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Note

Separate continuity of the Lempert function of the spectral ball ☆

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ABSTRACT

We find all matrices A from the spectral unit ball Ω_n such that the Lempert function $l_{\Omega_n}(A,\cdot)$ is continuous.

The characteristic polynomial of an $n \times n$ complex matrix A is

$$P_A(t) := \det(tI_n - A) =: t^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A) t^{n-j},$$

where I_n is the unit matrix. Let $r(A) := \max\{|\lambda|\colon\ P_A(\lambda) = 0\}$ be the spectral radius of A. The spectral unit ball is the pseudoconvex domain $\Omega_n := \{A\colon\ r(A) < 1\}$.

Let $\sigma(A) := (\sigma_1(A), \dots, \sigma_n(A))$. The *symmetrized polydisc* is the bounded domain $\mathbb{G}_n := \sigma(\Omega_n) \subset \mathbb{C}^n$, which is hyperconvex (see [3]) and hence taut.

We are interested in two-point Nevanlinna–Pick problems with values in the spectral unit ball, so let us consider the Lempert function of a domain $D \subset \mathbb{C}^m$: for $z, w \in D$,

$$l_D(z,w) := \inf\bigl\{|\alpha|\colon\ \exists \varphi \in \mathcal{O}(\mathbb{D}, D)\colon\ \varphi(0) = z,\ \varphi(\alpha) = w\bigr\},$$

where $\mathbb{D} \subset \mathbb{C}$ is the unit disc. For general facts about this function, see for instance [4]. The Lempert function is symmetric in its arguments, upper semicontinuous and decreases under holomorphic maps, so for $A, B \in \Omega_n$,

$$l_{\Omega_n}(A,B) \geqslant l_{\mathbb{G}_n}\bigl(\sigma(A), \sigma(B)\bigr). \tag{1}$$

The domain $\mathbb{G}_n$ is taut, so its Lempert function is continuous.

The systematic study of the relationship between Nevanlinna–Pick problems valued in the symmetrized polydisc or spectral ball began with [1]. In particular, it showed that when both A and B are *cyclic* (or non-derogatory) matrices, i.e. they admit a cyclic vector (see other equivalent properties in [5]), then equality holds in (1). It follows that l_{Ω_n} is continuous on $\mathcal{C}_n \times \mathcal{C}_n$, where $\mathcal{C}_n$ denotes the (open) set of cyclic matrices. On the other hand, in general, if equality holds

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in (1) at (A,B), then l_{Ω_n} is continuous at (A,B) (see [6, Proposition 1.2]). The converse is also true, since l_{Ω_n} is an upper semicontinuous function, $l_{\mathbb{G}_n}$ is a continuous function and (1) holds.

The goal of this note is to study the continuity of l_{Ω_n} separately with respect to each argument. In [6], the authors looked for matrices B such that $l_{\Omega_n}(A,.)$ is continuous at B for any A. They conjecture that this holds for any $B \in \mathcal{C}_n$, and prove it for $n \leqslant 3$ [6, Proposition 1.4], and the converse statement for all dimensions (see [6, Theorem 1.3]).

In the present paper, we ask for which A the function $l_{\Omega_n}(A,.)$ is continuous at B for any B (or simply, continuous on the whole Ω_n). By [5, Proposition 4], for any matrix $A \in \mathcal{C}_n$ with at least two different eigenvalues, the function $l_{\Omega_n}(A,\cdot)$ is not continuous at any scalar matrix. On the other hand, $l_{\Omega_n}(0,B) = r(B)$ and hence $l_{\Omega_n}(A,\cdot)$ is a continuous function for any scalar matrix A (since the automorphism $\Phi_\lambda(X) = (X-\lambda I)(I-\bar{\lambda}X)^{-1}$ of Ω_n maps λI_n to 0, where $\lambda \in \mathbb{D}$).

We have already mentioned that if $A \in \Omega_n$ ($n \geqslant 2$), then the following conditions are equivalent:

(i) the function l_{Ω_n} is continuous at (A,B) for any $B \in \Omega_n$;
(ii) $l_{\Omega_n}(A,\cdot) = l_{\mathbb{G}_n}(\sigma(A),\sigma(\cdot))$.

Consider also the condition:

(iii) $A \in \mathcal{C}_2$ has two equal eigenvalues.

