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On a classical renorming construction of V. Klee

A.J. Guirao [a,1], V. Montesinos [a,*,2], V. Zizler [b,c,3]

[a] Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, C/ Vera, s/n, 46022 Valencia, Spain

[b] Department of Mathematical and Statistical Sciences, University of Alberta, Central Academic Building, Edmonton, T6G2G1, Alberta, Canada

[c] Mathematical Institute, Czech Academy of Sciences, Žitná 25, 11567, Prague 1, Czech Republic

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ABSTRACT

We further develop a classical geometric construction of V. Klee and show, typically, that if X is a nonreflexive Banach space with separable dual, then X admits an equivalent norm $|\cdot|$ which is Fréchet differentiable, locally uniformly rotund, its dual norm $|\cdot|^*$ is uniformly Gâteaux differentiable, the weak* and the norm topologies coincide on the sphere of $(X^*, |\cdot|^*)$ and, yet, $|\cdot|^*$ is not rotund. This proves (a stronger form of) a conjecture of V. Klee.

1. Introduction

Differentiable norms on Banach spaces are most often obtained by constructing dual norms with rotundity properties. Indeed, a classical result of Šmulyan [14] implies that *if* $(X, \|\cdot\|)$ *is a Banach space and its dual norm* $\|\cdot\|^*$ *on* X^* *is rotund, then* $\|\cdot\|$ *is Gâteaux differentiable* (see also, e.g., [3, Corollary 7.23]). For sufficient conditions on a Banach space to have an equivalent norm such that its dual norm is rotund, and for characterizations of this property, see, e.g., [9,13,12,10].

The contribution of this note goes somehow in the opposite direction, exploring the failure of the converse to Šmulyan's result.

The first construction of a Gâteaux differentiable norm whose dual norm is not rotund was given in [7] and, independently, in [16]. Klee found, in [16], a geometric construction that, in the nonreflexive case, gave an application of Šmulyan's weak compactness result to the geometry of quotient spaces, *providing in every nonreflexive separable Banach space an equivalent norm that is Gâteaux differentiable and such that its dual norm is not rotund* ([7, Proposition 3.3], see also [3, Exercise 8.63]. The precise statement of Klee's result is slightly different. It reads: *suppose that* X *is a separable normed linear space and* L *is a nonreflexive closed subspace of* X, *of deficiency* $\geqslant 2$; *then* X *admits a norm under which* X *is smooth and* X/L *is not smooth.*) This in fact means that *a separable Banach space* X *is reflexive if and only if every equivalent Gâteaux differentiable norm on* X *has*

* Corresponding author.
E-mail addresses: anguisa2@mat.upv.es (A.J. Guirao), vmontesinos@mat.upv.es (V. Montesinos), zizler@math.cas.cz (V. Zizler).

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rotund dual norm. We extend Klee's result to spaces that admit an equivalent Gâteaux differentiable norm (Corollary 2) (note that every separable Banach space has this property [8], see, e.g., [3, Theorem 8.2]). A modification of Klee's construction is needed, as special "smooth" compact sets in X used by him are no longer available in the new setting.

In this note we develop this construction further, extending the range of its use in several directions – and proving, as a consequence, a stronger form of a conjecture of V. Klee in [3] (Klee conjectures there that its construction "can be sharpened to produce an admissible norm under which X is both smooth and rotund while X/L is neither".)

The Fréchet version of Šmulyan's result above says that *a dual locally uniformly rotund norm forces the predual norm on X to be Fréchet differentiable* (see, e.g., [3, Corollary 7.23]). Again, the converse fails, even up to renorming and asking only for the rotundity instead of local uniform rotundity of the dual norm: indeed, in [15] it was proved that, for any uncountable ordinal μ, the (nonseparable) Banach space $C[0,\mu]$ admits a *Fréchet differentiable* norm but *admits no norm whose dual norm is rotund* (see, e.g., [1, Theorems VII.5.2(ii) and VII.5.4]).

Recently, it was proved in [4] that $C[0,\mu]$ *admits an equivalent locally uniformly rotund norm that is Fréchet differentiable*. It seems to be unknown if the set of such norms is dense in the set of all equivalent norms on this space.

Our results include, too, a discussion of the failure of the Fréchet version of Šmulyan's result for separable spaces: it gives a relatively easy construction of a Fréchet differentiable and locally uniformly rotund norm on a *separable* space whose dual norm is not rotund.

Overall, we believe that the results in this note may help in providing some more insight in renorming theory, in the duality of smooth and rotund norms, and in the geometry of quotient spaces in general, in the case of nonreflexive spaces. For example, a natural byproduct is that, even in the class of separable Asplund spaces, the rotundity of the dual norm of X^* is a relatively quite strong notion, in the sense that it is not implied, in general, even by combined Fréchet differentiability, local uniform rotundity and weak uniform rotundity of its predual norm of X. This should be compared with the fact that every separable Asplund space admits an equivalent norm that is Fréchet differentiable, locally uniformly rotund, weakly uniformly rotund and whose dual norm is locally uniformly rotund (see, e.g., [3, Chapter 8]).

As the main result of this paper we formulate the following theorem, that shows the main practical applications of the construction. Later we shall discuss how to obtain further variants of this result.

