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Group classification of systems of non-linear reaction–diffusion equations with general diffusion matrix. II. Generalized Turing systems

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Abstract

Group classification of systems of two coupled non-linear reaction–diffusion equation with a diagonal diffusion matrix is carried out. Symmetries of diffusion systems with singular diffusion matrix and additional first order derivative terms are described.

Keywords: Reaction–diffusion systems; Quasilinear parabolic systems; Turing systems; Group classification; Symmetries

1. Introduction

Coupled systems of non-linear reaction–diffusion equations form the basis of many models of mathematical biology. These systems are widely used in mathematical physics, chemistry and also in social sciences and many other fields. Such reach spectrum of applications stimulates numerous thorough investigations of fundamentals of these equations theory.

In the present paper we continue group classification of systems of reaction–diffusion equations with general diffusion matrix

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$$u_t - \Delta_m\big(A^{11}u + A^{12}v\big) = f^1(u,v),$$
$$v_t - \Delta_m\big(A^{21}u + A^{22}v\big) = f^2(u,v), \tag{1}$$

where u and v are function of $t, x_1, x_2, \ldots, x_m$, A^{11}, A^{12}, A^{21} and A^{22} are real constants and Δ_m is the Laplace operator in R^m.

Up to linear transformations of functions u, v and f^1, f^2 it is sufficient to restrict ourselves to such diffusion matrices (i.e., matrices whose elements are $A^{11}, \ldots, A^{22}$) which are diagonal, triangular, or are sums of the unit and antisymmetric matrices. In the last case (1) can be reduced to a single equation for a complex function (generalized complex Ginzburg–Landau (CGL) equation) whose group classification was carried out in paper [1].

In the present paper we classify Eqs. (1) with a diagonal diffusion matrix. Without loss of generality such equations can be written as

$$u_t - \Delta_m u = f^1(u,v),$$
$$v_t - a\Delta_m v = f^2(u,v), \tag{2}$$

where a is a constant. The related diffusion matrix is $A = \left(\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right)$.

Just equations of type (2) are the most popular models of reaction–diffusion systems first introduced by Turing in 1952 [2]. It is practically impossible to enumerate all fields of applications of such equations. We restrict ourselves to few examples only.

- The complex Ginzburg–Landau (CGL) equation

$$W_t - (1+i\beta)\Delta_2 W = W - (1+i\alpha)|W|^2 W \tag{3}$$

  can be presented as a system (1) where u and v are the real and the imaginary part of W. In particular case $\beta = 0$ this system takes the form (2).
- The primitive predator-prey system which can be defined by [12]

$$u_t - Du_{xx} = -uv, \qquad v_t - \lambda D v_{xx} = uv \tag{4}$$

  also appears as an particular subject of our analysis.
- The λ–ω reaction–diffusion system [13]

$$u_t = \Delta_2 u + \lambda(R)u - \omega(R)v, \qquad v_t = \Delta_2 v + \omega(R)u + \lambda(R)v, \tag{5}$$

  where $R^2 = u^2 + v^2$, is widely used in studies of reaction–diffusion models, in particular, to describe spiral waves phenomena [14].
  Symmetries of Eqs. (5) were studied in paper [15]. We shall add the results [15] in the following.
- The Jackiw–Teitelboim model of two dimension gravity with the non-relativistic gauge [10] appears as a particular ((1 + 1)-dimensional) case of the following system:

$$u_t - \Delta_m u = 2ku - 2u^2 v = 0, \qquad v_t + \Delta_m v = 2uv^2 - 2kv = 0. \tag{6}$$

  Symmetries of Eqs. (6) for $m = 1$ were investigated in paper [11]. In the following we complete the results obtained in [11].

Apparently the first attempt of group classification of Eqs. (2) was made by Danilov [4]. But the results presented in [4] are rather incomplete.

Group classification of Eqs. (2) with general non-degenerated diffusion matrix was announced in [5] and presented in [6]. However, the equivalence relations where not used systematically there to simplify the equations which resulted in rather cumbersome form of the classification results. Moreover due to typographical errors the tables with classification results present in [6] are poorly readable (see [1] for additional comments).

Group classification of systems of heat equations

$$\begin{aligned} &u_t - \big(k_1(u)u_x\big)_x = Q_1(u,v), \\ &v_t - \big(k_2(v)v_x\big)_x = Q_2(u,v) \end{aligned} \tag{7}$$

has been performed in paper [7]. However the analysis presented in [7] was restricted to the cases $k_1 \neq$ and (or) $k_2 \neq$ const and so does not include equations of Turing type (2).

Symmetries of systems of reaction–diffusion equations with a diagonal diffusion matrix (i.e., of systems (2)) where studied in papers [8,9]. We will show in the following that the classification results obtained in [8,9] are incomplete and include many equivalent cases treated as non-equivalent ones.

The problem of group classification of Eqs. (2) is still relevant and we will present its solution here. In addition, we classify Eqs. (2) with non-invertible diffusion matrix (i.e., Eqs. (2) when parameter a is equal to zero) and also the following equations with first order derivative terms:

$$\begin{aligned} &u_t - \Delta u = f^1(u,v), \\ &v_t - p_\mu u_{x_\mu} = f^2(u,v), \end{aligned} \tag{8}$$

where $u_{x_\mu} = \frac{\partial u}{\partial x_\mu}$, p_μ are arbitrary constants and summation from 1 to m is imposed over the repeated index μ. Moreover, without loss of generality one can set

$$p_1 = p_2 = \cdots = p_{m-1} = 0, \quad p_m = p. \tag{9}$$

In the case $p \equiv 0$ Eq. (8) reduces to (2) with $a = 0$. We notice that Eq. (2) is used in such popular models of mathematical biology as the Fitzhung–Naguno [16] and Rinzel–Keller [17] ones.

2. Equivalence transformations

The problem of group classification of Eqs. (2), (8) will be solved up to equivalence transformations. Clear definition of these transformations is one of the main points of any classification procedure.

We say the equations

$$\begin{aligned} &\tilde{u}_t - \Delta_m \tilde{u} = \tilde{f}^1(\tilde{u},\tilde{v}), \\ &\tilde{v}_t - a\Delta_m \tilde{v} = \tilde{f}^2(\tilde{u},\tilde{v}) \end{aligned} \tag{10}$$

are equivalent to (2) if there exist invertible transformations $u \to \tilde{u} = G(t,x,u,v)$, $v \to \tilde{v} = \Phi(t,x,u,v)$, $t \to \tilde{t} = T(t,x,u,v)$, $x \to \tilde{x} = X(t,x,u,v)$ and $f^\alpha \to \tilde{f}^\alpha = F^\alpha(u,t,x,f^1,f^2)$ which connects (2) with (10). In other words the equivalence transformations should keep the general form of Eq. (2) but can change the concrete realization of non-linear terms f^1 and f^2.

The group of equivalence transformations for Eq. (2) can be found using the classical Lie approach and treating f^1 and f^2 as supplemental dependent variables. In addition to the obvious symmetry transformations

$$t \to t' = t + a, \qquad x_\mu \to x'_\mu = R_{\mu\nu} x_\nu + b_\mu, \tag{11}$$

where a, b_μ and $R_{\mu\nu}$ are arbitrary parameters satisfying $R_{\mu\nu} R_{\mu\lambda} = \delta_{\mu\lambda}$, this group includes the following transformations

$$\begin{aligned} &u_b \to K^{bc} u_c + b_b, \qquad f^b \to \lambda^2 K^{bc} f^c, \\ &t \to \lambda^{-2} t, \qquad x_b \to \lambda^{-1} x_b \end{aligned} \tag{12}$$

and

$$\begin{aligned} &u_b \to \widetilde{K}^{bc} u_c, \qquad f^b \to a^{-1} \widetilde{K}^{bc} f^c, \\ &t \to a^{-1} t, \qquad x_b \to x_b, \quad a \neq 0, \end{aligned} \tag{13}$$

where indices b, c take values 1 and 2, K^{bc} and $\widetilde{K}^{bc}$ are elements of invertible constant matrices K and $\widetilde{K}$, respectively, moreover, K commutes with A and $\widetilde{K} A(a) \widetilde{K}^{-1} = a A(\frac{1}{a})$; $\lambda \neq 0$ and b_a are arbitrary constants, and we use the temporary notations $u = u_1$, $v = u_2$.

If parameter a is equal to 1 then K is an arbitrary 2×2 invertible matrix, and equivalence transformations (13) are trivial. If $a \neq 1$ then K and $\widetilde{K}$ are arbitrary non-degenerated diagonal and anti-diagonal matrices, respectively.

Transformations (13) reduce to the change $a \to 1/a$ in the related matrix A, i.e., to scaling the free parameter a. Thus without loss of generality we can restrict ourselves to the following values of a:

$$1. \quad a = 0, \qquad 2. \quad -1 \leqslant a < 0,\ 0 < a < 1, \qquad 3. \quad a = 1. \tag{14}$$

It is possible to show that for $a \neq 0$ there are no more extended equivalence relations valid for arbitrary non-linearities f^1 and f^2. If $a = 0$ there exist powerful equivalence relations $u \to u$, $v \to \varphi(v)$ with an arbitrary function $\varphi(v)$. However for some particular functions f^1 and f^2 the invariance group can be more extended. In addition to transformations (12) and (13) it includes symmetry transformations which do not change the form of Eq. (2). Moreover, for some classes of functions f^1, f^2, Eq. (2) admits additional equivalence transformations (AET) which belong neither to symmetry transformations nor to transformations of kind (12), (13).

In spite of the fact that we search for AET *after* description of symmetries of Eqs. (2) and specification of functions f^1, f^2, for convenience we present the list of the additional equivalence transformations in the following formulae:

$$\begin{aligned}
&1. \quad u \to e^{\rho t} u, \qquad v \to e^{\rho t} v, \\
&2. \quad u \to u + \omega t, \qquad v \to v, \\
&3. \quad u \to u, \qquad v \to v + \rho t, \\
&4. \quad u \to u + \mu\rho t, \qquad v \to e^{-\rho t} v, \\
&5. \quad u \to e^{\rho t} u, \qquad v \to v - \kappa\rho t, \\
&6. \quad u \to u, \qquad v \to v + \rho t u, \\
&7. \quad u \to e^{2\omega t} u, \qquad v \to v + \omega t^2, \\
&8. \quad u \to u + \omega t^2, \qquad v \to v e^{2\omega t}, \\
&9. \quad u \to u, \qquad v \to v - 2\rho t u + \rho\delta t^2, \\
&10. \quad u \to e^{2\rho t} u, \qquad v \to e^{2\rho t}\big(v + \omega t u + \rho t^2 u\big),
\end{aligned}$$

$$
\begin{aligned}
&11.\quad u\to u+\eta\rho t,\qquad v\to v-\rho t,\\
&12.\quad u\to e^{\kappa t}u,\qquad v\to e^{\kappa t}(v-\nu\kappa t u),\\
&13.\quad u\to u+2\rho t,\qquad v\to v+2\rho t u+2\rho^2t^2,\\
&14.\quad u\to e^{\omega t}u,\qquad v\to e^{\rho t}v,\\
&15.\quad u\to e^{\nu\omega t}\big(u\cos(\omega\sigma t)+v\sin(\omega\sigma t)\big),\\
&\qquad\ v\to e^{\nu\omega t}\big(v\cos(\omega\sigma t)-u\sin(\omega\sigma t)\big),\\
&16.\quad u\to e^{2\omega t}\big(u\cos\big(\sigma\omega t^2\big)-v\sin\big(\sigma\omega t^2\big)\big),\\
&\qquad\ v\to e^{2\omega t}\big(v\cos\big(\sigma\omega t^2\big)+u\sin\big(\sigma\omega t^2\big)\big),\\
&17.\quad u\to e^{\lambda\omega t^2}\big(u\cos(2\omega t)+v\sin(2\omega t)\big),\\
&\qquad\ v\to e^{\lambda\omega t^2}\big(v\cos(2\omega t)-u\sin(2\omega t)\big),\\
&18.\quad u\to e^{\nu\omega t}u,\qquad v\to e^{\nu\omega t}(v-\sigma\omega t u),\\
&19.\quad u\to e^{\lambda\omega t^2}u,\qquad v\to e^{\lambda\omega t^2}(v+2\omega t u),\\
&20.\quad u\to e^{2\omega t}u,\qquad v\to e^{\varepsilon\omega t^2}v,\\
&21.\quad u\to u+3\omega t,\qquad v\to v+3\omega t^2u+3\omega^2t^3+\rho t+3\omega\rho t^2,\\
&22.\quad u\to u+\rho x_m,\qquad v\to v.
\end{aligned}
\tag{15}
$$

Here the Greek letters denote parameters whose values are either arbitrary or specified in the tables presented below. Equivalence transformations (15) are valid only for particular non-linearities which will be specified in the following.

