Journal of Differential Equations **174**, 200–236 (2001)
doi:10.1006/jdeq.2000.3936, available online at http://www.idealibrary.com on IDEAL®

Convergence to Strong Nonlinear Diffusion Waves for Solutions of p-System with Damping

Huijiang Zhao[1]

Young Scientist Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, People's Republic of China[2]; and Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2–4, 34014 Trieste, Italy

Received January 5, 2000; published online June 13, 2001

This paper is concerned with the p-system with frictional damping and our main [illegible]

$L^p(2 \leqslant p \leqslant \infty)$ decay rates, to the corresponding nonlinear diffusion wave $(\bar{v}(t, x), \bar{u}(t, x))$ governed by the classical Darcy's law provided that the corresponding prescribed initial error function $(V_0(x), U_0(x))$ lies in $(H^3 \times H^2)(\mathrm{R}) \cap (L^1 \times L^1)(\mathrm{R})$. Compared with former results in this direction obtained by L. Hsiao and T.-P. Liu (1992, *Comm. Math. Phys.* **143**, 599–605), K. Nishihara (1996, *J. Differential Equations* **131**, 171–188), and K. Nishihara, W.-K. Wang, and T. Yang (2000, *J. Differential Equations* **161**, 191–218), our main novelty lies in the facts that the nonlinear diffusion wave $(\bar{v}(t, x), \bar{u}(t, x))$ need not to be weak and $(V_0(x), U_0(x))$ can be chosen arbitrarily large. Secondly, we show that the nonlinear diffusion waves $(\bar{v}(t, x), \bar{u}(t, x))$ are nonlinear stable provided that the strength of the nonlinear diffusion waves is weak and that the initial disturbance $(V_0(x), U_0(x))$ satisfies the assumption that $\|V_{0xx}(x)\|_{L^\infty} + \|U_{0x}(x)\|_{L^\infty}$ is sufficiently small. We also show that the smallness assumption imposed on the strength of the diffusion waves is a necessary condition to guarantee the nonlinear stability result and compared with the corresponding results obtained by L. Hsiao and T.-P. Liu (1992, *Comm. Math. Phys.* **143**, 599–605), the smallness conditions we imposed on the initial disturbance are much more weaker.

1. INTRODUCTION AND THE STATEMENT OF OUR MAIN RESULTS

This paper is concerned with the Cauchy problem to the following hyperbolic conservation laws with frictional damping

[1] Supported in part by a Postdoctoral Fellowship of SISSA (1998–1999), a grant from National Natural Science Foundation of China, and a grant from the Chinese Academy of Sciences (Yin Jin Guo Wai Jie Chu Ren Cai Ji Jing).

[2] Current address. E-mail: hjzhao@wipm.whcnc.ac.cn.

0022-0396/01 $35.00

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(v)_x = -\alpha u, \qquad \alpha > 0, \qquad p'(v) < 0, \end{aligned} \tag{1.1}$$

with initial data

$$(v(t, x), u(t, x))|_{t=0} = (v_0(x), u_0(x)), \tag{1.2}$$

which satisfies

$$(v_0(x), u_0(x)) \to (v_\pm, u_\pm) \qquad \text{as} \qquad x \to \pm\infty. \tag{1.3}$$

Here $v(t, x)$, $v_\pm > 0$ and $u(t, x)$ represent the specific volume and velocity, respectively, the pressure $p(v)$ is assumed to be a smooth function of v on $v > 0$, and α is a positive constant. The system (1.1) can be viewed as the isentropic Euler equations in Lagrangian coordinates with frictional term $-\alpha u$ in the momentum equation. Thus it models the compressible flow through porous media. Since the commonly called porous media equation can be obtained by approximating the second equation with Darcy's law

$$\begin{aligned} v_t &= -\frac{1}{\alpha} p(v)_{xx}, \\ p(v)_x &= -\alpha u, \end{aligned} \tag{1.4}$$

it is natural to expect that Darcy's law can be obtained from the more complete Eq. (1.1) time-asymptotically. That is, solutions of (1.1) tend to those of (1.4) as the time tends to infinity.

The first rigorously mathematical justification of the above expectation was carried out by Hsiao and Liu in [5] (see also [3, 4, 6–8, 10, 13, 14, 16–19] for related subsequent results in this direction). Before stating their results precisely, following [5], we first remind the approach to this problem.

Denote the self-similar solution, the so-called nonlinear diffusion wave, of (1.4) in the form of $\varphi(x/\sqrt{t+1})$ by $\bar{v}(t, x)$ with the same end states as $v_0(x)$:

$$\bar{v}(0, \pm\infty) = v_\pm, \tag{1.5}$$

and set

$$\bar{u}(t, x) = -\frac{1}{\alpha} p(\bar{v}(t, x))_x. \tag{1.6}$$

To eliminate the values of $u(t, x)$ at $x = \pm\infty$, the following auxiliary functions are introduced in [5]

$$
\begin{aligned}
\hat{v}(t, x) &= -\frac{u_+ - u_-}{\alpha} \exp(-\alpha t)\, m_0(x), \\
\hat{u}(t, x) &= \exp(-\alpha t) \left(u_- + (u_+ - u_-) \int_{-\infty}^{x} m_0(y)\, dy \right),
\end{aligned}
\tag{1.7}
$$

where $m_0(x)$ is a smooth function with compact support and satisfies

$$
\int m_0(x)\, dx = 1.
$$

Under the above notations, if we let

$$
\begin{aligned}
V(t, x) &= \int_{-\infty}^{x} (v(t, y) - \bar{v}(t, y + x_0) - \hat{v}(t, y))\, dy, \\
U(t, x) &= u(t, x) - \bar{u}(t, x + x_0) - \hat{u}(t, x),
\end{aligned}
\tag{1.8}
$$

with x_0 uniquely determined by

$$
\int (v_0(x) - \bar{v}(0, x + x_0))\, dx = -\frac{u_+ - u_-}{\alpha}, \tag{1.9}
$$

then from (1.1)–(1.9), we can deduce that $(V(t, x), U(t, x))$ solves the following Cauchy problem

$$
\begin{aligned}
&V_t - U = 0, \\
&U_t + [p(V_x + \bar{v} + \hat{v}) - p(\bar{v})]_x + \alpha U = \frac{1}{\alpha} p(\bar{v})_{xt}, \\
&(V(t, x), U(t, x))|_{t=0} = (V_0(x), U_0(x)) \\
&\qquad\qquad = (V_0(x), V_t(0, x)) \to 0 \qquad \text{as} \quad x \to \pm\infty,
\end{aligned}
\tag{1.10}
$$

and the results obtained by Hsiao and Liu in [5] can be restated as

THEOREM 1.1 (Hsiao and Liu [5]). *If* $V_0(x) \in H^3(\mathbb{R})$, $U_0(x) \in H^2(\mathbb{R})$, *and*

$$
|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_3 + \|U_0\|_2 \leqslant \varepsilon_0 \tag{1.11}
$$

for some sufficiently small $\varepsilon_0 > 0$, *then there exists a global in time solution* $V(t, x) \in L^\infty([0, \infty), H^3(\mathrm{R}))$, $U(t, x) \in L^\infty([0, \infty), H^2(\mathrm{R}))$ *of* (1.10), *which satisfies*

$$\|(V_x, U)(t)\| + \|(V_x, U)(t)\|_{L^\infty} = O(1)\, \varepsilon_0 (1+t)^{-1/2}. \tag{1.12}$$