Theorem 1. *Let X be a subspace of a weakly compactly generated nonreflexive Banach space. Then*:

(a) *There exists an equivalent locally uniformly rotund and Gâteaux differentiable norm on X such that its dual norm on X^* is not rotund.*
(b) *If X is moreover an Asplund space, then there exists an equivalent Fréchet differentiable and locally uniformly rotund norm on X such that its dual norm on X^* is not rotund but the weak* and the norm topology on its dual unit sphere coincide.*
(c) *If X^* is separable, then there exists a Fréchet differentiable, locally uniformly rotund and weakly uniformly rotund equivalent norm on X whose dual norm is not rotund but the weak* and the norm topologies on its dual unit sphere coincide.*

As we mentioned above, part (a) of Theorem 1 solves Klee's conjecture positively. The following corollary extends the result of the same author in [7, Proposition 3.3], who proved it for separable spaces.

Corollary 2. *A Banach space X with a Gâteaux differentiable norm is reflexive if and only if any equivalent Gâteaux differentiable norm on X has rotund dual norm.*

Our notation is standard. Given a Banach space X, we denote by B_X and S_X the closed unit ball and unit sphere of X, respectively. The action of an element $x^* \in X^*$ on an element $x \in X$ will be denoted, indistinctly, by $\langle x^*, x\rangle$ or by $x^*(x)$. If $\|\cdot\|$ is the norm of a Banach space X, we denote by $\|\cdot\|^*$ the corresponding dual norm on X^*. Put $\Gamma(S)$ for the absolutely convex hull (i.e., the convex and symmetric hull) of a set $S \subset X$, and $\overline{\Gamma}(S)$ for the closed absolutely convex hull of S. Recall that the *Minkowski functional* p_B of a symmetric convex body $B \subset X$ is defined by $p_B(x) = \inf\{\lambda > 0\colon\ x \in \lambda B\}$, for $x \in X$. The convex body B is said to be *Gâteaux* (*Fréchet*) *smooth* whenever p_B is Gâteaux (respectively, Fréchet) differentiable in $X \setminus \{0\}$. Given a set $S \subset X$, the (absolute) polar set S° is the subset of X^* defined by $S^\circ = \{x^* \in X^*\colon\ |\langle x^*, x\rangle| \leqslant 1,\ \text{for all } x \in S\}$. Observe that $p_{S^\circ}(x^*) = \sup\{|x^*(s)|\colon\ s \in S\}$. A Banach space X is called *weakly compactly generated* (WCG, in short) if there is a weakly compact set $K \subset X$ so that the closed linear hull of K equals X. Let $(X, \|\cdot\|)$ be a Banach space. The norm $\|\cdot\|$ is called *rotund* (also called *strictly convex*) if $x = y$ whenever $\|x\| = \|y\| = \|(x+y)/2\| = 1$. The norm $\|\cdot\|$ is called *locally uniformly rotund* (in short LUR) if $\|x_n - x\| \to 0$ whenever $x_n, x \in S_X$ are such that $\|x_n + x\| \to 2$. The norm $\|\cdot\|$ is called *weakly uniformly rotund* (in short WUR) if $x_n - y_n \to 0$ in the weak topology of X whenever $x_n, y_n \in S_X$ are such that $\|x_n + y_n\| \to 2$. Note that it follows from Šmulyan's lemma that a norm is WUR if, and only if, its dual norm is uniformly Gâteaux differentiable (see, e.g., [1, Theorem II.6.7]). A Banach space X is called an *Asplund space* if every separable subspace of X has separable dual. For other concepts not defined here, we refer, e.g., to [3].

2. A modification of Klee's construction

Let $(X, |\cdot|_0)$ be a Banach space such that $|\cdot|_0^*$ is rotund. Fix $x_0 \in X$ such that $|x_0|_0 = 1$ and let x_0^* be the (unique) element in X^* such that $|x_0^*|_0^* = 1$ and $\langle x_0^*, x_0\rangle = 1$. (See Fig. 1.)

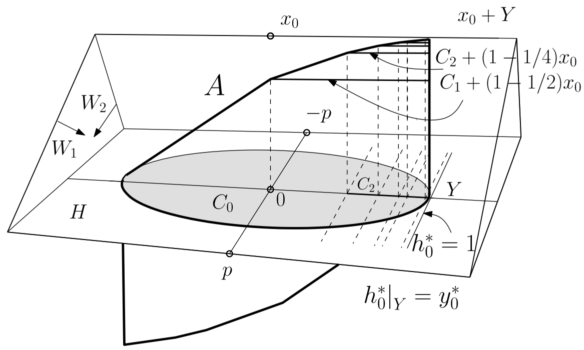

Fig. 1. Construction of the set A.

Let $H := \{x \in X\colon\ \langle x_0^*, x\rangle = 0\}$ (a closed hyperplane of X), and let Y be a closed hyperplane of H. Observe that $X = H \oplus \operatorname{span}\{x_0\}$ (both algebraically and topologically). Let $P : X \to H$ and $Q : X \to \operatorname{span}\{x_0\}$ be the corresponding canonical projections onto H and $\operatorname{span}\{x_0\}$, respectively, associated to the decomposition $X = H \oplus \operatorname{span}\{x_0\}$.