3. Symmetries and classifying equations

We search for symmetries of Eqs. (2) and (8) with respect to continuous groups of transformations using the infinitesimal approach. Applying the Lie algorithm or its specific formulation proposed in [6] one can find the determining equations for coordinates $\eta, \xi^a, \pi^1, \pi^2$ of generator X of the symmetry group:

$$X=\eta\partial_t+\xi^a\partial_{x_a}-\pi^1\partial_u-\pi^2\partial_v \tag{16}$$

and classifying equations for non-linearities f^1 and f^2. We will not reproduce the related routine calculations but present the general form of symmetry X for Eq. (2) with $a\neq 0, 1$ [1]:

$$
\begin{aligned}
X={}&\lambda K+\sigma^\mu G^\mu+\omega^\mu\widehat{G}^\mu+\mu D-C^1u\partial_u-C^2v\partial_v-B^1\partial_u-B^2\partial_v+\Psi^{\mu\nu}x_\mu\partial_{x_\nu}\\
&+\nu\partial_t+\rho^\mu\partial_{x_\mu},
\end{aligned}
\tag{17}
$$

where the Greek letters denote arbitrary constants, B^1, B^2 and C^1, C^2 are functions of t, x and t, respectively, and

$$
\begin{aligned}
&K=2t(t\partial_t+x_\mu\partial_{x_\mu})-\frac{x_2}{2}\left(u\partial_u+\frac{1}{a}v\partial_v\right)-tm(u\partial_u+v\partial_v),\\
&D=t\partial_t+\frac{1}{2}x_\mu\partial_{x_\mu},\qquad G_\mu=t\partial_{x_\mu}-\frac{1}{2}x_\mu\left(u\partial_u+\frac{1}{a}v\partial_v\right),\\
&\widehat{G}_\mu=e^{\gamma t}\left(\partial_{x_\mu}-\frac{1}{2}\gamma x_\mu\left(u\partial_u+\frac{1}{a}v\partial_v\right)\right).
\end{aligned}
\tag{18}
$$

If $a=0$ then the related generator X again has the form (17) where however $\lambda=\sigma^\mu=\omega^\mu=C^2=0$ and B^2 is a function of t, x and u.

For $a=1$ the symmetry group generator (which we denote by $\widetilde{X}$) is more extended and has the following form:

$$\widetilde{X}=X+C^3u\partial_v+C^4v\partial_u, \tag{19}$$

where X is given in (17) and C^3, C^4 are functions of t.

Equation (2) admits symmetry (19) iff the following classifying equations for f^1 and f^2 are satisfied [1]:

$$\begin{aligned}
&\left(\lambda(m+4)t+\mu+\left(\frac{1}{2}\lambda x^2+\sigma^\mu x_\mu+\gamma e^{\gamma t}\omega^\mu x_\mu\right)+C^1\right)f^1+C^4f^2+C^1_tu\\
&\qquad +C^4_tv+B^1_t-\Delta_m B^1\\
&\quad=\bigg(B^1\partial_u+B^2\partial_v+C^1u\partial_u+C^2v\partial_v+C^3u\partial_v+C^4v\partial_u+\lambda mt(u\partial_u+v\partial_v)\\
&\qquad +\left(\frac{1}{2}\lambda x^2+\sigma^\mu x_\mu+\gamma e^{\gamma t}\omega^\mu x_\mu\right)\left(u\partial_u+\frac{1}{a}v\partial_v\right)\bigg)f^1,\\
&\left(\lambda(m+4)t+\mu+\frac{1}{a}\left(\frac{1}{2}\lambda x^2+\sigma^\mu x_\mu+\gamma e^{\gamma t}\omega^\mu x_\mu\right)+C^2\right)f^2+C^3f^1+C^2_tv\\
&\qquad +C^3_tu+B^2_t-a\Delta_m B^2\\
&\quad=\bigg(B^1\partial_u+B^2\partial_v+C^1u\partial_u+C^2v\partial_v+C^3u\partial_v+C^4v\partial_u+\lambda mt(u\partial_u+v\partial_v)\\
&\qquad +\left(\frac{1}{2}\lambda x^2+\sigma^\mu x_\mu+\gamma e^{\gamma t}\omega^\mu x_\mu\right)\left(u\partial_u+\frac{1}{a}v\partial_v\right)\bigg)f^2.
\end{aligned} \tag{20}$$

In other words, to make group classification of systems (2) means to find all non-equivalent solutions of Eqs. (20) and to specify the related symmetries (17) [6]. We note that Eqs. (20) can be decoupled equating terms multiplied by the same variables x_μ or their powers.

Consider now Eq. (8) and the related symmetry operator (16). The determining equations for η, ξ^μ and π^a are easily obtained using the standard Lie algorithm:

$$\begin{aligned}
&\eta_{tt}=\eta_{x_\mu}=\eta_u=\eta_v=0, \qquad \xi^\mu_t=\xi^\mu_u=\xi^\mu_v=0,\\
&\pi^1_{uu}=p\pi^2_{vv}=0, \qquad \pi^a_{x_\mu u}+\pi^a_{x_\mu v}=0, \qquad \pi^1_v=\pi^2_u=0, \qquad p\left(\pi^1_u-\pi^2_v-\frac{1}{2}\eta_t\right)=0,\\
&\xi^\mu_{x_\nu}+\xi^\nu_{x_\mu}=-\delta^{\mu\nu}\eta_t, \quad \mu\neq m,
\end{aligned} \tag{21}$$

where subscripts denote derivatives w.r.t. the corresponding independent variable, i.e., $\eta_t=\frac{\partial\eta}{\partial t}$, $\xi^\mu_{x_\nu}=\frac{\partial\xi^\mu}{\partial x_\nu}$, etc.

Integrating system (21) we obtain the general form of operator X:

$$X=\nu\partial_t+\rho_\nu\partial_{x_\nu}+\Psi^{\sigma\nu}\partial_\nu x_\sigma+\mu D-B^1\partial_u-B^2\partial_v-Fu\partial_u-Gv\partial_v; \tag{22}$$

$$\mu=2(F-G) \quad \text{if } p\neq 0, \tag{23}$$

where B^1, B^2 are functions of (t,x), F and G are functions of t and summation over the indices σ, ν is assumed with $\sigma,\nu=1,2,\ldots,n-1$.

The classifying equations for f^1 and f^2 reduce to the following system:

$$(\mu+F)f^1+F_tu+(\partial_t-\Delta)B^1=\big(B^1\partial_u+B^2\partial_v+Fu\partial_u+Gv\partial_v\big)f^1, \tag{24}$$
$$(\mu+G)f^2+G_tv+B_t^2-pB^1_{x_m}=\big(B^1\partial_u+B^2\partial_v+Fu\partial_u+Gv\partial_v\big)f^2. \tag{25}$$

Solving (24), (25) we shall specify both the coefficients of infinitesimal operator (22) and the related non-linearities f^1 and f^2.

It is obvious that the widest spectrum of symmetries corresponds to the case when the parameter a is equal to 1 since the corresponding generator $\widetilde{X}$ (19) includes two additional terms $C^3u\partial_v$ and $C^4v\partial_u$. Quite the contrary, Eqs. (2) with $a\neq 1$ and especially (8) admit relatively small variety of symmetries.

4. Classification of symmetries

Following [1] we specify basic, main and extended symmetries for the analyzed systems of reaction–diffusion equations.

Basic symmetries are nothing but generators of transformations (11) forming the kernel of a symmetry group, i.e.,

$$P_0=\partial_t,\qquad P_\mu=\partial_{x_\mu},\qquad J^{\mu\nu}=x_\mu\partial_{x_\nu}-x_\nu\partial_{x_\mu}. \tag{26}$$

Main symmetries form an important subclass of general symmetries (17) and have the following form

$$\widetilde{X}=-\mu D+C^1u\partial_u+C^2v\partial_v+C^3u\partial_v+C^4v\partial_u+B^1\partial_u+B^2\partial_v \tag{27}$$

(if $a\neq 1$ then $C^3=C^4=0$).

In accordance with the analysis present in [1] the complete description of general symmetries (17) can be obtained using the following steps:

- Find all main symmetries (27), i.e., solve Eqs. (20) for $\Psi^{\mu\nu}=\nu=\rho^\nu=\sigma^\nu=\omega^\nu=0$:

$$\begin{aligned}&\big(\mu+C^1\big)f^1+C^4f^2+C^1_tu+C^4_tv+B^1_t-\Delta_mB^1\\&\quad=\big(C^1u\partial_u+C^2v\partial_v+C^3u\partial_v+C^4v\partial_u+B^1\partial_u+B^2\partial_v\big)f^1,\\&\big(\mu+C^2\big)f^2+C^3f^1+C^2_tv+C^3_tu+B^2_t-a\Delta_mB^2\\&\quad=\big(C^1u\partial_u+C^2v\partial_v+C^3u\partial_v+C^4v\partial_u+B^1\partial_u+B^2\partial_v\big)f^2.\end{aligned} \tag{28}$$

- Specify all cases when the main symmetries can be extended, i.e., at least one of the following systems is satisfied:

$$af^1=(au\partial_u+v\partial_v)f^1,\qquad f^2=(au\partial_u+v\partial_v)f^2; \tag{29}$$
$$a\big(f^1+\gamma u\big)=(au\partial_u+v\partial_v)f^1,\qquad f^2+\gamma v=(au\partial_u+v\partial_v)f^2 \tag{30}$$

  or if Eq. (29) is satisfied together with the following conditions:

$$\begin{aligned}&(m+4)f^c+\mu^{c1}f^1+\mu^{c2}f^2\\&\quad=\big(\big(\mu^{11}u+\mu^{12}v+mu\big)\partial_u+\big(\mu^{21}u+\mu^{22}v+mv\big)\partial_v\big)f^c,\\&\nu^{c1}f^1+\nu^{c2}f^2+\mu^{c1}u+\mu^{c2}v=\big(\big(\nu^{11}u+\nu^{12}v\big)\partial_u+\big(\nu^{21}u+\nu^{22}v\big)\partial_v\big)f^c,\end{aligned} \tag{31}$$

  where $c=1,2$, μ^{cb} and ν^{cb} are constants satisfying $(a-1)\mu^{cb}=(a-1)\nu^{cb}=0$.

If relations (29), (30) or (31) are valid then the system (2) admits symmetry G^α, $\widehat{G}^\alpha$ or the conformal symmetry $K - (t\mu^{11} + \nu^{11})u\partial_u - (t\mu^{12} + \nu^{12})v\partial_u - (t\mu^{21} + \nu^{21})u\partial_v - (t\mu^{22} + \nu^{22})v\partial_v$ correspondingly.
- When classifying Eqs. (8) for $p \neq 0$ the second step is not needed since in accordance with (22) these equations admit only basic and main symmetries.