Recently, Nishihara [16] and Nishihara, Wang, and Yang [18] improved the estimates (1.12) [16] and obtained the optimal $L^p(2 \leqslant p \leqslant \infty)$ decay estimates [18], i.e.,

THEOREM 1.2 (Nishihara [16]; Nishihara, Wang, and Yang [18]). *In addition to the assumptions that* $V_0(x) \in H^3 \cap L^1(\mathrm{R})$, $U_0(x) \in H^2 \cap L^1(\mathrm{R})$, *if we assume further that*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_3 + \|V_0\|_{L^1} + \|U_0\|_2 + \|U_0\|_{L^1} \leqslant \varepsilon_0 \tag{1.13}$$

for some sufficiently small $\varepsilon_0 > 0$, *then there exists a global solution* $(V(t, x), U(t, x))$ *of* (1.10), *which satisfies*

$$\begin{aligned} V(t, x) &\in W^{\bar{k}, \infty}([0, \infty); H^{3-\bar{k}}(\mathrm{R})), \\ U(t, x) &\in W^{k, \infty}([0, \infty); H^{2-k}(\mathrm{R})) \end{aligned} \tag{1.14}$$

for $\bar{k} = 0, 1, 2, 3$; $k = 0, 1, 2$, *and*

$$\begin{aligned} \|\partial_x^k V_x(t)\|_{L^p} &= O(1)\, \varepsilon_0 (1+t)^{-(p-1)/2p - (k+1)/2}, \\ \|\partial_x^k U(t)\|_{L^p} &= O(1)\, \varepsilon_0 (1+t)^{-(p-1)/2p - (k+2)/2}, \end{aligned} \tag{1.15}$$

for any $k \leqslant 2$ *if* $p = 2$ *and* $k \leqslant 1$ *if* $p \in (2, +\infty]$.

It is easy to see that in the above two results, they all asked that both the initial error and the strength of the nonlinear diffusion waves under their considerations to be small. From this facts and the Sobolev's inequality, one can easily deduce that the C^1-norm of the corresponding initial data $(v_0(x), u_0(x))$ to the system (1.1) should also be small. These assumptions seem reasonable since although the damping term has some dissipative effect, it can not guarantee the smoothness of solutions of the corresponding Cauchy problem with arbitrarily large initial data. However, from the systematical study of the influence of the damping term on the formation of the shock waves to the solution of the corresponding Cauchy problem, especially those on the *p*-system with damping, (a complete literature in this direction is beyond the scope of this paper, however we want to mention [3, 4, 9, 15, 21, 22, 23, 24, 26, 27, 28] and the references cited therein), one can find out that the smallness assumptions imposed on the initial data can indeed be relaxed in certain sense while the corresponding global smooth solvability result can still be established. The motivation of this paper is based on the above observation and our main purpose is two-fold: First, we try to deal with the asymptotic behavior of solutions to (1.1), (1.2) with prescribed initial data, i.e.,we want to show that for certain class of given large initial data $(v_0(x), u_0(x))$, the Cauchy problem (1.1), (1.2) admits a unique global smooth solution $(v(t, x), u(t, x))$ and such a solution tends time-asymptotically, at the optimal $L^p(2 \leqslant p \leqslant \infty)$ decay rates, to the corresponding nonlinear diffusion wave $(\bar{v}(t, x), \bar{u}(t, x))$ governed by the classical Darcy's law provided that the corresponding initial error function $(V_0(x), U_0(x))$ lies in $(H^3 \times H^2)(\mathrm{R}) \cap (L^1 \times L^1)(\mathrm{R})$ but without any further smallness assumptions on the strength of the nonlinear diffusion wave and the initial error. Compared with former results in this direction obtained by Hsiao and Liu [5], Nishihara [16], and Nishihara, Wang, and Yang [18], this allows us to deal with the problem of the convergence to the strong nonlinear diffusion waves for solutions of p-system with damping.

Secondly, we consider the problem of the nonlinear stability of the nonlinear diffusion waves $(\bar{v}(t, x), \bar{u}(t, x))$ defined by $(1.4)_1$, (1.5), (1.6). To this end, we first give the following definition on the nonlinear stability of the diffusion waves $(\bar{v}(t, x), \bar{u}(t, x))$ defined above

DEFINITION 1.1. The nonlinear diffusion waves $(\bar{v}(t, x), \bar{u}(t, x))$ defined by $(1.4)_1$, (1.5), (1.6), which is uniquely determined by its two end values $v_\pm$, is called *nonlinear stable* if for each $(v_0(x), u_0(x))$ which is a perturbation of $(\bar{v}(0, x + x_0) + \hat{v}(0, x), \bar{u}(0, x + x_0) + \hat{u}(0, x))$, the Cauchy problem (1.1), (1.2) admits a unique global smooth solution $(v(t, x), u(t, x))$ and such a solution tends to $(\bar{v}(t, x + x_0), \bar{u}(t, x + x_0))$ as t tends to infinity.

For such a problem, the results obtained by Hsiao and Liu in [5] showed that the nonlinear diffusion waves defined above are nonlinear stable provided that the strength of the nonlinear diffusion waves is weak and the initial disturbance $(V_0(x), U_0(x))$ is small in $(H^3 \times H^2)(\mathrm{R})$ and our main purpose is trying to show that the smallness assumption imposed on the strength of the nonlinear diffusion waves is necessary to guarantee their nonlinear stability and that the same stability result can still be obtained but only under a rather weaker smallness assumption on the initial disturbance $(V_0(x), U_0(x))$.

Before stating our main results, we first list some further notations and assumptions.

Our first assumption is on the regularity property of $p(v)$, i.e., we assume that $p(v)$ satisfies one of the following two requirements

$$(\mathrm{H}_1)\quad p(v)\in C^4(0,\infty),\qquad p'(v)<0,\quad p''(v)>0,\quad 4p'(v)\,p'''(v)\geqslant 5(p''(v))^2,$$

for all $v>0$ and

$$(\mathrm{H}_2)\qquad \lim_{v\to 0}\int_v^1\sqrt{-p'(\tau)}\,d\tau=+\infty,\qquad \lim_{v\to\infty}p'(v)=0;$$
$$p(v)\in C^4(\mathrm{R}),\qquad p'(v)<0,$$

for all $v\in\mathrm{R}$.

It is easy to see that under the above assumptions, (1.1) is hyperbolic with its two eigenvalues are

$$\lambda(v)=-\sqrt{-p'(v)},\qquad \mu(v)=\sqrt{-p'(v)},\tag{1.16}$$

and the corresponding Riemann invariants are respectively

$$r(v,u)=u+\Phi(v),\qquad s(v,u)=u-\Phi(v),\tag{1.17}$$

where

$$\Phi(v)=\int_{\underline{v}}^{v}\mu(\tau)\,d\tau.\tag{1.18}$$

Here $\underline{v}\in(0,\infty)$ is any fixed constant.

Under the above notations, our result on the asymptotic behavior of solutions to the Cauchy problem (1.1), (1.2) with prescribed initial data $(v_0(x),u_0(x))$ can be surmmarized as in the following

THEOREM 1.3 (Asymptotic behavior of solutions). *Let the assumption* (H_1) (*respectively* (H_2)) *holds and for arbitrarily given positive constants* v_1, v_2 (*respectively* M_1) *and* M_2, *there exists a sufficiently small positive constant* M_3 *such that* $(r_0(x),s_0(x))\in C_b^1(\mathrm{R})$ *with*

$$\begin{aligned}&v_1\leqslant v_0(x)\leqslant v_2\qquad (\text{respectively } |v_0(x)|\leqslant M_1),\quad |u_0(x)|\leqslant M_2,\\&|r_0'(x)|\leqslant \alpha M_3,\qquad |s_0'(x)|\leqslant\alpha M_3,\end{aligned}\tag{1.19}$$

where

$$r_0(x)=u_0(x)+\Phi(v_0(x)),\qquad s_0(x)=u_0(x)-\Phi(v_0(x)).\tag{1.20}$$

Then the Cauchy problem (1.1), (1.2) *admits a unique global smooth solution* $(v(t,x), u(t,x))$.