The norm $\|\cdot\|$

We may define an equivalent norm $\|\cdot\|$ on X by the formula

$$\|x\|^2 := |Px|_0^2 + |Qx|_0^2, \quad \text{for all } x \in X. \tag{1}$$

Observe that $\|\cdot\|$ agrees with $|\cdot|_0$ both on H and on $\operatorname{span}\{x_0\}$. This new norm is introduced to make some of the forthcoming computations easier. It is simple to check that

$$\left(\|x^*\|^*\right)^2 := \left(\left|x^*\restriction_H\right|_0^*\right)^2 + \left(\left|x^*\restriction_{\operatorname{span}\{x_0\}}\right|_0^*\right)^2, \quad \text{for all } x^* \in X^*. \tag{2}$$

The sets A and B, and the norm $|\!|\!|\cdot|\!|\!|$

Let $p \in H$ be such that $\operatorname{dist}(p, Y) \geqslant 2$. Denote by x_1^* and x_2^* the continuous linear functionals in $Y^\perp$ ($\subset X^*$) defined by

$$\langle x_1^*, x_0\rangle = \langle x_1^*, p\rangle = 1, \quad \text{and} \quad \langle x_2^*, x_0\rangle = \langle x_2^*, -p\rangle = 1. \tag{3}$$

By using (2), it is simple to prove that

$$\left(\|x_1^*\|^*\right)^2 = \left(\|x_2^*\|^*\right)^2 = \frac{1}{\operatorname{dist}(p, Y)^2} + 1 =: M^2. \tag{4}$$

The set $x_0 + Y$ together with the point p define a translate of a hyperplane, precisely $(x_1^*)^{-1}(1)$. Put $W_1 := (x_1^*)^{-1}(-\infty, 1]$. Analogously, $x_0 + Y$ together with $-p$ define a translate of a hyperplane, precisely $(x_2^*)^{-1}(1)$. Put $W_2 := (x_2^*)^{-1}(-\infty, 1]$.

Proposition 3. *There exists a bounded symmetric closed convex body B in X such that $B \subset W_1 \cap W_2$, $\operatorname{dist}(x_0 + Y, B) = 0$, and $(x_0 + Y) \cap B = \emptyset$.*

Proof. The construction of B is done in two steps. First, since Y is not reflexive, we may find, by James' weak compactness theorem, an element

$$y_0^* \in S_{(Y^*, \|\cdot\|^*)} \tag{5}$$

not attaining its norm on $B_{(Y,\|\cdot\|)}$. For $n \in \mathbb{N}$ let $C_n := \{y \in B_{(Y,\|\cdot\|)}\colon\ \langle y_0^*, y\rangle \geqslant 1 - 1/n\}$. We obtain in this way a decreasing sequence $\{C_n\}$ of closed convex subsets of $B_{(Y,\|\cdot\|)}$ with the property that $\bigcap_{n=1}^\infty C_n = \emptyset$.

Put $C_0 := B_{(H,\|\cdot\|)}$ and let (see Fig. 1)

$$A := \overline{\Gamma}\left(\bigcup_{n=0}^{\infty}\left(C_n + \left(1 - 2^{-n}\right)x_0\right)\right).$$

This set is bounded, closed and absolutely convex. It is clear that A has a nonempty interior. Moreover, $(x_0 + Y) \cap A = \emptyset$. This can be seen as follows. Assume that for some $y \in Y$ we have $x := x_0 + y \in A$. Then $\langle x_0^*, x\rangle = 1$. Find a sequence $\{x_n\}$ in $\Gamma(\bigcup_{n=0}^\infty (C_n + (1 - 2^{-n})x_0))$ that converges to x. For $n \in \mathbb{N}$, put $x_n = \sum_{i=0}^{m_n} \gamma_{n,i}(c_{n,i} + (1 - 2^{-i})x_0)$, where $c_{n,i} \in C_i$ for all

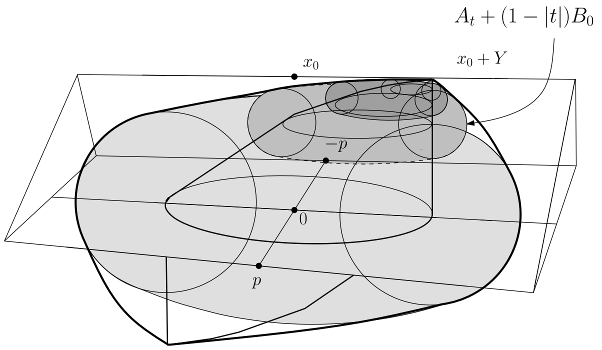

Fig. 2. Construction of the set B.

$i = 0, 1, 2, \dots, m_n$ and $\sum_{i=0}^{m_n} |\gamma_{n,i}| \leqslant 1$. Note that

$$\langle x_0^*, x_n\rangle = \sum_{i=0}^{m_n} \gamma_{n,i}\big(1 - 2^{-i}\big)\langle x_0^*, x_0\rangle = \sum_{i=0}^{m_n} \gamma_{n,i}\big(1 - 2^{-i}\big) \to 1, \quad \text{as } n \to \infty. \tag{6}$$

Fix $k \in \mathbb{N}$ and observe that, due to (6),

$$\sum_{\{i \leqslant m_n:\ i \leqslant k \text{ or } \gamma_{n,i} < 0\}} |\gamma_{n,i}| \to 0, \quad \text{as } n \to \infty.$$

This allows us to introduce small perturbations $\gamma'_{n,i}$ of the coefficients $\gamma_{n,i}$ in such a way that we have $\gamma'_{n,i} \geqslant 0$ for all $i = 0, 1, 2, \dots, m_n$, $n \in \mathbb{N}$, $\sum_{i=0}^{m_n} \gamma'_{n,i} = 1$, and, by passing eventually to a subsequence, $d_n \in C_n$ and $\{d_n\}_{n=1}^{\infty}$ converges, where $d_n := \sum_{i=0}^{m_n} \gamma'_{n,i} c_{n,i}$ for all $n \in \mathbb{N}$. This contradicts the fact that $\bigcap_{n=1}^{\infty} C_n = \emptyset$.