In the following sections we find main and extended symmetries for the classified equations. For clarity we start with group classification of systems (8) with $p \neq 0$ which is more simple technically and presents rather detailed calculations. Then we consider Eqs. (2) and present classification results without technical details.

5. Algebras of main symmetries for Eq. (8)

To describe main symmetries we use the trick discussed in [1], i.e., make a priori classification of low dimension algebras of these symmetries. In accordance with (22) any symmetry generator extending algebra (26) has the following form

$$X = \mu D - B^1\partial_u - B^2\partial_v - Fu\partial_u + \left(\frac{\mu}{2} - F\right)v\partial_v. \tag{32}$$

Let X^1 and X^2 be operators of the form (32) then the commutator $[X^1, X^2]$ is also a symmetry whose general form is given by (32). Thus operators (32) form a Lie algebra which we denote as $\mathcal{A}$.

Let us specify algebras $\mathcal{A}$ which can appear in our classification procedure. First consider one-dimensional $\mathcal{A}$, i.e., suppose that Eq. (8) admits the only symmetry of the form (32). Then any commutator of operator (26) with (32) should be equal to a linear combination of operators (26) and (32). Using this condition we come to the following possibilities only:

$$\begin{aligned}
&X = X^1 = \mu D - \alpha^1\partial_u - \alpha^2\partial_v - \beta u\partial_u - \left(\beta - \frac{\mu}{2}\right)v\partial_v,\\
&X = X^2 = e^{\nu t}\big(\alpha^1\partial_u + \alpha^2\partial_v + \beta u\partial_u + \beta v\partial_v\big),\\
&X = X^3 = e^{\nu t + \rho\cdot x}\big(\alpha^1\partial_u + \alpha^2\partial_v\big),
\end{aligned}\tag{33}$$

where the Greek letters again denote arbitrary parameters and $\rho \cdot x = \rho^\mu x_\mu$.

The next step is to specify all non-equivalent sets of arbitrary constants in (33) using the equivalence transformations (12).

If the coefficient for $u\partial_u$ (or $v\partial_v$) is non-zero then translating u (or v) we reduce to zero the related coefficient α^1 (α^2) in X^1 and X^2; then scaling u (v) we can reduce to ± 1 all non-zero α^a in (33). In addition, all operators (33) are defined up to constant multipliers. Using these simple arguments we come to the following non-equivalent versions of operators (33):

$$\begin{aligned}
&X_1^{(1)} = 2\mu D - u\partial_u + (\mu - 1)v\partial_v,\\
&X_1^{(2)} = 2D + v\partial_v + \nu\partial_u, \qquad X_1^{(3)} = 2D - u\partial_u - \partial_v,\\
&X_2^{(\nu)} = e^{\nu t + \rho\cdot x}(u\partial_u + v\partial_v),\\
&X_3^{(1)} = e^{\sigma^1 t + \rho^1\cdot x}(\partial_u + \partial_v), \qquad X_3^{(2)} = e^{\sigma^2 t + \rho^2\cdot x}\partial_u, \qquad X_3^{(3)} = e^{\sigma^3 t + \rho^3\cdot x}\partial_v.
\end{aligned}\tag{34}$$

To describe *two-dimensional* algebras $\mathcal{A}$ we represent one of the related basis element X in the general form (32) and calculate the commutators

$$Y = \left[P^0, X\right] - 2\mu P^0, \qquad Z = \left[P^0, Y\right], \qquad W = [X, Y],$$

where P^0 is operator given in (26). After simple calculations we obtain

$$\begin{aligned} &Y = F_t(u\partial_u + v\partial_v) + B_t^1\partial_u + B_t^2\partial_v, \qquad Z = F_{tt}(u\partial_u + v\partial_v) + B_{tt}^1\partial_u + B_{tt}^2\partial_v, \\ &W = 2\mu t Z + \mu x_b\left(B_{tx_b}^1\partial_u + B_{tx_b}^2\partial_v\right). \end{aligned} \tag{35}$$

By definition, Y, Z and W belong to $\mathcal{A}$. Let $F_t \neq 0$ then we obtain from (35):

$$\mu \neq 0: \quad B_{tt}^a = F_{tt} = B_{tb}^a = 0, \tag{36}$$

$$\mu = 0: \quad F_{tt} = \alpha F_t + \gamma^a B_t^a, \qquad B_{tt}^a = \gamma^a F_t + \beta^{ab} B_t^b. \tag{37}$$

Starting with (36) we conclude that up to translations of t the coefficients F and B^a have the following form

$$F = \sigma t \quad \text{or} \quad F = \beta; \qquad B^a = \nu^a t + \alpha^a \quad \text{if } \mu \neq 0.$$

If $F = \sigma t$ then the change

$$u_a \to u_a e^{-\sigma t} - \frac{\nu^a}{\mu} t \tag{38}$$

reduces the related operator (22) to X^1 of (33) for $\beta = 0$.

The choice $F = \beta$ corresponds to the following operator (32)

$$X = X^4 = X^1 - 2t\left(\alpha^1\partial_u + \alpha^2\partial_v\right), \tag{39}$$

where X^1 is given in (33).

Thus if one of basis elements of two dimension algebra $\mathcal{A}$ is of general form (32) with $\mu \neq 0$ then it can be reduced to X^1 with $\beta = 0$ or to generator (39). We denote such basis element as e^1. Without loss of generality the second basis element e^2 of $\mathcal{A}$ is a linear combination of operators $X_2^{(\nu)}$ and $X_3^{(a)}$ (34). Going over possible pairs (e^1, e^2) and requiring $[e^1, e^2] = \alpha^1 e^1 + \alpha^2 e^2$ we come to the following two-dimensional algebras:

$$\begin{aligned} &A_1 = \left\langle 2D + v\partial_v, X_2^{(0)}\right\rangle, \qquad A_2 = \left\langle X_1^{(2)}, X_3^{(3)}\right\rangle, \\ &A_3 = \left\langle X_1^{(3)}, X_{(3)}^3\right\rangle, \qquad A_4 = \left\langle X_1^{(1)}, X_3^{(3)}\right\rangle, \\ &A_5 = \left\langle X_{(1)}^1, X_3^{(3)}\right\rangle, \qquad A_6 = \left\langle 2D + 2v\partial_v + u\partial_u + \nu t\partial_v, X_3^{(2)}\right\rangle, \\ &A_7 = \left\langle 2D + 2u\partial_u + 3v\partial_v + 3\nu t\partial_u, X_{(1)}^3\right\rangle. \end{aligned} \tag{40}$$

The form of basis elements in (40) is defined up to transformations (12), (38).

If $\mathcal{A}$ does not include operators (32) with non-trivial parameters μ then in accordance with (38) its elements are of the following form

$$e_a = F^{(a)}(u\partial_u + v\partial_v) + B_{(a)}^1\partial_u + B_{(a)}^2\partial_v, \quad a = 1, 2, \tag{41}$$

where $F^{(\alpha)}$ and $B_{(a)}^1$, $B_{(a)}^2$ are solutions of (37).

Formulae (40), (41) define all non-equivalent two-dimensional algebras $\mathcal{A}$ which have to be considered as possible symmetries of Eqs. (8). We will see that asking for invariance of (8) w.r.t. these algebras the related arbitrary functions f^a are defined up to arbitrary constants, and it is impossible to make further specification of these functions by extending algebra $\mathcal{A}$.

6. Group classification of Eqs. (8)

We suppose that parameter p in (8) be non-zero. Then, scaling independent variables t, x we can reduce it to $p = 1$.

To classify equations (8) which admit one- and two-dimensional extensions of the basis invariance algebra (26) it is sufficient to solve determining equations (24) for f^a with *known* coefficient functions B^a and F of symmetries (32). These functions are easily found comparing (22) with (34), (40) and (41).

Let us present an example of such calculation which corresponds to algebra A_1 whose basis elements are $X^1 = 2t\partial_t + x_a\partial_{x_a} + v\partial_v$ and $X_2^{(0)} = u\partial_u + v\partial_v$, refer to (40). Operators X^1 and $X_2^{(0)}$ generate the following Eqs. (24), (25):

$$f^1 = -uf_u^1, \qquad f^2 = -\frac{1}{2}uf_u^2 \tag{42}$$

and

$$f^a = (u\partial_u + v\partial_v)f^a, \quad a = 1, 2. \tag{43}$$

General solution of (43) is: $f^1 = uF^1(\frac{v}{u})$, $f^2 = uF^2(\frac{v}{u})$ where F^1 and F^2 are arbitrary functions of $\frac{v}{u}$. Solving (42) for such functions f^1 and f^2 we obtain

$$f^1 = \alpha u^3 v^{-2}, \qquad f^2 = \lambda u^2 v^{-1}. \tag{44}$$

Thus Eq. (8) admits symmetries $X_0^{(2)}$ and X^1 provided f^1 and f^2 are functions given in (44). These symmetries are defined up to arbitrary constants α and λ. If one of these constants is non-zero, than it can be reduced to $+1$ or -1 by scaling independent variables.

In analogous way we solve Eq. (24) corresponding to other symmetries presented in (34) and (40). At that we do not consider functions f^1 and f^2 which are either linear in u, v or correspond to decoupled systems (8) (i.e., when f^1 and f^2 depend only on u and v correspondingly). The classification results are presented in Table 1.

In the fourth column of the table symmetries of the related equations (8), (9) are presented together with the additional equivalence transformations (AET) which are listed in formula (15); the numbers of AET from the list (15) are given in square brackets. Greek letters denote arbitrary real parameters which in particular can be equal to zero. Moreover, without loss of generality we restrict ourselves to $\eta = 0, 1$, $\delta = 0, \pm 1$, $\varepsilon = \pm 1$.

In Table 1 D is the dilatation operator given in (18), $\tilde{x} = (x_1, x_2, \ldots, x_{m-1})$, $\Psi(x)$ is an arbitrary function of spatial variables; $\Psi_\mu(\tilde{x}, x_m - \eta t)$ and $\Phi_\mu(t, \tilde{x})$ are solutions of the Laplace and linear heat equations:

$$\Delta_m \Psi_\mu = \mu \Psi_\mu, \qquad \left(\frac{\partial}{\partial t} - \Delta_{m-1}\right)\Phi_\mu = \mu \Phi_\mu.$$

7. Group classification of Eqs. (2)

In this section we present the classification results for coupled systems of equations (2). The related classifying equations are given by relations (20).