Moreover, if $V_0(x)\in H^3(\mathrm{R})$, $U_0(x)\in H^2(\mathrm{R})$, *we can get that, as the time* t *goes to infinity, such a solution* $(v(t,x), u(t,x))$ *tends to the similarity solution* $(\bar{v}(t,x+x_0), \bar{u}(t,x+x_0))$ *of* (1.4) *and the following decay estimates hold*

$$\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2+\sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k U(t)\|^2+(1+t)^4\|\partial_t U(t)\|^2$$
$$+\int_0^t\left[\sum_{j=1}^{3}(1+\tau)^{j-1}\|\partial_x^j V(\tau)\|^2+\sum_{j=0}^{2}(1+\tau)^{j+1}\|\partial_x^j U(\tau)\|^2\right]d\tau$$
$$\leqslant O(1)(\|V_0\|_3^2+\|U_0\|_2^2+1), \tag{1.21}$$

and

$$(t+1)^5(\|U_{tt}(t)\|^2+\|U_{xt}(t)\|^2)+\int_0^t(\tau+1)^5\|U_{tt}(\tau)\|^2\,d\tau$$
$$\leqslant O(1)(\|V_0\|_3^2+\|U_0\|_2^2+1). \tag{1.22}$$

Furthermore, under the additional assumption that $V_0(x)\in L^1(\mathrm{R})$, $U_0(x)\in L^1(\mathrm{R})$, *the following optimal* $L^p(2\leqslant p\leqslant\infty)$ *decay estimates are true*

$$\begin{aligned}\|\partial_x^k V_x(t)\|_{L^p}&=O(1)(1+t)^{-(p-1)/2p-(k+1)/2},\\ \|\partial_x^k U(t)\|_{L^p}&=O(1)(1+t)^{-(p-1)/2p-(k+2)/2},\end{aligned} \tag{1.23}$$

for any $k\leqslant 2$ *if* $p=2$ *and* $k\leqslant 1$ *if* $p\in(2,+\infty]$.

Remark 1.1. Since we can easily construct smooth function $f(x)\in C_b^1(\mathrm{R})$, for example, for a sufficiently large constant $A>0$, we can take $f(x)$ as

$$f(x)=A\left(1-\int_{-\infty}^{x/A^2}g(y)\,dy\right)$$

with

$$g(x)\in C_0^\infty(\mathrm{R}),\qquad |g(x)|\leqslant 1,\qquad \int g(x)\,dx=1,$$

which satisfies the following properties

(i) $\lim_{x\to\pm\infty} f(x)=f_\pm$ exist and $f_\pm$ are finite;

(ii) $\|f'(x)\|_{L^\infty}$ is sufficiently small while $|f_+-f_-|$ can be chosen arbitrarily large, we can conclude that in our Theorem 1.3, $|v_+-v_-|$ can be chosen arbitrarily large and, hence, in our results, we can indeed deal with the strong nonlinear diffusion waves. Notice also that in our results, we do not ask the initial error $(V_0(x), U_0(x))$ to be small.

Remark 1.2. When the pressure function $p(v)$ satisfies the γ-law, the assumption (H_1) corresponds to the case $1\leqslant\gamma\leqslant 3$.

Remark 1.3. We will only prove Theorem 1.3 for the case when the assumption (H_1) is satisfied in the following, the other case is less complicated and the details will be omitted.

Now we turn to consider the nonlinear stability of the nonlinear diffusion waves $(\bar{v}(t,x), \bar{u}(t,x))$ defined by $(1.4)_1$, (1.5), (1.6). For results in this direction, we have

THEOREM 14 (Nonlinear stability result). *Suppose that $p(v)$ satisfies (H_2) and the initial disturbance $(V_0(x), U_0(x))$ lies in $(H^3\times H^2)(\mathbb{R})$, then the nonlinear diffusion waves $(\bar{v}(t,x), \bar{u}(t,x))$ defined by $(1.4)_1$, (1.5), (1.6) are nonlinear stable provided that the strength of the nonlinear diffusion waves is weak, i.e., $|u_+-u_-|+|v_+-v_-|$ is suitably small and that $\|V_{0xx}(x)\|_{L^\infty}+\|U_{0x}(x)\|_{L^\infty}$ is sufficiently small.*

Furthermore, if we assume further that $(V_0(x), U_0(x))$ belongs to $(L^1\times L^1)(\mathbb{R})$, the optimal $L^p(2\leqslant p\leqslant\infty)$ decay estimates (1.23) are also true.

Remark 1.4. In this remark, we show that to guarantee a nonlinear diffusion wave to be nonlinear stable, a necessary condition is that its strength should be small. To see this, we know from Definition 1.1 that for the Cauchy problem (1.1), (1.2), the global smooth solvability result must hold for each initial data $(v_0(x), u_0(x))$ belonging to a neighborhood of $(\bar{v}(0,x+x_0)+\hat{v}(0,x), \bar{u}(0,x+x_0)+\hat{u}(0,x))$ and consequently the case $(v_0(x), u_0(x))=(\bar{v}(0,x+x_0)+\hat{v}(0,x), \bar{u}(0,x+x_0)+\hat{u}(0,x))$ must be included. For such an initial data, from the study of formation of singularity of solutions of the Cauchy problem (1.1), (1.2) obtained by Lin and Zheng [9], Slemrod [21], and Zheng [26, 27], we conclude that a necessary condition to guarantee the global smooth solvability of the corresponding Cauchy problem is that $\|v_{0x}(x)\|_{L^\infty}+\|u_{0x}(x)\|_{L^\infty}$ should be small. Due to the fact that

$$\|v_{0x}(x)\|_{L^\infty}+\|u_{0x}(x)\|_{L^\infty}=O(1)(|u_+-u_-|+|v_+-v_-|),$$

we conclude that the strength of the nonlinear diffusion waves should be small.

Remark 1.5. Note that in [5], in addition to the assumption that the strength of the nonlinear diffusion waves is small, the authors also asked that the initial disturbance $(V_0(x), U_0(x))$ be small in $(H^3 \times H^2)(\mathrm{R})$. But in our Theorem 1.4, we only ask $\|V_{0xx}(x)\|_{L^\infty} + \|U_{0x}(x)\|_{L^\infty}$ to be small which is weaker than those needed in [5]. Thus in this sense, Theorem 1.4 improves the nonlinear stability results obtained in [5].

Remark 1.6. In Theorem 1.4, to simplify the presentation of result, we assume that $p(v)$ satisfies the assumption (H_2). In fact, similar result also holds when $p(v)$ satisfies the assumption (H_1).