By [2, Theorem 8], (iii) implies (ii). Theorem 1 below says that the scalar matrices and the matrices satisfying (iii) are the only cases when $l_{\Omega_n}(A,\cdot)$ is a continuous function. Then the mentioned above result [5, Proposition 4] shows that for $n=2$ (i) implies (iii) and hence the conditions (i)–(iii) are equivalent.

Theorem 1. *If $A \in \Omega_n$, then $l_{\Omega_n}(A,\cdot)$ is a continuous function if and only if either A is scalar or $A \in \mathcal{C}_2$ has two equal eigenvalues.*

Proof. Using an automorphism of Ω_n of the form $X \to P^{-1}XP$, where P is an invertible matrix, we may assume that A is in Jordan form. Using an automorphism Φ_λ, we may assume $s_1 \geqslant \cdots \geqslant s_k$ are the numbers of Jordan blocks corresponding to each of the pairwise different eigenvalues $\lambda_1 = 0, \lambda_2, \ldots, \lambda_k$.

It is enough to prove that $l_{\Omega_n}(A,\cdot)$ is not a continuous function if A has at least one non-zero eigenvalue or $A \in \Omega_n$ is a non-zero nilpotent matrix and $n \geqslant 3$.

In the first case, we shall prove that $l_{\Omega_n}(A,\cdot)$ is not continuous at 0. It is easy to see that A can be represented as blocks $A_1,\ldots,A_l$ (with sizes $n_1,\ldots,n_l$) such that the eigenvalues of A_1 are equal to zero and the other blocks are cyclic with at least two different eigenvalues values (A_1 is omitted if $s_1 = s_2$). By [5, Proposition 4], we know that there are $(A_{i,j})_j \to 0$, $1 \leqslant i \leqslant l$, such that $\sup_{i,j} l_{\Omega_{n_i}}(A_i, A_{i,j}) := m < r(A)$. Taking A_j to be with blocks $A_{1,j},\ldots,A_{l,j}$, it is easy to see $l_{\Omega_n}(A,A_j) \leqslant \max_i l_{\Omega_{n_i}} l(A_i,A_{i,j}) \leqslant m < l_{\Omega_n}(A,0)$ which implies that $l_{\Omega_n}(A,\cdot)$ is not continuous at 0.

Let now $A \neq 0$ be a nilpotent matrix. Then $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ with $a_{ij} = 0$ unless $j = i+1$. Let $r = \operatorname{rank}(A) \geqslant 1$. Following the proof of Proposition 4.1 in [6], let

$$F_0 := \{1\} \cup \bigl\{j \in \{2,\ldots,n\}\colon a_{j-1,j} = 0\bigr\} := \{1 = b_1 < b_2 < \cdots < b_{n-r}\},$$

and $b_{n-r+1} := n+1$. We set $d_i := 1 + \#(F_0 \cap \{(n-i+2),\ldots,n\})$. The hypotheses on A imply that we can choose its Jordan form so that $a_{n-1,n} = 1$, so $1 = d_1 = d_2 \leqslant d_3 \leqslant \cdots \leqslant d_n = \#F_0 = n-r$, $d_{j+1} \leqslant d_j + 1$.

Corollary 4.3 and Proposition 4.1 in [6] show that for any $C \in \mathcal{C}_n$,

$$l_{\Omega_n}(A,C) = h_{\mathbb{G}_n}\bigl(0,\sigma(C)\bigr) := \inf\bigl\{|\alpha|\colon \exists \psi \in \mathcal{H}(\mathbb{D},\mathbb{G}_n)\colon \psi(\alpha) = \sigma(C)\bigr\},$$

where

$$\mathcal{H}(\mathbb{D},\mathbb{G}_n) = \bigl\{\psi \in \mathcal{O}(\mathbb{D},\mathbb{G}_n)\colon \operatorname{ord}_0 \psi_j \geqslant d_j,\ 1 \leqslant j \leqslant n\bigr\}.$$

Note that $d_j \leqslant j-1$ for $j \geqslant 2$. Let $m := \min_{j\geqslant 2} \frac{d_j}{j-1}$ and choose a k such that $\frac{d_k}{k-1} = m$. If $m = 1$, then $d_j = j-1$ for all $j \geqslant 2$, and if furthermore $n \geqslant 3$, we can take $k = 3$.

With k chosen as above, let λ be a small positive number, $b = k\lambda^{k-1}$ and $c = (k-1)\lambda^k$. Then λ is a double zero of the polynomial $\Lambda(z) = z^{n-k}(z^k - bz + c)$ with zeros in $\mathbb{D}$. Let B be a diagonal matrix such that its characteristic polynomial is $P_B(z) = \Lambda(z)$.