Like in Section 5 we first describe all non-equivalent low dimension algebras of the main symmetries for Eqs. (2). Non-equivalent realizations of these algebras (together with detailed

Table 1
Non-linearities and symmetries for Eqs. (8), (9) with $p=1$

No	Non-linearities	Arguments of F^1, F^2	Symmetries and AET Eq. (15)
1	$f^1=u^{2\nu+1}F^1$, $f^2=u^{\nu+1}F^2$	$vu^{\nu-1}$	$2\nu D-u\partial_u+(\nu-1)v\partial_v$ [AET 1 if $\nu=0$]
2	$f^1=F^1v^{-2}$, $f^2=F^2v^{-1}$	$u-\eta\ln v$	$2D+v\partial_v+\eta\partial_u$
3	$f^1=u(F^1+\varepsilon\ln u)$, $f^2=v(F^2+\varepsilon\ln u)$	$\frac{v}{u}$	$e^{\varepsilon t}(u\partial_u+v\partial_v)$
4	$f^1=u^3F^1$, $f^2=u^2F^2$	$v-\ln u$	$2D-u\partial_u-\partial_v$
5	$f^1=F^1+\mu u$, $f^2=F^2+\eta u$	v	$e^{-\eta x_m}\Phi_\mu(t,\tilde{x})\partial_u$ [AET 2, 22 if $\eta=\mu=0$]
6	$f^1=F^1+\eta(\mu-\nu)v$, $f^2=F^2+\nu v$	$u+\eta v$	$e^{\nu t}\Psi_\mu(\tilde{x},x_m-\eta t)(\partial_v-\eta\partial_u)$ [AET 11 if $\nu=\mu=0$]
7	$f^1=\nu u^3v^{-2}$, $f^2=\mu u^2v^{-1}$		$2D+v\partial_v,\ u\partial_u+v\partial_v$ [AET 1]
8	$f^1=\nu e^{-2u}$, $f^2=\eta e^{-u}$		$2D+v\partial_v+\partial_u,\ \Psi(x)\partial_v$ [AET 3]
9	$f^1=\eta e^{3v}$, $f^2=\delta e^{2v}$		$2D-u\partial_u-\partial_v,\ \Phi_0(t,\tilde{x})\partial_u$ [AET 2, 22]
10	$f^1=\mu u^{2\nu+1}$, $f^2=\eta u^{\nu+1}$		$2\nu D-u\partial_u+(\nu-1)v\partial_v,\ \Psi(x)\partial_v$ [AET 3]
11	$f^1=\eta v^{3\nu-2}$, $f^2=\mu v^{2\nu-1}$		$2(\nu-1)D-\nu u\partial_u-v\partial_v,\ \Phi_0(t,\tilde{x})\partial_u$ [AET 2, 22]
12	$f^1=\frac{v}{u},\ f^2=\ln u$		$2D+2v\partial_v+u\partial_u+t\partial_v,\ \Psi(x)\partial_v$ [AET 3, and 7 if $\nu=0$]
13	$f^1=\ln v$, $f^2=\nu v^{\frac{1}{3}}$		$2D+2u\partial_u+3v\partial_v+3t\partial_u,\ \Phi_0(t,\tilde{x})\partial_u$ [AET 2, 22, and 8 if $\nu=0$]

calculations) are present in Appendix A. Using found realizations of algebras $\mathcal{A}$ and solving the related classifying equations (28) we easily complete the group classification of Eqs. (2).

We will not reproduce here the related routine calculations but present the results of group classification in Tables 2–10. Besides symmetries and the related non-linearities, the additional equivalence transformations which are admissible by particular classes of Eqs. (2) are indicated there. The symbols D, G^μ, $\widehat{G}^\nu$ and K denote generators listed in (18), ψ_μ, $\tilde{\psi}_\mu$ and $\Psi_\mu=\Psi_\mu(x)$ are arbitrary solutions of the linear heat equations and Laplace equation:

$$\partial_t\psi_\mu-\Delta_m\psi_\mu=\mu\psi_\mu,\qquad \partial_t\tilde{\psi}_\mu-a\Delta_m\tilde{\psi}_\mu=\mu\tilde{\psi}_\mu,\qquad \Delta_m\Psi_\mu=\mu\Psi_\mu,$$

and $\Psi(x)$ is an arbitrary function of x. The Greek letters denote arbitrary parameters. Moreover, up to equivalence transformations we restrict ourselves to $\varepsilon=\pm1$, $\eta=0,1$ and $\delta=0,\pm1$.

In Table 3 Δ denotes the characteristic determinant, $\Delta=\frac{1}{4}(\mu-\nu)^2+\lambda\sigma$. Additional equivalence transformations are specified in the third column.

Classification results present in Tables 4 and 5 are related to systems (2) with arbitrary values of a presented by Eq. (14) (if not specified in the second columns of the tables).

Table 2
Non-linearities with arbitrary functions and symmetries for Eqs. (2) with arbitrary $a \neq 0$

No	Non-linear terms	Arguments of F^1, F^2	Symmetries	AET Eq. (15)
1	$f^1 = u^{\nu+1}F^1$, $f^2 = u^{\nu-\mu}F^2$	vu^{μ}	$\nu D - u\partial_u + \mu v\partial_v$ for any μ, ν, and G_α for $\nu = 0$, $a\mu = 1$	14, $\rho = \mu\omega$ if $\nu = 0$
2	$f^1 = u(F^1 + \varepsilon \ln u)$, $f^2 = v(F^2 + \varepsilon \ln v)$	vu^{μ}	$e^{\varepsilon t}(u\partial_u - \mu v\partial_v)$ for any μ, and $\widehat{G}_\alpha$ if $a\mu = -1$	
3	$f^1 = v^{\nu}F^1$, $f^2 = v^{\nu+1}F^2$	$u - \ln v$	$\nu D - v\partial_v - \partial_u$	4, $\mu = -1$ if $\nu = 0$
4	$f^1 = F^1 + \varepsilon u$, $f^2 = F^2 v + \varepsilon uv$	$u - \ln v$	$e^{\varepsilon t}(v\partial_v + \partial_u)$	
5	$f^1 = 0$, $f^2 = F^2$, $a \neq 1$	u	$D + v\partial_v$, $\tilde{\psi}_0\partial_v$	3
6	$f^1 = F^1$, $f^2 = F^2 + \delta v$	u	$\tilde{\psi}_\delta\partial_v$	3 if $\delta = 0$; 6 if $a = 1$, $F^1 = \delta u$
7	$f^1 = F^1 + \delta u$, $f^2 = F^2 + \sigma v$, $a \neq 1$	$v - u$	$e^{\kappa t}\Psi_\mu(x)(\partial_u + \partial_v)$, $\mu = \frac{\sigma-\delta}{(1-a)}$, $\kappa = \sigma + a\mu$	
8	$f^1 = e^u F^1$, $f^2 = e^u F^2$, $\eta = 0$ if $a = 1$	$v - \eta u$	$D - \partial_u - \eta\partial_v$	

Table 3
Symmetries of Eqs. (2) with arbitrary $a \neq 0$ and non-linearities $f^1 = u(\mu \ln u + \lambda \ln v)$, $f^2 = v(\nu \ln v + \sigma \ln u)$

No	Conditions	Symmetries and AET Eq. (15)	Additional symmetries
1	$\lambda = 0$, $\sigma = \varepsilon$, $\mu = \nu$	$e^{\mu t}v\partial_v$, $e^{\mu t}(u\partial_u + \varepsilon t v\partial_v)$ [AET 20 if $\mu = 0$]	none
2	$\lambda = 0$, $\sigma = \varepsilon$ $\mu \neq \nu$, $(a-1)^2 + \nu^2 \neq 0$	$e^{\mu t}((\mu - \nu)u\partial_u + \sigma v\partial_v)$, $e^{\nu t}v\partial_v$ [AET 14, $\omega = -\varepsilon\nu\rho$ if $\mu\nu = 0$]	G_α if $\nu = -a\sigma$, $\mu = 0$; $\widehat{G}_\alpha$ if $\mu \neq 0$, $\mu - \nu = a\sigma$
3	$\Delta = 0$, $\lambda\sigma \neq 0$, $\lambda^2 + \sigma^2 = 1$, $\mu + \nu = 2\Omega$	$X_2 = e^{\Omega t}(2\lambda u\partial_u + (\nu - \mu)v\partial_v)$, $2e^{\Omega t}v\partial_v + tX_2$ [AET 14, $\sigma\omega = -\nu\rho$ if $\mu + \nu = 0$]	G_α if $\mu = -\nu$, $\lambda = a\nu$; $\widehat{G}_\alpha$ if $\nu \neq -\mu$, $2\lambda = a(\nu - \mu)$
4	$\lambda\sigma \neq 0$, $\Delta = 1$, $\omega_\pm = \Omega \pm 1$	$X_\pm = e^{\omega_\pm t}(\lambda u\partial_u + (\omega_\pm - \mu)v\partial_v)$ [AET 14, $\sigma\omega = -\nu\rho$ if $\mu\nu = \lambda\sigma$]	G_α if $\nu\mu = \lambda\sigma$, $\lambda = -a\mu$; $\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma$, $\lambda = a(\nu - \mu + a\sigma)$
5	$\Delta = -1$	$e^{\Omega t}(2\lambda \cos tu\partial_u + ((\nu - \mu)\cos t - 2\sin t)v\partial_v)$, $e^{\Omega t}(2\lambda \sin tu\partial_u + ((\nu - \mu)\sin t + 2\cos t)v\partial_v)$	none

Table 4
Non-linearities with arbitrary parameters and extendible symmetries for Eqs. (2) with any a

No	Non-linear terms	Main symmetries	Additional symmetries	AET Eq. (15)
1	$f^1=\varepsilon u^{\nu+1}v^{\mu}$, $f^2=\sigma u^{\nu}v^{\mu+1}$, $a\neq 0$	$\mu D-v\partial_v$, $\nu D-u\partial_u$	G_α if $a\nu=-\mu$, and K if $\nu m(1-a)=4$	14, $\nu\omega+\mu\rho=0$
			$\psi_0\partial_u$ if $\sigma=0$, $\nu=-1$, and G_α if $\mu=a$, and K if $a=1+\frac{m}{4}$	2, 14, $\omega=\mu\rho$
2	$f^1=\varepsilon u^{\nu+1}$, $f^2=u^{\nu+\mu}$, $\nu^2+(a-1)^2\neq 0$	$\nu D-u\partial_u-\mu v\partial_v$, $\tilde{\psi}_0\partial_v$	G_α if $\nu=0$, $a\mu=1$	3, and 14, $\rho=\mu\omega$ if $\nu=0$
3	$f^1=\delta$, $f^2=\ln u$, $a\neq 1$	$D+u\partial_u+v\partial_v+t\partial_v$, $\tilde{\psi}_0\partial_v$	$u\partial_u+t\partial_v$ if $\delta=0$	3, 7, 9
4	$f^1=\delta u\ln u$, $f^2=\nu v+\ln u$, $a\neq 1$, $\nu^2+\delta^2\neq 0$	$\tilde{\psi}_\nu\partial_v$	$e^{\nu t}(u\partial_u+t\partial_v)$ if $\nu=\delta$	
			$e^{\delta t}((\delta-\nu)u\partial_u+\partial_v)$ if $\nu\neq\delta$	5, $\kappa=\frac{1}{\nu}$ if $\delta=0$
5	$f^1=\delta e^{\nu u}$, $f^2=e^{(\nu+1)u}$	$\nu D-v\partial_v-\partial_u$, $\tilde{\psi}_0\partial_v$	$(u-\delta t)\partial_v$ if $\nu=0$, $a=1$	3, and 9, 4, $\mu=-1$ if $\nu=0$

The non-linearities given in Table 4 are defined up to arbitrary parameters. For some values of these parameters the related equation (2) admits extended symmetries indicated in column 4 of the table.

In Table 5 non-linearities for Eq. (2) are classified whose symmetries are fixed for all admitted values of parameters.

In Tables 6–9 and 10 the additional symmetries are presented which correspond to the specific values $a=1$ and $a=0$ of the diffusion coefficient. We use the following notations here: $R=\sqrt{u^2+v^2}$, $z=\tan^{-1}\frac{v}{u}$.

In Table 6 the AET are given in square brackets and placed in the last column.

In the following Tables 7 and 8 $\Delta=\frac{1}{4}(\mu-\nu)^2+\lambda\sigma$. Symmetries and additional equivalence transformations are specified in the third column; AET are given in square brackets. In the last columns additional symmetries are specified which are valid for some particular values of parameters defining non-linearities.

Symmetries presented in Table 9 are valid for Eqs. (1) with the unit diffusion matrix only.