Before concluding this section, we give the main ideas used in deducing our main results. Unlike those arguments developed by Hsiao and Liu [5] and Nishihara [16], we first consider the global smooth solvability of the Cauchy problem (1.1), (1.2) and deduce some time independent L^∞ *a priori* estimates on the global smooth solution obtained above. Based on the above results and some delicate energy estimates, we can get the decay estimates (1.21) and (1.22). Our main contribution in this paper is to get these estimates without any smallness conditions on the initial error and on the strength of the nonlinear diffusion waves. To give the main ideas, we will use more words to explain it in the following. In fact, compared with former arguments developed in [5, 16, 18], the main new ingredients in our analyses lie in the following:

(i) Unlike those in [5] and [16] in which they consider the energy estimates and the decay estimates (1.21) and (1.22) simutaneously, we first deduce certain energy estimates. Due to the time-independent L^∞ *a priori* estimates obtained in Corollary 2.1 and by employing the dissipative effect of the nonlinear diffusion waves fully, we succeed in getting the desired energy estimates in Theorem 3.2 without any smallness conditions on the initial error and on the strength of the nonlinear diffusion waves.

(ii) The second new ingredient in our analyses lies in how to get the decay estimates (1.21) and (1.22). Note that the analyses performed in [16] are based on the *a priori* assumption

$$N(T) := \sup_{0\leqslant t\leqslant T} \left\{ \sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(t)\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k U(t)\|^2 \right\} \leqslant \varepsilon_0 \tag{1.24}$$

with ε_0 sufficiently small. Since we do not ask the initial error and the strength of the nonlinear diffusion waves to be small, such a technique can not be used any longer. However, through some delicate energy estimates, we find that if we can get the following estimates

$$\begin{aligned} \|V_{xx}(t)\|_{L^\infty} &\leqslant O(1)(1+t)^{-1/2}, \\ \|V_{xt}(t)\|_{L^\infty} &\leqslant O(1)(1+t)^{-1}, \end{aligned} \tag{1.25}$$

then, after slight modifications, Nishihara's techniques can still be used to deduce the decay estimates (1.21) and (1.22) even without the *a priori* assumption (1.24). Based on the above observation and by employing the estimates (3.1.1), which are direct corollary of the energy estimates obtained in Theorem 3.2, we can indeed prove (1.25) and, hence, (1.21) and (1.22) follow from the techniques developed by Nishihara in [16].

(iii) The last new ingredient in our analyses lies in the way to get the optimal $L^p(2 \leqslant p \leqslant \infty)$ decay estimates (1.23). In [18], the authors used the approximate Green function $G(t, x; s, y)$ to get an integral representation of the solution to (1.10), then based on some elegant estimates, they can get

$$M(t) \leqslant O(1)(\varepsilon_0 + M^2(t)), \tag{1.26}$$

where

$$\begin{aligned} M(t) = &\sup_{p>2,\, 0\leqslant s\leqslant t,\, l+k\leqslant 2,\, l\leqslant 1} B_p^{l,k}(s)\, \|\partial_t^l \partial_x^k V(s)\|_{L^p} \\ &+ \sup_{0\leqslant s\leqslant t,\, l+k=3,\, l\leqslant 1} B_2^{l,k}(s)\, \|\partial_t^l \partial_x^k V(s)\|, \\ B_p^{l,k}(t) = &(1+t)^{(1/2)(1-1p)+l+k/2}, \\ &\varepsilon_0 = |u_+ - u_-| + |v_+ - v_-| + \|V_0\|_3 + \|U_0\|_2 + \|V_0\|_{L^1} + \|U_0\|_{L^1}, \end{aligned} \tag{1.27}$$

and their results follow provided that ε_0 is sufficiently small. Compared with [18], our new observation in this step is that by employing the same integral representation as in [18], we can deduce two formulas for $\partial_x^i V_x(t,x)$ and $\partial_x^k U(t,x)$ respectively, then by exploiting the decay estimates (1.21) and (1.22) obtained above and by mimicing the techniques developed in [18], we can obtain that the L^2-norm of each term on the righthand side of the two integral representations for $\partial_x^i V_x(t,x)$ and $\partial_x^k U(t,x)$ are bounded by $O(1)(1+t)^{-(2i+3)/4}$ and $O(1)(1+t)^{-(2k+5)/4}$ respectively. These results and Sobolev's inequality immediately deduce (1.23).

Finally, we point out that similar ideas have been used to study the nonlinear stability of the strong planar rarefaction waves to the solutions of the relaxation approximation to scalar conservation laws in several space dimensions [25] and we believe that they can also be used to study the convergence of the solutions of the following hyperbolic system with relaxation

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= \frac{1}{\varepsilon}(f(v) - u) \end{aligned} \tag{1.28}$$

to the corresponding suitably defined diffusion waves which are not necessary to be weak. Such a problem will be considered in a forthcoming paper.

Notations. Throughout this paper ε will always be used to denote a sufficiently small positive constant and the symbol $O(1)$ will be used to represent a generic constant which is independent of x and t, both of them may vary from line to line. $H^m(\mathrm{R})$ is the usual Sobolev space with the norm

$$\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\|, \qquad \|f\| = \|f\|_{L^2}.$$

Moreover, the integral domain R will be abbreviated without causing confusions.

2. PRELIMINARIES

2.1. *Properties of the Nonlinear Diffusion Wave*

In this subsection, we collect some fundamental properties of the nonlinear diffusion wave $\bar{v}(t, x)$ to $(1.4)_1$, (1.5). What we are interested in is the dissipative nature of the nonlinear diffusion wave, particularly the convergence rate of $\partial_x^k \varphi(x/\sqrt{t+1})$ at $x = \pm\infty$ and as $t \to \infty$.

Notice

$$\bar{v}(t, x) = \varphi\left(\frac{x}{\sqrt{t+1}}\right) \equiv \varphi(\xi), \qquad -\infty < \xi < \infty, \tag{2.1.1}$$

and $\varphi(\xi)$ satisfies

$$\begin{aligned} \varphi''(\xi) + \frac{p''(\varphi(\xi))\,\varphi'(\xi) - \frac{\alpha}{2}\xi}{p'(\varphi(\xi))}\varphi'(\xi) &= 0, \\ \varphi(\pm\infty) &= v_\pm, \end{aligned} \tag{2.1.2}$$

one has [5]

$$\sum_{k=1}^{4} |\partial_\xi^k \varphi(\xi)| + |\varphi(\xi) - v_+|_{\xi>0} + |\varphi(\xi) - v_-|_{\xi<0}$$
$$= O(1)\,|v_+ - v_-| \exp(-C_1 \alpha \xi^2), \tag{2.1.3}$$

where C_1 is a positive constant independent of t and x.

Furthermore, according to the form of the function $\bar{v}(t, x)$, we have

$$\bar{v}_x(t,x) = \frac{\varphi'(\xi)}{\sqrt{t+1}}, \qquad \bar{v}_t(t,x) = -\frac{\xi\varphi'(\xi)}{2(t+1)}, \qquad \bar{v}_{xx}(t,x) = \frac{\varphi''(\xi)}{t+1},$$

$$\bar{v}_{xt}(t,x) = \frac{\varphi'(\xi) + \xi\varphi''(\xi)}{2(t+1)\sqrt{t+1}}, \qquad \bar{v}_{tt}(t,x) = \frac{3\xi\varphi'(\xi) + \xi^2\varphi''(\xi)}{4(t+1)^2},$$

$$\bar{v}_{xtt}(t,x) = \frac{3\varphi'(\xi) + 5\xi\varphi''(\xi) + \xi^2\varphi'''(\xi)}{4(t+1)^2\sqrt{t+1}},$$

$$\bar{v}_{xxt}(t,x) = -\frac{\varphi''(\xi) + \xi\varphi'''(\xi)}{(t+1)^2}, \tag{2.14}$$

$$\bar{v}_{xxx}(t,x) = \frac{\varphi'''(\xi)}{(t+1)\sqrt{t+1}}, \qquad \bar{v}_{xxxt}(t,x) = -\frac{2\varphi'''(\xi) + \xi\varphi''''(\xi)}{(t+1)^2\sqrt{t+1}},$$

$$\bar{v}_{xttt}(t,x) = -\frac{15\varphi'(\xi) + 33\xi\varphi''(\xi) + 12\xi^2\varphi'''(\xi) + \xi^3\varphi''''(\xi)}{8(t+1)^3\sqrt{t+1}}.$$