For the second step, let $A_t := A \cap (x_0^*)^{-1}(t)$ for $t \in (-1, 1)$. Let

$$B_0 := \frac{1}{2M} B_{(X, \|\cdot\|)}, \tag{7}$$

where M was defined in (4).

Put (see Fig. 2)

$$B := \bigcup_{t \in (-1,1)} \overline{A_t + \big(1 - |t|\big)B_0}, \tag{8}$$

where B_0 was defined in (7).

We claim that $B \subset W_1 \cap W_2$.

In order to prove the claim, observe first that $\ker(x_1^*\restriction_H) = Y$, hence $2 \leqslant \operatorname{dist}(p, Y) = |x_1^*(p)|/\|x_1^*\restriction_H\| = 1/\|x_1^*\restriction_H\|$, and so

$$\big\|x_1^*\restriction_H\big\| \leqslant 1/2. \tag{9}$$

Let $x = \sum_{i=0}^{m} \gamma_i(c_i + (1 - 2^{-i})x_0)$, where $c_i \in C_i$ for $i = 0, 1, \dots, m$. Fix $t \in (0, 1)$. Assume that $t = x_0^*(x)$. Then $t = \sum_{i=0}^{m} \gamma_i(1 - 2^{-i}) = \sum_{i=1}^{m} \gamma_i(1 - 2^{-i})$.

Let $y \in (1 - |t|)B_0$. Note that $|x_1^*(y)| \leqslant \frac{1}{2}(1 - |t|)$. Moreover, in view of (9), we have $|x_1^*(c_0)| \leqslant \frac{1}{2}$.

Additionally, $|\gamma_0| + |t| \leqslant |\gamma_0| + \sum_{i=1}^{m} |\gamma_i| = \sum_{i=0}^{m} |\gamma_i| \leqslant 1$. Thus

$$\big|x_1^*(x + y)\big| \leqslant \big|\gamma_0 x_1^*(c_0) + t\big| + \frac{1}{2}\big(1 - |t|\big) \leqslant \frac{1}{2}\big(1 - |t|\big) + |t| + \frac{1}{2}\big(1 - |t|\big) = 1.$$

To finish, we pass to closures. This shows that $B \subset W_1$. The same argument applies to x_2^* and so $B \subset W_2$. This proves the claim.

That B has a nonempty interior is clear, since it contains A. To check that B is convex and symmetric is easy; it is enough to deal with elements in sets of the form $A_t + (1 - |t|)B_0$, $t \in (-1, 1)$.

Let us prove now that B is indeed closed. To this end, let $x \in \overline{B}$, and let $\{x_n\}$ be a sequence in B that converges to x. For $n \in \mathbb{N}$, let $t_n \in (-1, 1)$ be such that $x_n \in \overline{A_{t_n} + (1 - |t_n|)B_0}$. Without loss of generality we may assume that $x_n \in A_{t_n} + (1 - |t_n|)B_0$, say $x_n = a_{t_n} + (1 - |t_n|)b_n$, where $a_{t_n} \in A_{t_n}$ and $b_n \in B_0$ for all $n \in \mathbb{N}$, and that $\{t_n\}$ converges to some $t \in [-1, 1]$.

We shall consider two cases.

1. Suppose first that $t \in (-1,1)$. If $t_n \leqslant t$ infinitely often, we may assume that $t_n \leqslant t$ for all $n \in \mathbb{N}$, and we fix $z \in A$ such that $\langle x_0^*, z\rangle > t$. Otherwise, we may assume that $t_n > t$ for all $n \in \mathbb{N}$, and we fix $z \in A$ such that $\langle x_0^*, z\rangle < t$. For $n \in \mathbb{N}$ and $\lambda_n \in [0,1]$, put $y_n := \lambda_n a_{t_n} + (1-\lambda_n)z$ in such a way that $\langle x_0^*, y_n\rangle = t$. This implies that $\lambda_n t_n + (1-\lambda_n)x_0^*(z) = t$ for all $n \in \mathbb{N}$, so $\lambda_n \to 1$. The element y_n, as a convex combination of two elements in A, belongs to A, too, so it belongs to A_t. It follows that $y_n + (1-|t|)b_n \in A_t + (1-|t|)B_0$ for $n \in \mathbb{N}$. As it is easy to show, the sequence $\{y_n + (1-|t|)b_n\}$ converges to x, so $x \in \overline{A_t + (1-|t|)B_0}(\subset B)$.
2. Suppose now that $t \in \{-1,1\}$, say $t = 1$. It follows that $a_{t_n} \to x$, so $x \in A$, and $\langle x_0^*, a_{t_n}\rangle \to \langle x_0^*, x\rangle$, hence $\langle x_0^*, x\rangle = 1$. By the first part of the proof, this is a contradiction with the fact that $\bigcap_{n=1}^{\infty} C_n = \emptyset$. The argument for $t = -1$ is similar. $\square$

Define an equivalent norm $|||\cdot|||$ on X by

$$|||\cdot||| := p_B, \tag{10}$$

where p_B is the Minkowski functional of the set B defined in (8).