We did not consider decoupled systems (2) whose symmetries can be easily found using the classification results of Dorodnitsyn [18] for a single diffusion equation. We also did not specify the case of linear systems (2) when

$$f^1=\nu u+\mu v+\alpha,\qquad f^2=\sigma u+\lambda v+\omega. \tag{45}$$

Equivalence transformations (12) and 1–3 of (15) make it possible to specify values of parameters in (45) by imposing the following conditions:

$$\alpha=\omega=\lambda=0;\qquad \mu\sigma=0\quad\text{or } \mu=\pm\sigma. \tag{46}$$

Table 5
Non-linearities with arbitrary parameters and non-extendible symmetries for Eqs. (2) with arbitrary a

No	Non-linear terms	Symmetries	AET Eq. (15)
1	$f^1=\delta(u+v)^{\nu+1}$, $f^1=\mu(u+v)^{\nu+1}$, $a\neq 1$	$\nu D-u\partial_u-v\partial_v$, $\Psi_0(x)(\partial_u-\partial_v)$	$11, \eta=1$
2	$f^1=e^v$, $f^2=\varepsilon e^v$, $a\neq 0$	$D-\partial_v$, $\psi_0\partial_u$	2
3	$f^1=\delta e^{u+v}$, $f^2=\sigma e^{u+v}$, $a\neq 1$	$D-\partial_v$, $\Psi_0(x)(\partial_u-\partial_v)$	$11, \eta=1$
4	$f^1=\varepsilon v^{\mu}e^u$, $f^2=\sigma v^{\mu+1}e^u$, $a\neq 0, \sigma^2+\mu^2\neq 0$	$D-\partial_u$, $v\partial_v-\mu\partial_u$	4 if $\sigma=0$
5	$f^1=\varepsilon e^u$, $f^2=u$	$D+v\partial_v-\partial_u-t\partial_v$, $\tilde{\psi}_0\partial_v$	3
6	$f^1=\varepsilon\ln(u+v)$, $f^2=\nu\ln(u+v)$, $a\neq 1$	$\Psi_0(x)(\partial_u-\partial_v)$, $\varepsilon(a-1)(D+u\partial_u+v\partial_v)$ $+((a+\varepsilon\nu)t+\frac{1+\varepsilon\nu}{2m}x^2)(\partial_u-\partial_v)$	$11, \eta=1$
7	$f^1=\varepsilon u^{\nu+1}$, $f^2=\ln u$, $\nu\neq -1$	$\nu(D+v\partial_v)-u\partial_u-t\partial_v$, $\tilde{\psi}_0\partial_v$	3, and 7 if $\nu=0$
8	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=-\nu^2\ln u+(\mu+\nu)v$	$X_3=e^{\mu t}(u\partial_u+\nu\partial_v)$, $tX_3+e^{\mu t}\partial_v$	$5, \kappa=-\nu$ if $\mu=0$
9	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=(1-\nu^2)\ln u+(\mu+\nu)v$	$X_4^{\pm}=e^{(\mu\pm 1)t}(u\partial_u+(\nu\pm 1)\partial_v)$	$5, \kappa=\mu-\nu$ if $\mu=\pm 1$
10	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=(\mu+\nu)v-(1+\nu^2)\ln u$	$e^{\mu t}(\cos t(u\partial_u+\nu\partial_v)-\sin t\partial_v)$, $e^{\mu t}(\sin t(u\partial_u+\nu\partial_v)+\cos t\partial_v)$	

Moreover, if the diffusion matrix A is proportional to the unit matrix then Eqs. (2), (46) can be reduced to the case $f^1=f^2=0$.

The classification results present in the tables are valid also for Eqs. (2) whose r.h.s. have the form (45), (46). However, to save a room we did not indicate the standard additional symmetries of linear equations, i.e., $U\partial_u$ and $V\partial_v$ where U and V satisfy the relations

$$U_t-\Delta U=\nu U, \qquad V_t-a\Delta V=\lambda V.$$

The following last table completes the classification results for the case of singular diffusion matrix.

In Table 10 ϕ is an arbitrary function of v. In addition to the equivalence transformations indicated in the fourth column, all the corresponding equations (2) admit the AET $u\to u$, $v\to\varphi(v)$ where φ is an arbitrary function of v.

8. Discussion

We have carried out the group classification of systems of coupled reaction–diffusion equations (2) with a diagonal diffusion matrix. The classification results are present in Tables 2–10. Moreover, symmetries of Eq. (8) with a singular diffusion matrix and additional first derivative terms are presented in Table 1.

The list of non-equivalent systems (2) appears to be rather extended, especially for the unit diffusion matrix. Equations (2) with invertible and non-unit diffusion matrix A have an essentially

Table 6
Additional non-linearities with arbitrary functions and symmetries for Eqs. (2) with $a=1$

No	Non-linear terms	Arguments of F^1, F^2, F	Symmetries and AET Eq. (15)
1	$f^1=uF^1+\delta\eta v$, $f^2=\delta\frac{v}{u}(u+\eta v)+uF^2+vF^1$, $\delta^2+\eta^2\neq 0$	$ue^{-\eta\frac{v}{u}}$	$e^{\delta t}(u\partial_v+\eta(u\partial_u+v\partial_v))$ and $\tilde{\psi}_\alpha\partial_v$ if $F^1=\alpha-\delta\neq 0$, $\eta=0$ [AET 3 if $F^1=-\delta$, $\eta=0$]
2	$f^1=u^{\nu+1}F^1$, $f^2=u^\nu(F^1v+F^2u)$	$ue^{-\eta\frac{v}{u}}$	$\eta(\nu D-u\partial_u-v\partial_v)-u\partial_v$ [AET 6 if $\eta=0$]
3	$f^1=uF^1+vF^2+\varepsilon z(\mu u-v)$, $f^2=vF^1-uF^2+\varepsilon z(\mu v+u)$	$Re^{-\mu z}$	$e^{\varepsilon t}(\mu R\partial_R+\partial_z)$
4	$f^1=e^{\eta u}F^1$, $f^2=e^{\eta u}(F^2+F^1u)$	$2v-u^2$	$\eta D-u\partial_v-\partial_u$ [AET 13 if $\eta=0$]
5	$f^1=\varepsilon u+F^1$, $f^2=\varepsilon u^2+F^1u+F^2$	$2v-u^2$	$e^{\varepsilon t}(u\partial_v+\partial_u)$
6	$f^1=Fu$, $f^2=Fv$	u	$v\partial_v$, $u\partial_v$ [AET 6, and 14, $\omega=0$]
7	$f^1=\eta$, $f^2=\delta v+F$	u	$\tilde{\psi}_\delta\partial_v$, $e^{\delta t}(u-\eta t)\partial_v$ and $D+v\partial_v$ if $\eta=\delta=0$ [AET 3 if $\delta=0$ and 6 if $\eta=0$]
8	$f^1=e^{\lambda z}(F^1v+F^2u)$, $f^2=e^{\lambda z}(F^2v-F^1u)$	$Re^{\nu z}$	$\lambda D+\nu(u\partial_u+v\partial_v)-u\partial_v+v\partial_u$ [AET 15, $\sigma=1$ if $\lambda=0$]
9	$f^1=e^{\frac{v}{u}}F_1u$, $f^2=e^{\frac{v}{u}}(F_1v+F_2)$	u	$D-u\partial_v$
10	$f^1=u^2$, $f^2=(u+\delta)v+F$	u	$e^{\delta t}u\partial_v$, $e^{\delta t}(\partial_v+tu\partial_v)$
11	$f^1=(u^2-1)$, $f^2=(u+\nu)v+F$	u	$e^{(\nu+1)t}(u\partial_v+\partial_v)$, $e^{(\nu-1)t}(u\partial_v-\partial_v)$
12	$f^1=(u^2+1)$, $f^2=(u+\nu)v+F$	u	$e^{\nu t}(\cos tu\partial_v-\sin t\partial_v)$, $e^{\nu t}(\sin tu\partial_v+\cos t\partial_v)$

shorter list of different symmetries. If the diffusion matrix is singular the number of inequivalent equations appears to be the smallest one which is caused by the powerful equivalence relations $u\to u$, $v\to\phi(v)$ where ϕ is an arbitrary function of v.

More exactly, if matrix A is of type 1, Eq. (14), then there exist 9 non-equivalent classes of Eqs. (2) defined up to arbitrary functions and 19 classes of such equations defined up to parameters. The related non-linearities and symmetries are presented in Tables 4, 5 and 10. The presented extensions of the basic symmetries (26) have dimensions from 1 up to 3 and include neither Galilei generators G_α nor conformal generators K.

In addition, in Table 1 thirteen classes of equations with a singular diffusion matrix and first derivative terms are presented.

For the case when matrix A is of type 2, Eq. (14), we indicate in Tables 2–5 ten classes of equations defined up to arbitrary functions and thirty five classes of equations defined up to arbitrary or fixed parameters. Among them there are 7 Galilei invariant systems, whose r.h.s. terms are given in Table 2, Item 2; Table 3, Items 2, 3, 4 and Table 4, Items 1, 2. In addition, there exist two systems of type (2) with a diagonal (but not unit) diffusion matrix, which are invariant

Table 7
Symmetries of Eqs. (2) with $a=1$ and non-linearities $f^1=(\mu u-\sigma v)\ln R+z(\lambda u-\nu v)$, $f^2=(\mu v+\sigma u)\ln R+z(\lambda v+\nu u)$

No	Conditions for coefficients	Symmetries and AET Eq. (15)	Additional symmetries
1	$\lambda=0$, $\mu=\nu=\delta$	$e^{\delta t}\partial_z$, $e^{\delta t}(R\partial_R+\sigma t\partial_z)$ [AET 16 if $\mu=0$]	$\widehat{G}_\alpha$ if $\sigma=0$, $\mu\neq 0$
2	$\lambda=0$, $\mu\neq\nu$, $\mu^2+\nu^2=1$	$e^{\nu t}\partial_z$, $e^{\mu t}(\sigma\partial_z+(\mu-\nu)R\partial_R)$ [AET 15 if $\mu\nu=0$]	G_α if $\mu=\sigma=0$, $\widehat{G}_\alpha$ if $\mu\neq 0$, $\sigma=0$
3	$\Delta=0$, $\lambda=\varepsilon$, $\mu+\nu=2\Omega$	$X_5=e^{\Omega t}(2\varepsilon R\partial_R+(\nu-\mu)\partial_z)$, $2e^{\Omega t}\partial_z+tX_5$ [AET 15 if $\nu+\mu=0$, and 1, 17 if $\mu=\nu=0$]	G_α if $\mu=\nu=0$, $\widehat{G}_\alpha$ if $\mu=\nu\neq 0$
4	$\lambda\neq 0$, $\Delta=1$ $\omega_\pm=\Omega\pm 1$	$e^{\omega_+ t}(\lambda R\partial_R+(\omega_+-\mu)\partial_z)$, $e^{\omega_- t}(\lambda R\partial_R+(\omega_--\mu)\partial_z)$ [AET 15 if $\mu\nu=\lambda\sigma$, and 1 if $\mu=\sigma=0$]	G_α if $\sigma=\mu=0$, $\widehat{G}_\alpha$ if $\sigma=0$, $\mu\neq 0$
5	$\Delta=-1$	$\exp(\Omega t)[2\lambda\cos tR\partial_R+((\nu-\mu)\cos t-2\sin t)\partial_z]$, $\exp(\Omega t)[2\lambda\sin tR\partial_R+((\nu-\mu)\sin t+2\cos t)\partial_z]$	none

