{2}\bar{r})$, we trivially have

$$\int_{B_{2s}(x_0)} |u|^2\,dx \leqslant \int_{B_{\vartheta\bar{r}}(x_0)} |u|^2\,dx;$$

next, by applying Theorem 3.1 with $r_1 = \vartheta^*\bar{r}$, $r_2 = \vartheta\bar{r}$, $r_3 = \bar{r}$, we have

$$\int_{B_{\vartheta\bar{r}}(x_0)} |u|^2\,dx \leqslant C \left(\int_{B_{\vartheta^*\bar{r}}(x_0)} |u|^2\,dx\right)^{\delta} \left(\int_{B_{\bar{r}}(x_0)} |u|^2\,dx\right)^{1-\delta} \tag{3.17}$$

with $\delta \in (0, 1)$, $C > 0$ only depending on $n$, $l$, $K_0$, $\frac{r_1}{r_3} = \vartheta^*$, $\frac{r_2}{r_3} = \vartheta$. Since $\vartheta$ only depends on $n$, $l$, $K_0$, we have that $C$ only depends on $n$, $l$, $K_0$, $\vartheta^*$. We obtain

$$\frac{\int_{B_{2s}(x_0)} |u|^2\,dx}{\int_{B_{\vartheta^*\bar{r}}(x_0)} |u|^2\,dx} \leqslant C \left(\frac{\int_{B_{\bar{r}}(x_0)} |u|^2\,dx}{\int_{B_{\vartheta^*\bar{r}}(x_0)} |u|^2\,dx}\right)^{1-\delta} \leqslant C \frac{\int_{B_{\bar{r}}(x_0)} |u|^2\,dx}{\int_{B_{\vartheta^*\bar{r}}(x_0)} |u|^2\,dx}, \tag{3.18}$$

and therefore, recalling that $\vartheta^*\bar{r} < s$, we have

$$\int_{B_{2s}(x_0)} |u|^2\,dx \leqslant C \frac{\|u\|^2_{L^2(B_{\bar{r}}(x_0))}}{\|u\|^2_{L^2(B_{\vartheta^*\bar{r}}(x_0))}} \int_{B_s(x_0)} |u|^2\,dx, \tag{3.19}$$

for every $s$, $\vartheta^*\bar{r} < s < \frac{\vartheta}{2}\bar{r}$.

Let us estimate $\widetilde{F}_{\mathrm{loc}}$. Let $\rho = \min\{R_0^4\bar{r}, \frac{\vartheta}{4}\bar{r}\}$. By applying Theorem 3.2, we have

$$\widetilde{F}_{\mathrm{loc}} \leqslant \frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^2(B_\rho(x_0))}} \leqslant \frac{1}{\sqrt{C_\rho}}, \tag{3.20}$$

with $C_\rho$ only depending on $n$, $l$, $K_0$, $r_0$, $M_0$, $|\Omega|$, $\bar{r}$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$.

Therefore $\vartheta^*$ and the constants $K$, $C$ appearing in (3.15), (3.19), respectively, only depend on the above constants. We can now estimate $\|u\|_{L^2(B_{\bar r}(x_0))}/\|u\|_{L^2(B_{\vartheta^*\bar r}(x_0))}$ with the same quantities by applying analogously Theorem 3.2. The thesis follows from (3.15) and (3.19). $\square$

**Proposition 3.5.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. Let the Lamé moduli $\mu, \lambda \in C^{2,1}(\overline{\Omega})$ satisfy the strong ellipticity conditions*

$$\mu(x) \geqslant \alpha_0 > 0, \qquad 2\mu(x) + \lambda(x) \geqslant \beta_0 > 0 \quad \text{for every } x \in \overline{\Omega} \tag{3.21}$$

*and the upper bound*

$$\|\mu\|_{C^{2,1}(\overline{\Omega})} + \|\lambda\|_{C^{2,1}(\overline{\Omega})} \leqslant M, \tag{3.22}$$

*where $\alpha_0$, $\beta_0$, $M$ are given positive constants. Then, there exists a positive constant $K_0$ only depending on $n, \alpha_0, \beta_0, M$ such that, for every solution $u \in H^4_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of the Lamé system*

$$\operatorname{div}\big(\mu\big(\nabla u + (\nabla u)^T\big)\big) + \nabla(\lambda \operatorname{div} u) = 0, \quad \text{in } \Omega, \tag{3.23}$$

*we also have*

$$\big|\Delta^2 u^i\big| \leqslant K_0 \sum_{|\alpha|=1}^{3} \big|D^\alpha u\big|, \quad i = 1, \ldots, n. \tag{3.24}$$

**Remark 3.6.** Let us notice that, being $\mu, \lambda \in C^{2,1}(\overline{\Omega})$, by interior regularity estimates for the Lamé system, we have that for any weak solution $u$ to (3.23) we also have $u \in H^4_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. See, for instance, [10]. Consequently (3.24) is indeed valid for any weak solution to the Lamé system.