Assuming that $l_{\Omega_n}(A,\cdot)$ is continuous at B, then

$$l_{\Omega_n}(A,B) = h_{\mathbb{G}_n}\bigl(0,\sigma(B)\bigr) =: \alpha.$$

Lemma 2. *If $l_{\Omega_n}(A,B) = \alpha$, then there is a $\psi \in \mathcal{H}(\mathbb{D},\mathbb{G}_n)$ with $\psi(\alpha) = \sigma(B)$ and*

$$\sum_{j=1}^{n} \psi_j'(\alpha)(-\lambda)^{n-j} = 0.$$

Proof. This is analogous to the proof of the necessary condition in Proposition 4.1 in [6]. Let $\varphi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ be such that $\varphi(0) = A$ and $\varphi(\tilde{\alpha}) = B$. Corollary 4.3 in [6] applied to A shows that $\tilde{\psi} := \sigma \circ \varphi \in \mathcal{H}(\mathbb{D}, \mathbb{G}_n)$.

Now we study $\sigma_n(\varphi(\zeta)) - \sigma_n(B) = \sigma_n(\varphi(\zeta))$ near $\zeta = \alpha$. We may assume that the first two diagonal coefficients of B are equal to λ. If we let $\varphi_\lambda(\zeta) := \varphi(\zeta) - \lambda I_n$, then the first two columns of $\varphi_\lambda(\alpha)$ vanish, so $\sigma_n \circ \varphi_\lambda = \det(\varphi_\lambda)$ vanishes to order 2 at α. On the other hand,

$$\det\bigl(-\varphi_\lambda(\zeta)\bigr) = \det\bigl(\lambda I_n - \varphi(\zeta)\bigr) = \lambda^n + \sum_{j=1}^{n} (-1)^j \lambda^{n-j} \tilde{\psi}_j(\zeta),$$

and since the derivative of the left-hand side vanishes at $\tilde{\alpha}$, the same holds for the right-hand side. It remains to let $\tilde{\alpha} \to \alpha$ and to use that $\mathbb{G}_n$ is a taut domain, providing the desired ψ. □

Lemma 3. *We have $\alpha^m \lesssim \lambda$; furthermore if $m = 1$ and $n \geqslant 3$, then $\alpha^{2/3} \lesssim \lambda$. So in all cases $\alpha \ll \lambda$.*

Proof. Note that there is an $\varepsilon > 0$ such that for $\lambda < \varepsilon$ the map $\zeta \to (0, \dots, 0, k(\varepsilon\zeta)^{d_k}, (k-1)\lambda(\varepsilon\zeta)^{d_k}, 0, \dots, 0)$ is a competitor for $h_{\Omega_n}(A, B)$. So $(\varepsilon\alpha)^{d_k} \leqslant \lambda^{k-1}$, that is, $\alpha^m \lesssim \lambda$.

If $m = 1$ and $n \geqslant k = 3$, then considering the map $\zeta \to (0, 3\lambda^{1/2}\varepsilon\zeta, 2(\varepsilon\zeta)^2, 0, \dots, 0)$ we see that $(\varepsilon\alpha)^2 \leqslant \lambda^3$. □

Setting $\psi_j(\zeta) = \zeta^{d_j}\theta(\zeta)$, the condition in Lemma 2 becomes

$$a\frac{(-\lambda)^n}{\alpha} + S = 0, \tag{2}$$

where $a = (k-1)d_k - kd_{k-1}$ and $S = \sum_{j=1}^n \alpha^{d_j}\theta_j'(\alpha)(-\lambda)^{n-j}$. Note that $a \neq 0$. Indeed, if $m < 1$, then $d_k = d_{k-1}$ and hence $a = -d_k$; if $m = 1$, then $a = (k-1)(k-1) - k(k-2) = 1$. Since $\mathbb{G}_n$ is bounded, $|\theta_j'(\alpha)| \lesssim 1$.

By Lemma 3 and the choice of k, for any j,

$$\alpha^{d_j} \lesssim \lambda^{(k-1)d_j/d_k} \leqslant \lambda^{j-1} \leqslant \lambda^{n-1}.$$

Thus $S \lesssim \lambda^{n-1}$. By Lemma 3 again, $\alpha \ll \lambda$, a contradiction with (2). □

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