Table 8
Symmetries of Eqs. (2) with $a=1$ and non-linearities $f^1=\lambda v+\mu u\ln u$, $f^2=\lambda\frac{v^2}{u}+(\sigma u+\mu v)\ln u+\nu v$, $\lambda^2+\sigma^2\neq 0$

No	Conditions for coefficients	Symmetries and AET Eq. (15)	Additional symmetries
1	$\lambda=0$, $\mu=\nu=\delta$	$e^{\delta t}u\partial_v$, $e^{\delta t}(R\partial_R+\sigma tu\partial_v)$ [AET 10 if $\mu=0$]	$\tilde{\psi}_0\partial_v$, $D+v\partial_v$, if $\mu=0$
2	$\lambda=0$, $\mu\neq\nu$, $\mu^2+\nu^2=1$	$e^{\mu t}((\mu-\nu)R\partial_R+\sigma u\partial_v)$, $e^{\nu t}u\partial_v$ [AET 18 if $\mu\nu=0$]	$\tilde{\psi}_\nu\partial_v$ if $\mu=0$
3	$\Delta=0$, $\lambda=\varepsilon$, $\mu+\nu=2\Omega$	$X^4=e^{\Omega t}(2\varepsilon R\partial_R+(\nu-\mu)u\partial_v)$, $2e^{\Omega t}u\partial_v+tX^4$ [AET 18 if $\mu=-\nu$, and 1, 19 if $\mu=\nu=0$]	$D+u\partial_u$, G_a if $\mu=\nu=0$, $\widehat{G}^a$ if $\mu=\nu\neq 0$
4	$\lambda\neq 0$, $\Delta=1$, $\omega_\pm=\Omega\pm 1$	$e^{\omega_+ t}(\lambda R\partial_R+(\omega_+-\mu)u\partial_v)$, $e^{\omega_- t}(\lambda R\partial_R+(\omega_--\mu)u\partial_v)$ [AET 18 if $\mu\nu=\lambda\sigma$, and 1 if $\mu=\sigma=0$]	G_a if $\sigma=\mu=0$, $\widehat{G}_\alpha$ if $\sigma=0$, $\mu\neq 0$
5	$\Delta=-1$,	$e^{\Omega t}[2\lambda\cos tR\partial_R+((\nu-\mu)\cos t-2\sin t)u\partial_v]$, $e^{\Omega t}[2\lambda\sin tR\partial_R+((\nu-\mu)\sin t+2\cos t)u\partial_v]$	none

w.r.t. extended Galilei algebra spanned on P^μ, $J^{\mu\nu}$ (26) dilatation operator and also generators G_α, K (18). These equations correspond to the non-linearities present in Table 4, Item 1 and have the following form

$$u_t-\Delta u=\lambda u\bigl(uv^{-a}\bigr)^{\frac{4}{m(1-a)}},\qquad v_t-a\Delta v=\sigma v\bigl(uv^{-a}\bigr)^{\frac{4}{m(1-a)}}$$

and

$$u_t-\Delta u=\lambda v^{\frac{4+m}{4}},\qquad v_t-\left(1+\frac{4}{m}\right)\Delta v=0.$$

Table 9
Additional non-linearities with arbitrary parameters and symmetries for Eqs. (2) with $a = 1$

No	Non-linear terms	Symmetries	AET Eq. (15)
1	$f^1 = \delta$, $f^2 = u^\nu$, $\delta = 0$ if $\nu = 2$	$D + u\partial_u + (\nu+1)v\partial_v$, $\tilde{\psi}_0\partial_v$, $(u - \delta t)\partial_v$, for any ν, δ, and $u\partial_u + \nu v\partial_v$ for $\delta = 0$ and $\partial_u + 2tu\partial_v$ for $\nu = 2$	3, 9, and 14, $\rho = \nu\omega$ if $\delta = 0$, and 21 if $\nu = 2$
2	$f^1 = \varepsilon u$, $f^2 = u^\nu$, $\nu \neq 0, 1$	$u\partial_u + \nu v\partial_v$, $\tilde{\psi}_0\partial_v$, $e^{-\varepsilon t}u\partial_v$ for any ν, and $e^{\varepsilon t}(u\partial_v + \varepsilon\partial_u)$ for $\nu = 2$	3, and 14, $\rho = \nu\omega$
3	$f^1 = \eta v$, $f^2 = -\frac{v^2}{u}$	$D - v\partial_v$, $u\partial_u + v\partial_v$, G_α, $K + (2-\eta)(t(\eta u\partial_u - (2+\eta)v\partial_v) - u\partial_v)$	1
4	$f^1 = \varepsilon u^{\nu+1}$, $f^2 = \varepsilon u^\nu v$, $\nu \neq 0$	$\nu D - u\partial_u$, $v\partial_v$, $u\partial_v$ for any ν, and $(1 + tu)\partial_v$ for $\nu = 1$	6, and 14, $\omega = 0$
5	$f^1 = \delta u^{\nu+1}$, $f^2 = u^\nu(\delta v + \mu u^\sigma)$, $\nu + \sigma \neq 0$, $\mu\nu \neq 0$	$\nu D - u\partial_u - \sigma v\partial_v$, $u\partial_v$	6
6	$f^1 = \delta(2v - u^2)^{\nu+\frac{1}{2}}$, $f^2 = \delta u(2v - u^2)^{\nu+\frac{1}{2}} + \mu(2v - u^2)^{\nu+1}$	$2\nu D - u\partial_u - 2v\partial_v$, $\partial_u + u\partial_v$ for any ν, and $e^{\mu t}(2t(u\partial_v + \partial_u) + \partial_v)$ for $\nu = \frac{1}{2}$, $\mu = 0$	13, and 14, $\rho = 2\omega$ if $\nu = 0$
7	$f^1 = \delta e^u$, $f^2 = ue^u$	$\delta(D - \partial_u) - u\partial_v$, $\tilde{\psi}_0\partial_v$, and $u\partial_v$ if $\delta = 0$	3
8	$f^1 = \eta e^{2v-u^2}$, $f^2 = (\eta u + \mu)e^{2v-u^2}$	$2D - \partial_v$, $\partial_u + u\partial_v$	13
9	$f^1 = \delta u^{\nu+1}e^{\frac{v}{u}}$, $f^2 = e^{\frac{v}{u}}(\delta v + \sigma u)u^\nu$	$D - u\partial_v$, $\nu D - u\partial_u - v\partial_v$ for any ν, and G_α for $\nu = 0$	12
10	$f^1 = e^{\nu z}R^\sigma(\delta u - \mu v)$, $f^2 = e^{\nu z}R^\sigma(\delta v + \mu u)$	$\sigma D - u\partial_u - v\partial_v$, $\nu D - u\partial_v + v\partial_u$ for any σ, and G_α for $\sigma = 0$	15
11	$f^1 = \varepsilon u \ln u$, $f^2 = \varepsilon v \ln u$	$e^{\varepsilon t}(u\partial_u + v\partial_v)$, $v\partial_v$, $u\partial_v$, $\widehat{G}^\alpha$	6, and 14, $\omega = 0$
12	$f^1 = \delta$, $f^2 = \ln u$	$D + u\partial_u + v\partial_v + t\partial_v$, $\tilde{\psi}_0\partial_v$ $(u - \delta t)\partial_v$ for any δ, and $u\partial_u + t\partial_v$ for $\delta = 0$	3, 9, and 6, 7 if $\delta = 0$
13	$f^1 = \varepsilon u^{\mu+1}$, $f^2 = \varepsilon u^\mu(v - \ln u)$, $\mu \neq 0$	$\mu D - u\partial_u - \partial_v$, $u\partial_v$ for any $\mu \neq 0$, and $\partial_v + tu\partial_v$ for $\mu = 1$	6
14	$f^1 = \delta \ln(2v - u^2)$, $f^2 = \sigma(2v - u^2)^{1/2} + \delta u \ln(2v - u^2)$	$D + u\partial_u + 2v\partial_v + 2\delta t(\partial_u + u\partial_v)$, $\partial_u + u\partial_v$	13
15	$f^1 = \varepsilon u^{\nu+1}$, $\nu \neq -1$, $f^2 = u^{\nu+1}\ln u$	$\nu D - (u\partial_u + v\partial_v + \varepsilon u\partial_v)$, $\tilde{\psi}_0\partial_v$	3
16	$f^1 = \varepsilon u^{\nu+1}$, $\nu \neq 1$, $f^2 = \varepsilon u^\nu v + u \ln u$	$\nu D - u\partial_u - tu\partial_v - (1-\nu)v\partial_v$, $u\partial_v$	6, and 5, $\kappa = \varepsilon$ if $\nu = 0$

(*continued on next page*)

Table 9 (*Continued*)

No	Non-linear terms	Symmetries	AET Eq. (15)
17	$f^1 = 2v - u^2$, $f^2 = (\mu + u)(2v - u^2) - \frac{\mu^2}{2}u$, $\mu \neq 0$	$X^1 = e^{\mu t}(2\partial_u + 2u\partial_v + \mu\partial_v)$, $tX^1 + e^{\mu t}\partial_v$	
18	$f^1 = 2v - u^2$, $f^2 = (\mu + u)(2v - u^2) + \frac{1-\mu^2}{2}u$	$X^{\pm} = e^{(\mu\pm 1)t}(2\partial_u + 2u\partial_v + (\mu \pm 1)\partial_v)$	13 if $\mu^2 = 1$
19	$f^1 = 2v - u^2$, $f^2 = -\frac{1+\mu^2}{2}u + (\mu + u)(2v - u^2)$	$e^{\mu t}(2\cos t(\partial_u + u\partial_v) + (\mu\cos t - \sin t)\partial_v)$, $e^{\mu t}(2\sin t(\partial_u + u\partial_v) + (\mu\sin t + \cos t)\partial_v)$	

Table 10
Additional non-linearities and symmetries for Eqs. (2) with $a = 0$

No	Non-linear terms	Arguments of F^1, F^2	Symmetries and AET Eq. (15) [in square brackets]
1	$f^1 = F^1 + (\delta - \mu)u$, $f^2 = F^2 + \delta v$	$v - u$	$e^{\delta t}\Psi_\mu(x)(\partial_u + \partial_v)$ [AET 11, $\eta = 1$ if $\mu = \delta = 0$]
2	$f^1 = e^u F^1$, $f^2 = e^u F^2$, $F^2 = 1$ if $\eta = 0$	$v - \eta u$	$D - \partial_u - \eta\partial_v$
3	$f^1 = F^1$, $f^2 = F^2 + \eta v$	u	$e^{\eta t}\Psi(x)\partial_v$ [AET 3 if $\eta = 0$]
4	$f^1 = vF^1$, $f^2 = F^2$	u	$D + v\partial_v$
5	$f^1 = uF^1 + \delta uv$, $f^2 = F^2 + \delta v$	$v - \ln u$	$e^{\delta t}(u\partial_u + \partial_v)$ [AET 5 if $\delta = 0$]
6	$f^1 = u^{\nu+1}F^1$, $f^2 = u^\nu F^2$, $\nu \neq 0$	$v - \ln u$	$\nu D - u\partial_u - \partial_v$
7	$f^1 = F^1 + \nu u$, $f^2 = \eta$	v	$\psi_\nu\partial_u$
8	$f^1 = u^{\nu+1}F^1$, $f^2 = 0$	v	$\nu D - u\partial_u$, and $\psi_0\partial_u$ if $\nu = -1$ [AET 2, and 14, $\rho = 0$ if $\nu = 0$]
9	$f^1 = v^{1+\lambda}$, $f^2 = \delta$		$D + v\partial_v - \lambda u\partial_u$, $\psi_0\partial_u$ [AET 2]
10	$f^1 = \delta e^u$, $f^2 = e^u$		$D - \partial_u$, $\Psi(x)\partial_v$ [3, and 4 if $\delta = 0$]
11	$f^1 = \ln v$, $f^2 = \varepsilon$		$D + u\partial_u + v\partial_v + t\partial_u$, $\psi_0\partial_u$ [AET 2]
12	$f^1 = \delta u^{\nu+1}v^{-1}$, $f^2 = u^\nu$		$D + v\partial_v$, $\nu D - u\partial_u$ [AET 14, $\nu\omega = \rho$]

Finally, if the diffusion matrix is the unit one then we indicate 98 non-equivalent classes of equations, among them 21 including arbitrary functions and 14 admitting Galilei generators. There is the only equation admitting extended Galilei algebra, the related non-linearities are given in Table 9, Item 3.