Consequently

Lemma 2.2. *For each $p \in [1, \infty]$, we have*

$$\|\bar{v}_x(t)\|_{L^p} = O(1)(1+t)^{-1/2+1/2p},$$
$$\|\bar{v}_t(t)\|_{L^p} + \|\bar{v}_{xx}(t)\|_{L^p} = O(1)(1+t)^{-1+1/2p},$$
$$\|\bar{v}_{xt}(t)\|_{L^p} + \|\bar{v}_{xxx}(t)\|_{L^p} = O(1)(1+t)^{-3/2+1/2p}, \tag{2.1.5}$$
$$\|\bar{v}_{tt}(t)\|_{L^p} + \|\bar{v}_{xxt}(t)\|_{L^p} = O(1)(1+t)^{-2+1/2p},$$
$$\|\bar{v}_{xtt}(t)\|_{L^p} + \|\bar{v}_{xxxt}(t)\|_{L^p} = O(1)(1+t)^{-5/2+1/2p},$$
$$\|\bar{v}_{xttt}(t)\|_{L^p} = O(1)(1+t)^{-7/2+1/2p}. \tag{2.1.5}$$

For $\hat{v}(t, x)$, the corresponding result is given in the following lemma

LEMMA 2.2. *For $k \geqslant 0$, $j \geqslant 0$, $p \in [1, \infty]$, we have*

$$\|\partial_x^k \partial_t^j \hat{v}(t, x)\|_{L^p} = O(1) \exp(-\alpha t). \tag{2.1.6}$$

2.2. *Global Smooth Solvability of the Cauchy Problem* (1.1), (1.2)

This subsection is devoted to the study of the Cauchy problem (1.1), (1.2). We will concentrate on the existence and uniqueness of global smooth solution to (1.1), (1.2) and derive some time-independent L^∞ *a priori* estimates on the global smooth solution obtained above. These time-independent L^∞ *a priori* estimates will play an essential role in our subsequent analyses.

The global smooth solvability results for the Cauchy problem (1.1), (1.2) has been considered by Yang and Zhu in [23]. Their results can be restated as in the following

THEOREM 2.1 (Yang and Zhu [23]). *Let the assumption* (H_1) *holds and for arbitrarily given positive constants v_1, v_2 and M_2, there exists a sufficiently small constant $M_3 > 0$ such that if $(r_0(x), s_0(x)) \in C_b^1(\mathrm{R})$ and the conditions stated in* (1.19) *are satisfied, then the Cauchy problem* (1.1), (1.2) *admits a unique global smooth solution $(v(t, x), u(t, x))$ which satisfies*

$$\begin{aligned} v_* \leqslant v(t, x) \leqslant v^*, &\qquad |u(t, x)| \leqslant M_2, \\ |r_x(t, x)| \leqslant \alpha M_4, &\qquad |s_x(t, x)| \leqslant \alpha M_4, \end{aligned} \tag{2.2.1}$$

where v_*, v^*, and M_4, which can be chosen sufficiently small, are positive constants depending only on v_1, v_2, M_2, and M_3 but independent of t and x.

As a direct consequency of Theorem 2.1, we have

COROLLARY 2.1 (Certain time-independent *a priori* estimates). *Under the conditions stated in Theorem* 1.3, *we have that the Cauchy problem* (1.10) *admits a unique global smooth solution $(V(t, x), U(t, x))$ satisfying*

$$|V_x(t, x)| \leqslant M_5, \qquad |U(t, x)| \leqslant M_5, \tag{2.2.2}$$

and

$$\begin{gathered} |\partial_x^i \partial_t^j (V_x(t, x) + \bar{v}(t, x + x_0) + \hat{v}(t, x))| \leqslant M_6, \\ |U_x(t, x) + \bar{u}_x(t, x + x_0) + \hat{u}_x(t, x)| \leqslant M_6, \\ i \geqslant 0, \qquad j \geqslant 0, \qquad i + j = 1. \end{gathered} \tag{2.2.3}$$

Here $M_i (i=5, 6)$ *are time-independent positive constants and* M_6 *can be chosen as small as we wanted.*

Remark 2.1. The fact that M_6 can be chosen sufficiently small will play an essential role in our proving Theorem 1.3.

3. THE PROOF OF OUR MAIN RESULTS

In this section, we prove our main results Theorem 1.3 and Theorem 1.4. Since Theorem 1.4 is a direct corollary of Theorem 1.3, we only prove Theorem 1.3 in the following. To make the proofs easy to read, we divide it into the following three steps:

3.1. *The First Step*: *Energy Estimates and Asymptotic Behaviors*

The main purpose of this subsection is to prove the following asymptotic behavior of the solution $(v(t, x), u(t, x))$ obtained above

THEOREM 3.1. *If the assumptions stated in Theorem* 1.3 *are satisfied, then the unique global smooth solution* $(V(t, x), U(t, x))$ *of the Cauchy problem* (1.10) *satisfies*

$$\lim_{t\to\infty} \sup_{x\in R} \{|\partial_x^i \partial_t^j V(t, x)|\} = 0, \qquad i \geqslant 0, \qquad j \geqslant 0, \qquad i+j \leqslant 2. \tag{3.1.1}$$

Before proving Theorem 3.1, we first cite the following fundamental result, whose proof can be found in [28],

LEMMA 3.1. *If there exists a constant* $C>0$, *which is independent of* x *and* t, *such that*

$$\int u^2(t, x)\, dx \leqslant C, \qquad \int u_x^2(t, x)\, dx \leqslant C,$$

$$\int_0^t \int u_x^2\, (t, x)\, dx\, dt \leqslant C, \qquad \int u_t^2(t, x)\, dx \leqslant C.$$

Then we have

$$\lim_{t\to\infty} \sup_{x\in R} \{|u(t, x)|\} = 0. \tag{3.1.2}$$

From Lemma 3.1, to prove Theorem 3.1, we only need to get the following energy estimates

THEOREM 3.2 (Energy estimates). *Under the assumptions of Theorem* 1.3, *we have that*

$$\|V(t)\|_3^2+\|V_t(t)\|_2^2+\|V_{tt}(t)\|_1^2+\|V_{ttt}(t)\|^2$$
$$+\int_0^t(\|V_x(s)\|_2^2+\|V_t(s)\|_2^2+\|V_{tt}(s)\|_1^2+\|V_{ttt}(s)\|^2)\,ds$$
$$\leqslant O(1)(\|V_0\|_3^2+\|U_0\|_2^2+1). \tag{3.1.3}$$

Theorem 3.2 will be proved by the following a series of lemmas. To this end, we first know from (1.10) that $V(t,x)$ satisfies the following Cauchy problem

$$\begin{aligned} &V_{tt}+[p(V_x+\bar{v}+\hat{v})-p(\bar{v})]_x+\alpha V_t=\frac{1}{\alpha}p(\bar{v})_{xt},\\ &V(0,x)=V_0(x),\qquad V_t(0,x)=U_0(x), \end{aligned} \tag{3.1.4}$$

and our first result is on the basic energy estimates

LEMMA 3.2 (Basic energy estimates). *Under the assumptions of Theorem* 1.3, *we have*

$$\|V(t)\|_1^2+\|V_t(t)\|^2+\int_0^t(\|V_x(s)\|^2+\|V_t(s)\|^2)\,ds$$
$$\leqslant O(1)(\|V_0\|_1^2+\|U_0\|^2+1). \tag{3.1.5}$$