Some more constructions

The norm $|||\cdot|||$ on X defined in (10) has the property that x_1^* and x_2^* introduced in (3) define two distinct supporting hyperplanes to $B_{(X/Y,|||\cdot|||)}$ at $x_0 + Y$, hence the dual norm $|||\cdot|||^*$ is not rotund. This was the conclusion reached in [7, Proposition 3.3]. To be a little bit more precise, observe that

$$\left[x_1^*, x_2^*\right] \subset S_{(X^*,|||\cdot|||^*)}. \tag{11}$$

This can be seen as follows:

(i) Since $B \subset W_1 \cap W_2$, we get $|||x_i^*|||^* \leqslant 1$, $i = 1,2$.
(ii) $x_0^* = \frac{1}{2}(x_1^* + x_2^*)(\in B_{(X^*,|||\cdot|||^*)})$.
(iii) For $n \in \mathbb{N}$, put

$$w_n := c_n + \left(1 - 2^{-n}\right)x_0, \tag{12}$$

where $c_n \in C_n$. Then $w_n \in A \subset B$, hence $|||w_n||| \leqslant 1$, for all $n \in \mathbb{N}$, and $x_0^*(w_n) = 1 - 2^{-n} \to 1$ as $n \to \infty$. This shows that $|||x_0^*||| \geqslant 1$. All together, we have proved (11).

The norm $|\cdot|$

Our last step in the construction of the sought norm is to use the equation

$$|\cdot|^2 = |||\cdot|||^2 + \|\cdot\|^2 \tag{13}$$

to define a new equivalent norm $|\cdot|$ on X. This is the norm with which we will test the announced result and its variants.

3. Proof of Theorem 1 (and of Corollary 2)

We prove here the main result of this note.

Proof of Theorem 1, part (a). First of all, every weakly compactly generated space admits an equivalent norm that is LUR and its dual norm is rotund (see, e.g., [1, Theorems II.4.1, VII.I.16 and Corollary VII.1.11]). This will be the norm $|\cdot|_0$ to start with in the construction done in Section 2.

From (2) it follows, by a standard convexity argument, that $\|\cdot\|$ is LUR and that $\|\cdot\|^*$ is rotund. By Šmulyan's lemma, $\|\cdot\|$ is Gâteaux differentiable.

Let us show that $|||\cdot|||$ defined in (10) is Gâteaux differentiable, too. To this end, assume that x^* and y^* are two non-zero functionals in X^* that support B at some point $x \in B$. By the definition of B in (8), there exists $t \in (-1,1)$ such that $x \in \overline{A_t + (1-|t|)B_0}$. Then x^* and y^* support $\overline{A_t + (1-|t|)B_0}$ at x. Since

$$(p_t :=)\ p_{(\overline{A_t+(1-|t|)B_0})^\circ} = p_{(A_t)^\circ} + p_{((1-|t|)B_0)^\circ}, \tag{14}$$

and $p_{((1-|t|)B_0)^\circ}$ is rotund, so is p_t, and we get $x^* = y^*$. This proves that $|||\cdot|||$ is Gâteaux differentiable.

It is straightforward then that $|\cdot|$, defined in (13), is Gâteaux differentiable, too. It is also LUR (see, e.g., [1, Fact II.2.3]).

In order to prove that $|\cdot|^*$ is not rotund we need some basic facts and some (easy) computations, that we record below for the sake of completeness. First of all, if $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are two Banach spaces, and

$$\left(X, \|\cdot\|\right) := \left(X_1, \|\cdot\|_1\right) \oplus_2 \left(X_2, \|\cdot\|_2\right),$$

then $(X^*, \|\cdot\|^*)$ is isometric to $(X_1^*, \|\cdot\|_1^*) \oplus_2 (X_2^*, \|\cdot\|_2^*)$. The isometry

$$\varphi : \left(X_1^*, \|\cdot\|_1^*\right) \oplus_2 \left(X_2^*, \|\cdot\|_2^*\right) \to \left(X^*, \|\cdot\|^*\right)$$

is given by

$$\varphi\left(x_1^*, x_2^*\right)(x_1, x_2) = \left\langle x_1^*, x_1\right\rangle + \left\langle x_2^*, x_2\right\rangle, \quad \text{for } x_1 \in X_1,\ x_2 \in X_2,\ x_1^* \in X_1^*,\ x_2^* \in X_2^*.$$

We shall identify from now on the two spaces $(X_1^*, \|\cdot\|_1^*) \oplus_2 (X_2^*, \|\cdot\|_2^*)$ and $(X^*, \|\cdot\|^*)$.

Consider, as a particular case, the two Banach spaces $(X, |\!|\!|\cdot|\!|\!|)$ and $(X, \|\cdot\|)$ defined above, and let $(Z, \|\cdot\|_2) := (X, \|\cdot\|) \oplus_2 (X, |\!|\!|\cdot|\!|\!|)$. Denote by Δ the diagonal of $X \times X$. Certainly, the space $(\Delta, \|\cdot\|_2)$ is isometric, via the mapping $D : \Delta \to X$ given by $D(x, x) = x$ for all $x \in X$, to the space $(X, |\cdot|)$, where $|\cdot|$ has been defined in (13); thus, $D^* : (X^*, |\cdot|^*) \to (Z^*, \|\cdot\|_2^*)/\Delta^{\perp}$ is again an isometry. Note that $(Z^*, \|\cdot\|_2^*) = (X^*, |\!|\!|\cdot|\!|\!|^*) \oplus_2 (X^*, \|\cdot\|^*)$. For $x^* \in X^*$, and being D^*x^* an element of a quotient space, we have