Consider examples of well-known reaction–diffusion equations which appear to be particular subjects of our analysis.

The CGL equation (3) with $\beta = 0$ can be rewritten as

$$\begin{aligned} u_t - \Delta_2 u &= u + \left(u^2 + v^2\right)(\alpha v - u), \\ v_t - \Delta_2 v &= v - \left(u^2 + v^2\right)(v + \alpha u), \end{aligned} \tag{47}$$

where u and v are real and imaginary components of the complex function W.

The r.h.s. of Eqs. (47) has the form presented in Item 8 of Table 6 (with $\lambda = \nu = 0$), and so in addition to basic symmetries $\langle \partial_0, \partial_1, \partial_2, x_1\partial_2 - x_2\partial_1 \rangle$ this system admits the symmetry

$$X = u\partial_v - v\partial_u. \tag{48}$$

Using the anzats

$$u = e^{i(x_1 \cos\theta + x_2 \sin\theta)}\tilde{u}, \qquad v = e^{i(x_1 \cos\theta + x_2 \sin\theta)}\tilde{v},$$

where $\tilde{u}$ and $\tilde{v}$ are functions of t and ω, $\omega = x_1 \sin\theta - x_2 \cos\theta$, θ is a parameter, the system (47) can be reduced to the form

$$\tilde{u} - \tilde{u}_{\omega\omega} = \left(\tilde{u}^2 + \tilde{v}^2\right)(\alpha\tilde{v} - \tilde{u}), \qquad \tilde{v}_t - \tilde{v}_{\omega\omega} = \left(\tilde{u}^2 + \tilde{v}^2\right)(\tilde{v} + \alpha\tilde{u}). \tag{49}$$

Main symmetries of the reduced equation (49) appear to be more extended then of the CGL one. As is indicated in Item 11 of Table 9 Eq. (49) admits symmetry (48) and also the following one:

$$X_2 = 2D - u\partial_u - v\partial_v.$$

The primitive predator-prey system (4) is a particular case of Eq. (1) with the non-linearities given in the first line of Table 2 where however $-\mu = \nu = 1$, $F^1 = -F^2 = \frac{u}{v}$. In addition to the basic symmetries $\langle \partial_t, \partial_x \rangle$ this equation admits the (main) symmetry:

$$X = D - u\partial_u - v\partial_v.$$

The λ–ω reaction–diffusion system (5) and its symmetries was studied in paper [15]. Our investigations confirm and complete the results of [15]. First we recognize that this system is a particular case of (1) with non-linearities given in Item 11 of Table 6 with $\mu = \nu = 0$. Hence it admits the five-dimensional Lie algebra generated by basic symmetries (26) with $\mu, \nu = 1, 2$ and also the symmetry (48). This is in accordance with results of paper [15] for arbitrary functions λ and ω. Moreover, using Table 9, Item 11 we find that for the cases when

$$\lambda(R) = \tilde{\lambda} R^\nu, \qquad \omega = \sigma R^\nu \tag{50}$$

Eq. (5) admits additional symmetry with respect to scaling transformations generated by the operator:

$$X = \nu D - u\partial_u - v\partial_v. \tag{51}$$

The other extensions of the basic symmetries correspond to the case when $\lambda(R) = \mu \ln(R)$, $\omega(R) = \sigma \ln(R)$, the related additional symmetries are given in Table 7, Items 1, 2, 5 where $\nu = \lambda = 0$.

Consider now the system (6). This system admits the equivalence transformation 1 (15) for $\rho = -\omega$. Choosing $\rho = 2k$ we transform Eq. (6) to the form (1) where $a = -1$, $f^1 = -2u^2v$ and $f^2 = 2v^2u$. The symmetries corresponding to these non-linearities are given in the first line of Table 4. For $m = 2$ the symmetries are the most extended and include two dilatations, two Galilei generators G_α, $\alpha = 1, 2$, and the conformal generator K. All these symmetries except K are valid for other numbers m of independent variables.

Symmetries of Eqs. (6) for $m = 1$ were investigated in paper [11] whose results are in accordance with our analysis.

The results of the present paper related to non-degenerated diffusion matrix can be compared with those of [4] and [8,9].

Paper [4] was apparently the first work were the problem of group classification of Eqs. (2) with a diagonal diffusion matrix was formulated and partially solved. However the classification results presented in [4] include only a small part of ones presented in Tables 2–10.

In papers [8,9] Lie symmetries of the same equations and also of systems of diffusion equations with the unit diffusion matrix were classified. The results present in those papers are much more advanced then the pioneer Danilov ones, nevertheless they are also incomplete. In particular, the cases presented above in Items 10–12 of Table 6 and Items 1, 2 of Table 7 and some other ones were not indicated in [9]. The classification results presented in [9] include a lot of arbitrary parameters which can be removed using equivalence transformations. Moreover, many of equations treated in [9] as non-equivalent ones, in fact are equivalent. For instance, all versions 14, 15, 18 and 20 from Table 4 presented in [9] are equivalent one to another.

Notice that the results related to the group classification of systems of non-linear systems of reaction–diffusion equations are presented in a very compressed form in the survey [19]. The principally new points of the present paper in comparison with [19] are the following ones:

- In the present paper we give the completed list of admissible equivalence transformations (15) for all classified equations (2) whereas in [19] only an a priori fixed subclass of equivalence transformations was discussed.
- For any particular system of equations (2) whose non-linear terms are given in the classification tables the admissible equivalence transformations are specified and presented explicitly at the same tables while in [19] the general (incomplete) list of such transformations was presented only.
- We use our knowledge of all admissible equivalence transformations to reduce the number of non-equivalent versions of systems (2) to absolute minimum. In particular many of quantities which define non-linearities and are treated in [19] as arbitrary parameters are reduced to $\delta = 0, \pm 1$, $\varepsilon = \pm 1$ or $\eta = 0, 1$ and possible values of parameter a in the diffusion matrix A are reduced to ones given by Eq. (14).
- Summarizing, in the present paper the problem of group classification of systems of reaction–diffusion equations (2) is solved completely whereas all previous publications [4–9] and [19] can be treated only as steps to the complete solution.

Thus we present group classification of reaction–diffusion systems with a diagonal diffusion matrix. Such systems with the square and triangular diffusion matrix have been classified in paper [1] and preprint [3], respectively. The results of papers [1,3] and the present one consist in the completed group classification of systems of two coupled diffusion equations with the general diffusion matrix.

Appendix A. Algebras of main symmetries

Following [1] we first specify all non-equivalent terms

$$N = C^{ab}u_b\partial_{u_a} + B^a\partial_{u_a}, \tag{A.1}$$

where summation from 1 to 2 is imposed over the repeated indices and we again use the notations $u_1 = u$, $u_2 = v$.

Let (A.1) be a basis element of a one-dimensional invariance algebra $\mathcal{A}$ then commutators of N with P^0 and P^a should be equal to a linear combination of N and operators (26). This condition presents the following three possibilities [1]:

$$\begin{aligned}
&1. \quad C^{ab} = \mu^{ab}, \qquad B^a = \mu^a,\\
&2. \quad C^{ab} = e^{\lambda t}\mu^{ab}, \qquad B^a = e^{\lambda t}\mu^a,\\
&3. \quad C^{ab} = 0, \qquad B^a = e^{\lambda t+\omega\cdot x}\mu^a,
\end{aligned} \tag{A.2}$$

where μ^{ab}, μ^a, λ, and ω are constants.

Like in [1] to classify all non-equivalent symmetries (A.2) we use their isomorphism with 3×3 matrices of the following form

$$g = \begin{pmatrix} 0 & 0 & 0\\ \mu^1 & \mu^{11} & \mu^{12}\\ \mu^2 & \mu^{21} & \mu^{12}\end{pmatrix}. \tag{A.3}$$

Equations (2) admit equivalence transformations (12). The corresponding transformations for matrix (A.3) are

$$g \to g' = UgU^{-1}, \quad U = \begin{pmatrix} 1 & 0 & 0\\ b^1 & K^{11} & K^{12}\\ b^2 & K^{21} & K^{22}\end{pmatrix}, \tag{A.4}$$

were K^{ab} are the same parameters as in (12), (13).

For the case of Eq. (2) with $a \neq 1$ matrices μ and K in (A.3), (A.4) are diagonal, and up to equivalence there exist three matrices (A.3), namely

$$g^1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda\end{pmatrix}, \qquad g^2 = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad g^3 = \begin{pmatrix} 0 & 0 & 0\\ \lambda & 0 & 0\\ 1 & 0 & 0\end{pmatrix}. \tag{A.5}$$

In accordance with (27), (A.1), (A.2) the related symmetry operator can be represented in one of the following forms

$$\begin{aligned}
&X^1_{(k)} = \mu D - 2(g^k)_{bc}\tilde{u}_c\partial_{\tilde{u}_b}, \qquad X^2_{(k)} = e^{\lambda t}(g^k)_{bc}\tilde{u}_c\partial_{\tilde{u}_b},\\
&X^3 = e^{\lambda t+\omega\cdot x}(\partial_{u_2} + \mu\partial_{u_1}), \quad k = 1, 2, 3.
\end{aligned} \tag{A.6}$$

Here $(g^k)_{bc}$ are elements of matrices (A.5), $b, c = 0, 1, 2$, $\tilde{u} = \text{column}(1, u_1, u_2)$.

Formulae (A.6) and (A.5) give the principal description of one-dimensional algebras $\mathcal{A}$ for Eq. (2) with $a \neq 1$.

To describe two-, three- and four-dimensional algebras $\mathcal{A}$ we first classify the corresponding algebras $A_{n,s}$ of matrices g (A.3) where index n indicates the dimension of the algebra and s is used to mark different algebras of the same dimension n. Choosing a basis element of $A_{2,s}$ in one of the forms given in (A.5) we find that up to equivalence transformations (A.3) there exist six two dimension algebras with basis elements $\langle e_1, e_2\rangle$:

$$\begin{aligned}
&A_{2,1}\colon\quad e_1 = g^1_{(0)},\quad e_2 = g^4; \qquad A_{2,2}\colon\quad e_1 = g^1_{(0)},\quad e_2 = g^3_{(0)};\\
&A_{2,3}\colon\quad e_1 = g^5,\quad e_2 = g^3_{(0)}, \qquad\qquad (A.7)\\
&A_{2,4}\colon\quad e_1 = g^1,\quad e_2 = g^5; \qquad A_{2,5}\colon\quad e_1 = g^1_{(1)},\quad e_2 = g^3;\\
&A_{2,6}\colon\quad e_1 = g^2,\quad e_3 = g^3_{(0)}, \qquad\qquad (A.8)
\end{aligned}$$

where $g^1_{(0)} = g^1|_{\lambda=0}$, $g^1_{(1)} = g^1|_{\lambda=1}$, $g^3_{(0)} = g^3|_{\lambda=0}$, and

$$g^4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad g^5 = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{A.9}$$

Algebras (A.7) are Abelian while algebras (A.8) are characterized by the following commutation relations:

$$[e_1, e_2] = e_2. \tag{A.10}$$

Using (A.7), (A.8) and applying arguments analogous to those which follow Eqs. (33) we find pairs of operators (27) forming Lie algebras. Denoting

$$\hat{e}_\alpha = (e_\alpha)_{ab}\tilde{u}_b\frac{\partial}{\partial\tilde{u}_a}, \quad \alpha = 1, 2,$$

we represent them as follows:

$$\begin{aligned}
&\langle\mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle, \qquad \langle\mu D + \hat{e}_2 + \nu t\hat{e}_1, \hat{e}_1\rangle,\\
&\langle\mu D - \hat{e}_1, \nu D - \hat{e}_2\rangle, \qquad \big\langle F^1\hat{e}_1 + G^1\hat{e}_2, F^2\hat{e}_1 + G^2\hat{e}_2\big\rangle
\end{aligned} \tag{A.11}$$

for e_1, e_2 belonging to algebras (A.7) and

$$\langle\mu D - \hat{e}_1, \hat{e}_2\rangle, \qquad \langle\mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2\rangle \tag{A.12}$$

for e_1, e_2 belonging to algebra (A.8).