**Proof.** In what follows we denote

$$\Pi = \begin{pmatrix} \mu \\ \lambda \end{pmatrix}, \tag{3.25}$$

and, for any function $v$ we denote by $D^k v$ the set of all derivatives of order $k$ of $v$. Moreover, we shall denote by $B_j(X; Y)$, $j = 1, 2, \ldots$ bilinear (vector valued) functions of the vectors (or tensors) $X$ and $Y$, their explicit expression shall vary from line to line.

We can rewrite (3.23) as follows

$$\mu \Delta u^j + (\mu + \lambda)(\operatorname{div} u)_{x_j} = B_j(D\Pi; Du), \quad j = 1, \ldots, n. \tag{3.26}$$

Differentiating by $x_j$ and summing up, we have

$$(2\mu + \lambda)\Delta(\operatorname{div} u) = B_1\big(D\Pi; D^2 u\big) + B_2\big(D^2\Pi; Du\big). \tag{3.27}$$

Differentiating once more into Eq. (3.27), we obtain, in the almost everywhere sense,

$$(2\mu + \lambda)\nabla\big(\Delta(\operatorname{div} u)\big) = B_1\big(D\Pi; D^3 u\big) + B_2\big(D^2\Pi; D^2 u\big) + B_3\big(D^3\Pi; Du\big). \tag{3.28}$$

By applying the Laplacian to (3.26) we also have

$$\mu \Delta^2 u + (\mu + \lambda)\nabla\big(\Delta(\operatorname{div} u)\big) = B_1\big(D\Pi; D^3 u\big) + B_2\big(D^2\Pi; D^2 u\big) + B_3\big(D^3\Pi; Du\big), \tag{3.29}$$

in the almost everywhere sense.

With the aid of the strong ellipticity conditions (3.21) we can eliminate the term $\nabla(\Delta(\operatorname{div} u))$ from Eqs. (3.28) and (3.29). Recalling the bounds (3.22) we arrive at (3.24). $\square$

**Theorem 3.7** *(Global doubling inequality). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with boundary of Lipschitz class with constants $r_0$, $M_0$. Let $u \in H^1(\Omega, \mathbb{R}^n)$ be a weak solution to the boundary value problem (2.6), (2.7) satisfying the normalization conditions (2.8). Let $\mu, \lambda \in C^{2,1}(\overline{\Omega})$ satisfy the regularity condition (3.22) and the strong ellipticity condition (2.4).*

*There exists a constant $\vartheta$, $0 < \vartheta < 1$, only depending on $n, \alpha_0, \gamma_0, M$, such that for every $\bar r > 0$ and for every $x_0 \in \Omega_{\bar r}$, we have*

$$\int_{B_{2r}} |u|^2\,dx \leqslant K \int_{B_r} |u|^2\,dx, \quad \text{for every } r,\ 0 < r \leqslant \frac{\vartheta}{2}\bar r, \tag{3.30}$$

*where the constant $K > 0$ only depends on $n$, $\alpha_0$, $\gamma_0$, $M$, $r_0$, $M_0$, $|\Omega|$, $\bar r$ and $\|\varphi\|_{H^{-1/2}(\partial\Omega)}/\|\varphi\|_{H^{-1}(\partial\Omega)}$.*

**Proof.** By applying Theorem 3.4 and Proposition 3.5 we infer that (3.30) holds with the constant $K$ only depending on $n, \alpha_0, \beta_0, M, |\Omega|, r_0, M_0$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$. By the weak formulation of the problem (2.6), (2.7) and by the normalization conditions (2.8) we have [22, §III]

$$\|u\|_{H^1(\Omega)} \leqslant C\|\varphi\|_{H^{-1/2}(\partial\Omega)}, \tag{3.31}$$

where $C > 0$ only depends on $n, r_0, M_0, |\Omega|, \alpha_0, \beta_0$. Moreover the following interpolation inequality holds

$$\|u\|^2_{H^{1/2}(\Omega)} \leqslant \|u\|_{H^1(\Omega)}\|u\|_{L^2(\Omega)}. \tag{3.32}$$

Let us now recall the trace inequality (see, for instance, [15, Theorem 1.5.1.10])

$$\|u\|^2_{L^2(\partial\Omega)} \leqslant C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}, \tag{3.33}$$

where $C$ only depends on $r_0$, $M_0$, $|\Omega|$, and the estimate of Lemma 4.10 in [5]

$$\|\varphi\|_{H^{-1}(\partial\Omega)} \leqslant C\|u\|_{L^2(\partial\Omega)}, \tag{3.34}$$

where $C > 0$ only depends on $|\Omega|, r_0, M_0, \alpha_0, \beta_0$ and $M$.

Therefore

$$\frac{\|u\|^2_{H^{1/2}(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \leqslant \frac{\|u\|_{H^1(\Omega)}}{\|u\|_{L^2(\Omega)}} \leqslant C\frac{\|u\|^2_{H^1(\Omega)}}{\|u\|^2_{L^2(\partial\Omega)}} \leqslant C\frac{\|\varphi\|^2_{H^{-1/2}(\partial\Omega)}}{\|\varphi\|^2_{H^{-1}(\partial\Omega)}}, \tag{3.35}$$

where $C > 0$ only depends on $n, r_0, M_0, |\Omega|, \alpha_0, \beta_0, M$. □

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