Proof. First, multiplying $(3.1.4)_1$ by V and integrating the results with respect to t and x over $[0,t]\times\mathrm{R}$, we can get

$$\frac{\alpha}{2}\int V^2\,dx-\int_0^t\int V_x[p(V_x+\bar{v}+\hat{v})-p(\bar{v})]\,dx\,dt$$
$$=\frac{\alpha}{2}\int V_0^2\,dx+\int_0^t\int V_t^2\,dx\,dt-\int VV_t\,dx+\int V_0U_0\,dx$$
$$+\frac{1}{\alpha}\int_0^t\int p(\bar{v})_{xt}V\,dx\,dt. \tag{3.1.6}$$

From Lemma 2.1, Lemma 2.2, and Cauchy–Schwarz's inequality, we have

$$
\begin{aligned}
&-\int_0^t \int V_x[p(V_x+\bar{v}+\hat{v})-p(\bar{v})]\,dx\,dt \\
&\quad = -\int_0^t \int p'(\theta(V_x+\hat{v})+\bar{v})\, V_x^2\,dx\,dt \\
&\qquad -\int_0^t \int p'(\theta(V_x+\hat{v})+\bar{v})\, V_x\hat{v}\,dx\,dt \qquad (\theta\in(0,1)) \\
&\quad \geqslant C_2\int_0^t \int V_x^2\,dx\,dt-\int_0^t \int p'(\theta(V_x+\hat{v})+\bar{v})\, V_x\hat{v}\,dx\,dt,
\end{aligned} \tag{3.1.7}
$$

$$
\begin{aligned}
&\frac{\alpha}{2}\int V_0^2\,dx+\int V_0U_0\,dx \leqslant O(1)(\|V_0\|^2+\|U_0\|^2), \\
&\qquad -\int VV_t\,dx \leqslant \frac{\alpha-\varepsilon}{2}\int V^2\,dx+\frac{1}{2(\alpha-\varepsilon)}\int V_t^2\,dx,
\end{aligned} \tag{3.1.8}
$$

$$
\begin{aligned}
&-\int_0^t \int p'(\theta(V_x+\hat{v})+\bar{v})\, V_x\hat{v}\,dx\,dt \\
&\quad \leqslant \frac{\varepsilon}{2}\int_0^t \int V_x^2\,dx\,dt+O(1)\int_0^t \int \hat{v}^2\,dx\,dt \\
&\quad \leqslant \frac{\varepsilon}{2}\int_0^t \int V_x^2\,dx\,dt+O(1),
\end{aligned} \tag{3.1.9}
$$

and

$$
\begin{aligned}
\frac{1}{\alpha}\int_0^t \int p(\bar{v})_{xt}V\,dx\,dt &= -\frac{1}{\alpha}\int_0^t \int p(\bar{v})_t\, V_x\,dx\,dt \\
&\leqslant \frac{\varepsilon}{2}\int_0^t \int V_x^2\,dx\,dt+O(1)\int_0^t \int \bar{v}_t^2\,dx\,dt \\
&\leqslant \frac{\varepsilon}{2}\int_0^t \int V_x^2\,dx\,dt+O(1),
\end{aligned} \tag{3.1.10}
$$

where

$$C_2 := \inf_{v \in M} \{ -p'(v) \} > 0,$$

$$M = [\min\{v_*, \min_{\xi \in \mathbf{R}} \varphi(\xi)\}, \max\{v^*, \max_{\xi \in \mathbf{R}} \varphi(\xi)\}].$$

Substituting (3.1.7)–(3.1.10) into (3.1.6) deduce

$$\frac{\varepsilon}{2} \int V^2\,dx + (C_2 - \varepsilon) \int_0^t \int V_x^2\,dx\,dt \leqslant O(1)(\|V_0\|^2 + \|U_0\|^2 + 1) + \int_0^t \int V_t^2\,dx\,dt + \frac{1}{2(\alpha - \varepsilon)} \int V_t^2\,dx. \tag{3.1.11}$$

Next, we multiply $(3.1.4)_1$ by V_t and integrate the results with respect to t and x over $[0, t] \times \mathrm{R}$, after some integrations by parts, we get

$$\frac{1}{2} \int V_t^2\,dx + \alpha \int_0^t \int V_t^2\,dx\,dt = \frac{1}{2} \int U_0^2\,dx + \frac{1}{\alpha} \int_0^t \int p(\bar{v})_{xt} V_t\,dx\,dt - \int_0^t \int V_t [p(V_x + \bar{v} + \hat{v}) - p(\bar{v})]_x\,dx\,dt. \tag{3.1.12}$$

We now estimate the righthand side of (3.1.12) term by term. First, we have from Lemma 2.1 and Cauchy–Schwarz's inequality that

$$\frac{1}{\alpha} \int_0^t \int p(\bar{v})_{xt} V_t\,dx\,dt \leqslant \varepsilon \int_0^t \int V_t^2\,dx\,dt + O(1) \int_0^t \int (\bar{v}_{xt}^2 + \bar{v}_x^2 \bar{v}_t^2)\,dx\,dt \leqslant \varepsilon \int_0^t \int V_t^2\,dx\,dt + O(1). \tag{3.1.13}$$

Now we deal with the last term of the righthand side of (3.1.12). Noticing

$$[p(V_x + \bar{v} + \hat{v}) - p(\bar{v})] V_{xt} = \frac{d}{dt} \left[\int_{\bar{v}}^{V_x + \bar{v} + \hat{v}} p(s)\,ds - p(\bar{v}) V_x \right] - [p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v}) V_x] \bar{v}_t - p(V_x + \bar{v} + \hat{v}) \hat{v}_t, \tag{3.1.14}$$

we have

$$
\begin{aligned}
-\int_0^t \int V_t[p(V_x+\bar{v}+\hat{v})-p(\bar{v})]_x \, dx \, dt
&= \int \left[\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s) \, ds - p(\bar{v}) \, V_x \right] dx \\
&\quad - \int \left[\int_{\bar{v}_0}^{V_{0x}+\bar{v}_0+\hat{v}_0} p(s) \, ds - p(\bar{v}_0) \, V_{0x} \right] dx \\
&\quad - \int_0^t \int [p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v}) \, V_x] \, \bar{v}_t \, dx \, dt \\
&\quad - \int_0^t \int p(V_x+\bar{v}+\hat{v}) \, \hat{v}_t \, dx \, dt \\
&:= \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.1.15}
$$