$$\left|x^*\right|^* = \left\|D^*x^*\right\|_2^* = \inf\left\{\left\|\left(z_1^*, z_2^*\right)\right\|_2^*\colon z_1^*, z_2^* \in Z^*,\ q\left(z_1^*, z_2^*\right) = D^*x^*\right\}, \tag{15}$$

where $q : Z^* \to Z^*/\Delta^{\perp}$ is the canonical quotient mapping. Observe, too, that $q(z_1^*, z_2^*) = D^*x^*$ if, and only if, $z_1^* + z_2^* = x^*$. So, (15) becomes

$$\left|x^*\right|^* = \inf\left\{\left(\left(\left\|z_1^*\right\|^*\right)^2 + \left(|\!|\!|z_2^*|\!|\!|^*\right)^2\right)^{1/2}\colon z_1^*, z_2^* \in Z^*,\ z_1^* + z_2^* = x^*\right\}. \tag{16}$$

Let $h_0^* (\in H^*)$ be a Hahn–Banach extension of y_0^* (recall that y_0^* was introduced in (5)) to H (this extension is unique, by a result of Phelps [11], although this is irrelevant here). Define an extension $z_0^* \in X^*$ of h_0^* to X by letting $\langle z_0^*, x_0\rangle = 0$. Observe that $\|z_0^*\|^* = 1$. Put

$$u^* := \frac{1}{\sqrt{2}}\left(x_0^* + z_0^*\right). \tag{17}$$

By using (2), we get

$$\left\|u^*\right\|^* = 1. \tag{18}$$

Put

$$v_i^* := \frac{x_i^* + \sqrt{2}u^*}{\sqrt{3}}\left(= \frac{x_i^* + x_0^* + z_0^*}{\sqrt{3}}\right), \quad \text{for } i = 1, 2. \tag{19}$$

We shall show that

$$\left[v_1^*, v_2^*\right] \subset S_{(X^*, |\cdot|^*)}. \tag{20}$$

This is a consequence of the following three observations:

(i) According to (11), (16), (18), and (19), we have

$$\left(\left|v_i^*\right|^*\right)^2 \leqslant \frac{(|\!|\!|x_i^*|\!|\!|^*)^2}{3} + \frac{2(\|u^*\|^*)^2}{3} = 1, \quad i = 1, 2,$$

so $[v_1^*, v_2^*] \subset B_{(X^*, |\cdot|^*)}$.

(ii) For w_n as in (12), note that

$$\begin{aligned} |w_n|^2 &= |\!|\!|w_n|\!|\!|^2 + \|w_n\|^2 \\ &\leqslant 1 + \|c_n\|^2 + \left\|\left(1 - 2^{-n}\right)x_0\right\|^2 \leqslant 1 + 1 + \left(1 - 2^{-n}\right)^2 \leqslant 3, \end{aligned}$$

hence $|w_n| \leqslant \sqrt{3}$ for all $n \in \mathbb{N}$.

(iii) Observe, too, that $\frac{1}{2}(v_1^* + v_2^*) = 2x_0^* + z_0^*$. Thus,

$$\begin{aligned} \frac{1}{2}\left(v_1^* + v_2^*\right)(w_n) &= \frac{1}{\sqrt{3}}\left(2x_0^* + z_0^*\right)\left(c_n + \left(1 - 2^{-n}\right)x_0\right) \\ &= \frac{1}{\sqrt{3}}\left(\left(1 - 2^{-n}\right)2x_0^*(x_0) + z_0^*(c_n)\right) \geqslant \frac{1}{\sqrt{3}}\left(2\left(1 - 2^{-n}\right) + \left(1 - n^{-1}\right)\right) \to \sqrt{3}. \end{aligned}$$

All together, we get (20), showing that $|\cdot|^*$ is not rotund. $\square$

Proof of Theorem 1, part (b). First of all, any Asplund weakly compactly generated Banach space admits an equivalent norm that is, together with its dual norm, LUR ([2], see also, e.g., [1, Theorem VII.1.14]). This will be now the norm $|\cdot|_0$ to start with in the construction done in Section 2.

By (2) and a standard convexity argument, it follows that both norms $\|\cdot\|$ and $\|\cdot\|^*$ are also LUR (in particular, $\|\cdot\|$ is Fréchet differentiable).

Let us show that $|||\cdot|||$ is Fréchet differentiable, too. Observe first that the rotund dual norm p_t on X^* defined in (14) has the property that w^* and the p_t-topology coincide on the unit sphere defined by p_t. Indeed, since A_t is bounded, $p_{A_t^\circ}$ is a $\|\cdot\|^*$-continuous seminorm on X^* and, by assumption, $p_{((1-|t|)B_0)^\circ}$ is an equivalent LUR norm on X^*. It is routine to check that any net $\{x^*_\alpha\} \subset S_{(X^*,\ p_t)}$ such that $x^*_\alpha \xrightarrow{\omega^*} x^* \in S_{(X^*,p_t)}$, will satisfy $p_{((1-|t|)B_0)^\circ}(x^*_\alpha - x^*) \to 0$. Therefore, $p_t(x^*_\alpha - x^*) \to 0$. Behind is the fact that, if the dual norm of a Banach space X is LUR, then X^* has the so-called w^*-Kadec–Klee property (see, e.g., [3, Exercise 8.45]).