Here $\{F^1, G^1\}$ and $\{F^2, G^2\}$ are fundamental solutions of the following system

$$F_t = \lambda F + \nu G, \qquad G_t = \sigma F + \gamma G \tag{A.13}$$

with arbitrary parameters λ, ν, σ, γ.

The list (A.11)–(A.12) does not includes two dimension algebras whose basis is $\langle F\hat{e}_\alpha, G\hat{e}_\alpha\rangle$ (with F, G satisfying (A.13)) or $\langle\mu D + \lambda e^{\nu t+\omega\cdot x}\hat{e}_\alpha, e^{\nu t+\omega\cdot x}\hat{e}_\alpha\rangle$ which are incompatible with classifying equations (20). In the following we ignore all algebras $\mathcal{A}$ which include such subalgebras.

Up to equivalence there exist three realizations of three dimension algebras of matrices (A.5), (A.9):

$$\begin{aligned}
&A_{3,1}\colon\quad e_1 = g^1_{(0)},\quad e_2 = g^4,\quad e_3 = g^3_{(0)},\\
&A_{3,2}\colon\quad e_1 = g^5,\quad e_2 = g^4,\quad e_3 = g^3_{(0)}, \qquad\qquad (A.14)\\
&A_{3,3}\colon\quad e_1 = g^1_{(1)},\quad e_2 = g^5,\quad e_3 = g^3_{(0)}. \qquad\qquad (A.15)
\end{aligned}$$

Non-zero commutators for matrices (A.14) and (A.15) are $[e_2, e_3] = e_3$ and $[e_1, e_\alpha] = e_\alpha$ ($\alpha = 2, 3$), respectively. The algebras of operators (27) corresponding to realizations (A.14) and (A.15) are of the following general forms:

$$\langle \mu D - \hat{e}_1, \nu D - \hat{e}_2, \hat{e}_3 \rangle, \qquad \langle \hat{e}_1, D + \hat{e}_2 + \mu t \hat{e}_3, \hat{e}_3 \rangle \tag{A.16}$$

and

$$\begin{aligned} &\langle \mu D - \hat{e}_1, \hat{e}_2, \hat{e}_3 \rangle, \qquad \langle D + \hat{e}_1 + \nu t \hat{e}_2, \hat{e}_2, \hat{e}_3 \rangle, \\ &\langle D + \hat{e}_1 + \nu t \hat{e}_3, \hat{e}_3, \hat{e}_2 \rangle, \qquad \langle \hat{e}_1, F^1 \hat{e}_2 + G^1 \hat{e}_3, F^2 \hat{e}_2 + G^2 \hat{e}_3 \rangle \end{aligned} \tag{A.17}$$

correspondingly.

In addition, we have the only four dimension algebra

$$\hat{A}_{4,1}: \quad e_1 = g^1_{(0)}, \quad e_2 = g^5, \quad e_3 = g^3_{(0)}, \quad e_4 = g^4, \tag{A.18}$$

which generates the following algebras of operators (27):

$$\begin{aligned} &\langle \mu D - \hat{e}_1, \nu D - \hat{e}_3, \hat{e}_2, \hat{e}_4 \rangle, \qquad \langle \hat{e}_1, D + \hat{e}_3 + \nu t \hat{e}_4, \hat{e}_2, \hat{e}_4 \rangle, \\ &\langle D + \hat{e}_1 + \nu t \hat{e}_2, \hat{e}_2, \hat{e}_3, \hat{e}_4 \rangle. \end{aligned} \tag{A.19}$$

Finally, it is necessary to take into account the special type of $(m + 2)$-dimensional algebras $\mathcal{A}$ generated by two dimension algebras (A.7), namely, algebras whose basis elements have the following general form: $\langle \mu D + \hat{e}_1 + (\alpha t + \lambda^{\sigma\rho} x_\sigma x_\rho)\hat{e}_2, x_\nu \hat{e}_2, \hat{e}_2 \rangle$ where ν, σ, ρ run from 1 to m. The related classifying equations generated by all symmetries $x_1\hat{e}_2, x_2\hat{e}_2, \ldots, x_m\hat{e}_2$ and $\hat{e}_2$ coincide and we have the same number of constrains for f^1, f^2 as in the case of two dimension algebras $\mathcal{A}$.

The case $a = 1$ appears to be much more complicated. The related matrices g are of general form (A.3) and defined up to the general equivalence transformation (A.4) with arbitrary K^{ab}. Namely there are seven non-equivalent matrices (A.3), including g^1, g^2 (A.5), g^5 (A.9) and also the following matrices

$$\begin{aligned} &g^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & -1 \\ 0 & 1 & \mu \end{pmatrix}, \qquad g^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \\ &g^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad g^9 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{A.20}$$

In addition, we have fifteen two dimension algebras of matrices (A.3),

$$\begin{aligned} &A_{2,1} = \langle g^1_{(0)}, g^4 \rangle, \qquad A_{2,2} = \langle g^1_{(0)}, g^3_{(0)} \rangle, \qquad A_{2,3} = \langle g^3_{(0)}, g^5 \rangle, \\ &A_{2,7} = \langle g^7, g^8 \rangle, \qquad A_{2,8} = \langle g^3_{(0)}, g^8 \rangle, \qquad A_{2,9} = \langle g^3_{(0)}, g^9 \rangle, \\ &A_{2,10} = \langle g^1_{(1)}, g^6 \rangle, \end{aligned} \tag{A.21}$$

$$\begin{aligned} &A_{2,4} = \langle g^1, g^5 \rangle, \qquad A_{2,5} = \langle g^1_{(1)}, g^3 \rangle, \qquad A_{2,6} = \langle g^2, g^3_{(0)} \rangle, \\ &A_{2,11} = \left\langle \frac{1}{\lambda - 1} g^1 \bigg|_{\lambda \neq 1}, g^8 \right\rangle, \qquad A_{2,12} = \langle -g^{10}, g^8 \rangle, \qquad A_{2,13} = \langle g^1_{(2)}, g^9 \rangle, \\ &A_{2,14} = \langle g^7, g^3_{(0)} \rangle, \end{aligned} \tag{A.22}$$

where

$$g^{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad g^1_{(2)} = g^1\big|_{\lambda=2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Algebras (A.21) are Abelian whereas algebras (A.22) are characterized by commutation relations (A.10). The corresponding algebras $\mathcal{A}$ are given by Eq. (A.11) and (A.12), respectively.

Three dimension algebras $A_{3,s}$ are the algebras $A_{3,1}$–$A_{3,3}$ given by relations (A.14), (A.15) where matrices g^1 and g^3 are of general form (A.5) with arbitrary λ (i.e., $g^1_{(0)} \to g^1$, etc.) and also algebras $A_{3,4}$–$A_{3,11}$ given below:

$$\begin{aligned}
&A_{3,4}: \quad e_1 = g^8, \quad e_2 = g^1_{(1)}, \quad e_3 = g^3_{(0)}, \\
&A_{3,5}: \quad e_1 = g^1, \quad e_2 = g^8, \quad e_3 = g^3_{(0)}, \\
&A_{3,6}: \quad e_1 = g^1_{(0)}, \quad e_2 = g^8, \quad e_3 = g^4, \\
&A_{3,7}: \quad e_1 = g^4, \quad e_2 = g^8, \quad e_3 = g^3_{(0)}, \\
&A_{3,8}: \quad e_1 = g^5, \quad e_2 = g^6, \quad e_3 = g^3_{(0)}, \\
&A_{3,9}: \quad e_1 = g^3_{(0)}, \quad e_2 = g^8, \quad e_3 = g^9, \\
&A_{3,10}: \quad e_1 = g^2, \quad e_2 = g^8, \quad e_3 = g^3_{(0)}, \\
&A_{3,11}: \quad e_1 = g^3_{(0)}, \quad e_2 = g^5, \quad e_3 = g^7.
\end{aligned}$$

Algebras $A_{3,4}$–$A_{3,6}$ and $A_{3,7}$ are isomorphic to $A_{3,1}$ and $A_{3,3}$, respectively. The related algebras $\mathcal{A}$ are given by Eqs. (A.16) and (A.17) correspondingly.

Algebra $A_{3,8}$ is isomorphic to $A_{3,3}$ and so generates algebra (A.17).

Algebras $A_{3,9}$ and $A_{3,10}$ are characterized by the following commutation relations

$$[e_2, e_3] = e_1 \tag{A.23}$$

(the remaining commutators are equal to zero); non-zero commutators for basis elements of $A_{3,11}$ are given below:

$$[e_1, e_2] = e_2, \qquad [e_1, e_3] = e_2 + e_3. \tag{A.24}$$

Using (A.23) and (A.24) we come to the following related three dimension algebras $\mathcal{A}$ generated by $A_{3,9}$ and $A_{3,10}$:

$$\begin{aligned}
&\langle \mu D - 2\hat{e}_2, \nu D - 2\hat{e}_3, e_1 \rangle, \qquad \langle e_1, D + 2e_\alpha + 2\nu t e_1, e_{\alpha'} \rangle, \\
&\langle e^{\nu t} e_1, e^{\nu t} e_\alpha, e_{\alpha'} \rangle, \quad \alpha, \alpha' = 2, 3, \ \alpha' \neq \alpha,
\end{aligned} \tag{A.25}$$

and algebras (A.26) generated by $A_{3,11}$:

$$\langle \mu D - 2e_1, e_2, e_3 \rangle, \qquad \langle e_1, e^{\nu t} e_2, e^{\nu t} e_3 \rangle. \tag{A.26}$$

Finally, four dimension algebras of matrices (A.4) are $A_{4,1}$ given by Eqs. (A.18) and also $A_{4,2}$–$A_{4,5}$ given below:

$$\begin{aligned}
&A_{4,2}: \quad e_1 = g^{1'}, \quad e_2 = g^6, \quad e_3 = g^3_{(0)}, \quad e_4 = g^5, \\
&A_{4,3}: \quad e_1 = g^3_{(0)}, \quad e_2 = g^5, \quad e_3 = g^{1'}, \quad e_4 = g^8, \\
&A_{4,4}: \quad e_1 = g^1, \quad e_2 = g^4, \quad e_3 = g^8, \quad e_4 = g^3, \\
&A_{4,5}: \quad e_1 = g^4, \quad e_2 = g^8, \quad e_3 = g^5, \quad e_4 = g^3.
\end{aligned}$$

We do not present the related algebras $\mathcal{A}$ because all possible non-linearities f^1 and f^2 will be fixed asking for invariance of Eq. (2) with respect to transformations generated by three-dimensional algebras.

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