Since

$$
\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s) \, ds = p(\bar{v})(V_x+\hat{v}) + \frac{p'(\theta(V_x+\hat{v})+\bar{v})}{2} (V_x+\hat{v})^2, \quad (\theta \in (0, 1)),
$$

we have

$$
\begin{aligned}
I_1 &\leqslant -\frac{C_2-\varepsilon}{2} \int V_x^2 \, dx + O(1) \int (\hat{v}^2+|\hat{v}|) \, dx \\
&\leqslant -\frac{C_2-\varepsilon}{2} \int V_x^2 \, dx + O(1),
\end{aligned} \tag{3.1.16}
$$

$$
I_2 \leqslant O(1)(\| V_{0x} \|^2+1),
$$

Furthermore

$$
I_4 \leqslant O(1) \int_0^t \int |\hat{v}_t| \, dx \, dt \leqslant O(1), \tag{3.1.17}
$$

and

$$I_3 = -\int_0^t \int p'(\theta_1(V_x+\hat{v})+\bar{v})\,\hat{v}\bar{v}_t\,dx\,dt \qquad (\theta_1 \in (0,1))$$

$$-\theta_1 \int_0^t \int p''(\theta_2(V_x+\hat{v})+\bar{v})\,V_x\bar{v}_t(V_x+\hat{v})\,dx\,dt \qquad (\theta_2 \in (0,1))$$

$$\leqslant O(1)\int_0^t \int |\hat{v}|\,|\bar{v}_t|\,dx\,dt + O(1)\int_0^t \int |\bar{v}_t|\,V_x^2\,dx\,dt$$

$$\leqslant O(1) + \varepsilon \int_0^t \int V_x^2\,dx\,dt + O(1)\int_0^t (1+s)^{-2}\,\|V_x(s)\|^2\,ds. \tag{3.1.18}$$

Inserting (3.1.16)–(3.1.18) into (3.1.15), we can conclude that

$$\frac{1}{2}\int V_t^2\,dx + (\alpha-\varepsilon)\int_0^t \int V_t^2\,dx\,dt + \frac{C_2-\varepsilon}{2}\int V_x^2\,dx$$

$$\leqslant O(1)(\|V_0\|_1^2 + \|U_0\|^2 + 1)$$

$$+\varepsilon \int_0^t \int V_x^2\,dx\,dt + O(1)\int_0^t (1+s)^{-2}\,\|V_x(s)\|^2\,ds. \tag{3.1.19}$$

Taking $k>0$ sufficiently large and choosing $\varepsilon>0$ suitably small such that

$$\begin{gathered}(\alpha-\varepsilon)\,k > 1,\\ C_2-\varepsilon > k\varepsilon,\\ \tfrac{1}{2}k > \varepsilon,\end{gathered} \tag{3.1.20}$$

we have from $(3.1.19)\times k + (3.1.11)$ that

$$\|V(t)\|_1^2 + \|V_t(t)\|^2 + \int_0^t (\|V_x(s)\|^2 + \|V_t(s)\|^2)\,ds$$

$$\leqslant O(1)(\|V_0\|_1^2 + \|U_0\|^2 + 1) + O(1)\int_0^t (1+s)^{-2}\,\|V_x(s)\|^2\,ds, \tag{3.1.21}$$

and the Gronwall's inequality implies (3.1.5). This completes the proof of Lemma 3.2.

Next, we consider the higher order energy estimates. For results in this direction, we can get

LEMMA 3.3 (Higher order energy estimates). *Under the assumptions of Theorem* 1.3, *we have*

$$\|V_{xx}(t)\|^2+\|V_{xt}(t)\|^2+\|V_{tt}(t)\|^2+\int_0^t(\|V_{xx}(s)\|^2+\|V_{xt}(s)\|^2+\|V_{tt}(s)\|^2)\,ds$$

$$\leqslant O(1)(\|V_0\|_2^2+\|U_0\|_1^2+1), \tag{3.1.22}$$

and

$$\|V_{xxx}(t)\|^2+\|V_{xxt}(t)\|^2+\|V_{xtt}(t)\|^2+\|V_{ttt}(t)\|^2$$

$$+\int_0^t(\|V_{xxx}(s)\|^2+\|V_{xxt}(s)\|^2+\|V_{xtt}(s)\|^2+\|V_{ttt}(s)\|^2)\,ds$$

$$\leqslant O(1)(\|V_0\|_3^2+\|U_0\|_2^2+1). \tag{3.1.23}$$

Proof. We only prove (3.1.22). (3.1.23) can be treated in the same way.

Differentiating $(3.1.4)_1$ with respect to x, multiplying the resulting equation by V_{xt}, and integrating the results with respect to t and x over $[0, t]\times \mathrm{R}$, we have after some integrations by parts that

$$\frac{1}{2}\int V_{xt}^2\,dx+\alpha\int_0^t\int V_{xt}^2\,dx\,dt=\frac{1}{2}\int U_{0x}^2\,dx+\frac{1}{\alpha}\int_0^t\int V_{xt}p(\bar{v})_{xxt}\,dx\,dt$$

$$-\int_0^t\int V_{xt}[\,p(V_x+\bar{v}+\hat{v})-p(\bar{v})\,]_{xx}\,dx\,dt$$

$$:=\sum_{i=5}^{7} I_i. \tag{3.1.24}$$

Similar to the proofs of Lemma 3.2, we have

$$I_5+I_6\leqslant O(1)(\|U_{0x}\|^2+1)+\frac{\varepsilon}{3}\int_0^t\int V_{xt}^2\,dx\,dt. \tag{3.1.25}$$

Now we turn to estimate I_7. Notice

$$
\begin{aligned}
V_{xxt}p(V_x+\bar{v}+\hat{v})_x = & \frac{d}{dt}\left[\frac{1}{2}\,p'(V_x+\bar{v}+\hat{v})\;V_{xx}^2\right] \\
& -\frac{1}{2}\,V_{xx}^2 p''(V_x+\bar{v}+\hat{v})(V_x+\bar{v}+\hat{v})_t \\
& +[\,V_{xt}(\bar{v}_x+\hat{v}_x)\,p'(V_x+\bar{v}+\hat{v})\,]_x \\
& -V_{xt}(\bar{v}_{xx}+\hat{v}_{xx})\,p'(V_x+\bar{v}+\hat{v}) \\
& -V_{xt}V_{xx}(\bar{v}_x+\hat{v}_x)\,p''(V_x+\bar{v}+\hat{v}) \\
& -V_{xt}(\bar{v}_x+\hat{v}_x)^2\,p''(V_x+\bar{v}+\hat{v}), \qquad (3.1.26)
\end{aligned}
$$

we get

$$
\begin{aligned}
I_7 = & \tfrac{1}{2}\int p'(V_x+\bar{v}+\hat{v})\;V_{xx}^2\,dx - \tfrac{1}{2}\int p'(V_{0x}+\bar{v}_0+\hat{v}_0)\;V_{0xx}^2\,dx \\
& -\tfrac{1}{2}\int_0^t\int V_{xx}^2\,p''(V_x+\bar{v}+\hat{v})(V_x+\bar{v}+\hat{v})_t\,dx\,dt \\
& -\int_0^t\int V_{xt}(\bar{v}_{xx}+\hat{v}_{xx})\,p'(V_x+\bar{v}+\hat{v})\,dx\,dt \\
& -\int_0^t\int V_{xt}V_{xx}(\bar{v}_x+\hat{v}_x)\,p''(V_x+\bar{v}+\hat{v})\,dx\,dt \\
& -\int_0^t\int V_{xt}(\bar{v}_x+\hat{v}_x)^2\,p''(V_x+\bar{v}+\hat{v})\,dx\,dt \\
& +\int_0^t\int V_{xt}[\,p'(\bar{v})\,\bar{v}_{xx}+p''(\bar{v})\,\bar{v}_x^2]\,dx\,dt \\
:= & \sum_{j=1}^{7} I_7^j. \qquad (3.1.27)
\end{aligned}
$$