Now, take $x \in B$ such that $|||x||| = 1$. Let $x^* \in X^*$ be such that $|||x^*|||^* = 1$ and $x^*(x) = 1$. For $n \in \mathbb{N}$, let $x^*_n \in X^*$ be such that $|||x^*_n|||^* = 1$ and $x^*_n(x) \to 1$. There exists $t \in (-1,1)$ such that $x \in \overline{A_t + (1-|t|)B_0}$. By convexity, we deduce that $p_t(x^*_n) \to 1$. Since p_t is rotund, its predual norm is Gâteaux differentiable and, by Šmulyan's lemma (see, e.g., [1, Theorem 1.4]), $(x^* - x^*_n) \xrightarrow{w^*} 0$. Since w^* and the p_t-topology coincide on the unit sphere defined by p_t, we deduce that $p_t(x^* - x^*_n) \to 0$, so $|||x^*_n - x^*|||^* \to 0$. The Fréchet differentiability of $|||\cdot|||$ at x follows by using Šmulyan's lemma again.

Since $\|\cdot\|$ and $|||\cdot|||$ are Fréchet differentiable, we may assert that $|\cdot|$ defined in (13) is also Fréchet differentiable. It is also LUR, due to the way it was defined and the fact that $\|\cdot\|$ is LUR. That $|\cdot|^*$ is not rotund was shown in the proof of Theorem 1, part (a).

To prove the statement on coincidence of the topologies, let $q : (Z^*, \|\cdot\|_2^*) \to (X^*, |\cdot|^*)$ be the canonical quotient mapping (see the construction at the fourth paragraph in the proof of part (a)). Assume that $\{x^*_i\}_{i\in I}$ is a net in $S_{(X^*,|\cdot|^*)}$ that w^*-converges to an element $x^* \in S_{(X^*,|\cdot|^*)}$. Choose elements $z^*_i \in S_{(Z^*,\|\cdot\|_2^*)}$ such that $q(z^*_i) = x^*_i$ for $i \in I$. Take an arbitrary subnet $\{z^*_{i_j}\}_{j\in J}$ of $\{z^*_i\}_{i\in I}$; it has a w^*-cluster point $z^* \in B_{(Z^*,\|\cdot\|_2^*)}$. Since $q(z^*) = x^*$, we get $\|z^*\|_2^* \geqslant 1$, hence $\|z^*\|_2^* = 1$ and so z^* is a Hahn–Banach extension of x^* to Z. Since $(Z^*, \|\cdot\|_2^*)$ is rotund (it is even LUR, see above in this proof), this extension is unique ([11], see also, e.g., [3, Exercise 7.69]). It follows that the net $\{z^*_i\}_{i\in I}$ is w^*-convergent to z^*. Due to the fact that $(Z^*, \|\cdot\|_2^*)$ is LUR, we get $\|z^*_i - z^*\|_2^* \to 0$ (see again, e.g., [3, Exercise 8.45]), hence $|x^*_i - x^*|^* \to 0$. □

Proof of Theorem 1, part (c). This follows from the fact (see, e.g., [1, Theorem II.7.1 (ii)]) that every Banach space with a separable dual has an equivalent LUR and WUR norm $|\cdot|_0$ such that $|\cdot|_0^*$ is LUR.

Then the desired properties are possessed by the norm $|\cdot|$ defined in (13) thanks to the way $\|\cdot\|$, $|||\cdot|||$, and $|\cdot|$, were defined, the use of [1, Propositions II.1.2 and II.1.3] for the LUR and rotundity properties respectively, and [1, Proposition II.6.2] for the WUR property. □

Proof of Corollary 2. If X is reflexive and $|\cdot|_0$ is an equivalent Gâteaux differentiable norm on X, then its dual norm is rotund by Šmulyan's lemma. Assume now that X is not reflexive. If $|\cdot|_0$ is a Gâteaux differentiable norm on X whose dual norm is not rotund, we are done. If, on the contrary, $|\cdot|_0^*$ is rotund, then (following the notation in the proof of Theorem 1), the norm $\|\cdot\|^*$ is also rotund, hence $\|\cdot\|$ is Gâteaux differentiable. The rest is the same as the proof of Theorem 1. □

Remarks.

1. By using the same method of proof, the following extension of Theorem 1 can be proved:

   Let $(X, |\cdot|_0)$ be a nonreflexive Banach space.

   (a) *If $|\cdot|_0^*$ is rotund, then there exists an equivalent Gâteaux differentiable norm $|\cdot|$ on X such that its dual norm on X^* is not rotund. If, in addition, X has a norm that is rotund, then $|\cdot|$ can even be taken to be rotund.*

   (b) *If $|\cdot|_0^*$ is LUR, then there exists an equivalent Fréchet differentiable and LUR norm $|\cdot|$ on X such that $|\cdot|^*$ is not rotund. Moreover, the norm and w^* topologies agree on $S_{(X^*,|\cdot|^*)}$.*

   To show (a), note that (i) in case the Banach space X has a dual rotund norm, then the set of all equivalent norms on X having a rotund dual norm is residual in the space of all equivalent norms on X (endowed with the metric of uniform convergence on the unit ball of X, a Baire space, see, e.g., [1, Section II.4]), and (ii) if X has a rotund norm, then the set of all rotund equivalent norms on X is residual in the space of all equivalent norms on X (for both results, see, e.g., [1, Theorem II.4.1]). Therefore, we may start the construction of the norm $|\cdot|$ from a norm $|\cdot|_0$ that is, simultaneously, rotund and having a rotund dual norm. It is clear that $|\cdot|$ so constructed is, also, rotund.

   Part (b) follows from the result of Haydon quoted in Remark 2. Indeed, we may assume then that both norms $|\cdot|_0$ and $|\cdot|_0^*$ are LUR. By (2) and a standard convexity argument, it follows that both norms $\|\cdot\|$ and $\|\cdot\|^*$ are also LUR (in particular, $\|\cdot\|$ is Fréchet differentiable). The rest of the proof is the same as to the proof of Theorem 1.

   Part (a) of the extension stated above applies, for example, to the class of weakly countably determined Banach spaces (see, e.g., [1, Theorems VII.1.16 and II.4.1]), since those spaces have always an equivalent norm that is LUR and such that its dual norm is rotund.

2. Note that Haydon showed in [6] that a Banach space X admits an equivalent LUR norm such that its dual norm is again LUR whenever X admits an equivalent norm whose dual is LUR. In [5], it is also proved that there exists a Banach space X such that the dual norm is rotund although no rotund equivalent norm can be found on X.
3. Observe that, by modifying the basic construction slightly, we may conclude that, in a nonreflexive Banach space X which admits an equivalent LUR (rotund) dual norm, the set of norms on X that are simultaneously Fréchet (respectively Gâteaux) differentiable, and that have a dual non-rotund norm, is dense in the set of all equivalent norms on X.
Indeed, assume first that $(X, |\cdot|_0)$ is a nonreflexive Banach space such that $|\cdot|_0^*$ is rotund. As it was mentioned in Remark 1, the set of equivalent norms on X whose dual norms are rotund is residual in the set of all equivalent norms on X endowed with the metric ρ of uniform convergence on the unit ball of the space. In particular, given an arbitrary equivalent norm $|\cdot|_1$ on X and $\varepsilon > 0$, we may find an equivalent norm (call it again $|\cdot|_0$) such that $\rho(|\cdot|_0, |\cdot|_1) < \varepsilon$ and that its dual norm is rotund. This time, instead of defining the norm $\|\cdot\|$ by using the projections P and Q (see Section 2), we just put $\|\cdot\| := |\cdot|_0$. Now we can build, instead of B, a set B_ε with the same properties there and such that $(1-\varepsilon)B_{(X,\|\cdot\|)} \subset B_\varepsilon \subset (1+\varepsilon)B_{(X,\|\cdot\|)}$. For this, B_ε should be constructed (we follow the notation in Section 2) by letting $C_n := \{y \in (\varepsilon/2)B_{(Y,\|\cdot\|)}\colon \langle y_0^*, y\rangle \geqslant (\varepsilon/2) - 1/n\}$, for $n \in \mathbb{N}$ big enough, putting
$$A_\varepsilon := \overline{\Gamma}\big(A \cup (1-\varepsilon)B_{(X,\|\cdot\|)}\big),$$
and
$$B_\varepsilon := \bigcup_{t\in(-1,1)} \overline{A_{\varepsilon,t} + \varepsilon\big(1-|t|\big)B_0},$$
where $A_{\varepsilon,t} = A_\varepsilon \cap (x_0^*)^{-1}(t)$, for $t \in (-1,1)$.
Observe that B_ε is not necessarily included in $W_1 \cap W_2$. However, for $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that the continuous functionals $x_{1,\varepsilon}^*$ and $x_{2,\varepsilon}^*$ in $Y^\perp$, given by $x_{1,\varepsilon}^*(x_0) = x_{2,\varepsilon}^*(x_0) = 1$ and $x_{2,\varepsilon}^*(-p) = x_{1,\varepsilon}^*(p) = n^{-1}$, define, analogously as on p. 460, sets W_1^ε and W_2^ε such that $B_\varepsilon \subset W_1^\varepsilon \cap W_2^\varepsilon$.
This set B_ε defines a norm $|||\cdot|||_\varepsilon := p_{B_\varepsilon}$. This is now the norm needed. The rest of the proof is similar to the former one. This time we do not obtain rotundity of $|||\cdot|||$.
For the Fréchet case, let us recall that in case that $(X^*, |\cdot|_0^*)$ is LUR, the set of equivalent norms in X that have a dual LUR norm is residual (see [1, Theorem II.4.1]). Since X has an equivalent LUR norm [6], the set of equivalent LUR norms in X is again residual [1, Theorem II.4.1]. An appeal to the Baire category theorem shows that the set of equivalent LUR norms in X that have a dual LUR norm is residual, too. This allows to take, given any equivalent norm $|\cdot|_1$ in X, an equivalent norm in this class (called again $|\cdot|_0$), as close to $|\cdot|_1$ as we wish, and start the construction above.
4. The results in this paper should be compared with the (simple) fact that if the norm $\|\cdot\|$ of X as well as its dual norm are both Fréchet differentiable, then the norm $\|\cdot\|$ as well as its dual norm are both LUR (see, e.g., [3, Exercise 8.5]).

Open problem *(S. Troyanski).* For the class of Banach spaces with (uncountable) unconditional basis, a characterization of those spaces admitting an equivalent norm whose dual norm is rotund was provided in [13]. It is not known whether a Banach space with an uncountable unconditional basis such that its norm is Gâteaux differentiable has an equivalent norm whose dual norm is rotund.

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