By employing Lemma 2.1, Lemma 2.2, Corollary 2.1, and Cauchy–Schwarz's inequality, we can get

$$I_7^3 \leqslant O(1)\, M_6 \int_0^t \int V_{xx}^2\, dx\, dt, \tag{3.1.28}$$

$$\begin{aligned} I_7^4 &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1) \int_0^t \int (\bar{v}_{xx}^2 + \hat{v}_{xx}^2)\, dx\, dt \\ &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1), \end{aligned} \tag{3.1.29}$$

$$I_7^5 \leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1) \int_0^t \int (\bar{v}_x^2 + \hat{v}_x^2)\, V_{xx}^2\, dx\, dt, \tag{3.1.30}$$

$$\begin{aligned} I_7^6 &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1) \int_0^t \int (\bar{v}_x^4 + \hat{v}_x^4)\, dx\, dt \\ &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1), \end{aligned} \tag{3.1.31}$$

and

$$\begin{aligned} I_7^7 &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1) \int_0^t \int (\bar{v}_x^4 + \bar{v}_{xx}^2)\, dx\, dt \\ &\leqslant \frac{\varepsilon}{6} \int_0^t \int V_{xt}^2\, dx\, dt + O(1). \end{aligned} \tag{3.1.32}$$

Consequently

$$\begin{aligned} I_7 \leqslant{}& \frac{2\varepsilon}{3} \int_0^t \int V_{xt}^2\, dx\, dt + O(1) \int_0^t \int (|\hat{v}_x|^2 + |\bar{v}_x|^2)\, V_{xx}^2\, dx\, dt \\ &+ O(1)\, M_6 \int_0^t \int V_{xx}^2\, dx\, dt + O(1)(\|V_0\|_2^2 + \|U_0\|^2 + 1) \\ &- \frac{C_2}{2} \int V_{xx}^2\, dx. \end{aligned} \tag{3.1.33}$$

From (3.1.24), (3.1.25), and (3.1.33), we deduce

$$\begin{aligned} &\frac{1}{2} \int V_{xt}^2\, dx + \frac{C_2}{2} \int V_{xx}^2\, dx + (\alpha - \varepsilon) \int_0^t \int V_{xt}^2\, dx\, dt \\ &\quad \leqslant O(1)(\|V_0\|_2^2 + \|U_0\|_1^2 + 1) + O(1) \int_0^t \int (|\hat{v}_x|^2 + |\bar{v}_x^2|)\, V_{xx}^2\, dx\, dt \\ &\quad + O(1)\, M_6 \int_0^t \int V_{xx}^2\, dx\, dt. \end{aligned} \tag{3.1.34}$$

Next, we multiply $(3.1.4)_1$ by $-V_{xx}$ and integrate the results with respect to t and x over $[0, t]\times \mathrm{R}$ to get

$$\begin{aligned}\frac{\alpha}{2}\int V_x^2\,dx &= \frac{\alpha}{2}\int V_{0x}^2\,dx-\int V_xV_{xt}\,dx+\int V_{0x}U_{0x}\,dx\\ &\quad+\int_0^t\int V_{xt}^2\,dx\,dt-\frac{1}{\alpha}\int_0^t\int V_{xx}p(\bar{v})_{xt}\,dx\,dt\\ &\quad+\int_0^t\int V_{xx}[p(V_x+\bar{v}+\hat{v})-p(\bar{v})]_x\,dx\,dt\\ &=\sum_{i=8}^{13} I_i. \end{aligned}\tag{3.1.35}$$

It is easy to see that

$$\begin{aligned} I_8+I_{10} &\leqslant O(1)(\|V_{0x}\|^2+\|U_{0x}\|^2),\\ I_9 &\leqslant \varepsilon\int V_{xt}^2\,dx+O(1)\int V_x^2\,dx \\ &\leqslant O(1)(\|V_0\|_1^2+\|U_0\|^2+1)+\varepsilon\int V_{xt}^2\,dx, \end{aligned}\tag{3.1.36}$$

$$\begin{aligned} I_{12} &\leqslant O(1)\int_0^t\int(|\bar{v}_{xt}|+|\bar{v}_x\bar{v}_t|)\,|V_{xx}|\,dx\,dt\\ &\leqslant \frac{\varepsilon}{2}\int_0^t\int V_{xx}^2\,dx\,dt+O(1), \end{aligned}\tag{3.1.37}$$

and

$$\begin{aligned} I_{13} &\leqslant \int_0^t\int p'(V_x+\bar{v}+\hat{v})\,V_{xx}^2\,dx\,dt\\ &\quad+O(1)\int_0^t\int[\,|\bar{v}_x|\,(|V_x|+|\hat{v}|)+|\hat{v}_x|\,]\,|V_{xx}|\,dx\,dt\\ &\leqslant\left(\frac{\varepsilon}{2}-C_2\right)\int_0^t\int V_{xx}^2\,dx\,dt+O(1)(\|V_0\|_1^2+\|U_0\|^2+1). \end{aligned}\tag{3.1.38}$$

Consequently

$$(C_2-\varepsilon)\int_0^t\int V_{xx}^2\,dx\,dt$$
$$\leqslant O(1)(\|V_0\|_1^2+\|U_0\|_1^2+1)+\varepsilon\int V_{xt}^2\,dx+\int_0^t\int V_{xt}^2\,dx\,dt. \quad (3.1.39)$$

Let k, ε be the positive constants chosen in Lemma 3.2, we have from $(3.1.39)+(3.1.34)\times k$ that

$$\left(\frac{k}{2}-\varepsilon\right)\int V_{xt}^2\,dx+\frac{kC_2}{2}\int V_{xx}^2\,dx+[(\alpha-\varepsilon)\,k-1]\int_0^t\int V_{xt}^2\,dx\,dt$$
$$+(C_2-\varepsilon)\int_0^t\int V_{xx}^2\,dx\,dt$$
$$\leqslant O(1)(\|V_0\|_2^2+\|U_0\|_1^2+1)+O(1)\int_0^t\int(|\hat{v}_x|^2+|\bar{v}_x^2|)\,V_{xx}^2\,dx\,dt$$
$$+O(1)\,M_6k\int_0^t\int V_{xx}^2\,dx\,dt.$$

Thus if we choose M_6 sufficiently small such that

$$C_2-\varepsilon>O(1)\,M_6k,$$

we can deduce that

$$\int(V_{xt}^2+V_{xx}^2)\,dx+\int_0^t\int(V_{xt}^2+V_{xx}^2)\,dx\,dt$$
$$\leqslant O(1)(\|V_0\|_2^2+\|U_0\|_1^2+1)+O(1)\int_0^t\int(|\hat{v}_x|^2+|\bar{v}_x^2|)\,V_{xx}^2\,dx\,dt. \quad (3.1.40)$$

Due to

$$\int_0^t\int(|\hat{v}_x|^2+|\bar{v}_x^2|)\,V_{xx}^2\,dx\,dt$$
$$\leqslant\varepsilon\int_0^t\int V_{xx}^2\,dx\,dt+O(1)\int_0^t\int(\hat{v}_x^4+\bar{v}_x^4)\,V_{xx}^2\,dx\,dt$$
$$\leqslant\varepsilon\int_0^t\int V_{xx}^2\,dx\,dt+O(1)\int_0^t(1+s)^{-2}\,\|V_{xx}(s)\|^2\,ds, \quad (3.1.41)$$

we can get

$$\int (V_{xt}^2 + V_{xx}^2)\, dx + \int_0^t \int (V_{xt}^2 + V_{xx}^2)\, dx\, dt$$
$$\leqslant O(1)(\|V_0\|_2^2 + \|U_0\|_1^2 + 1) + O(1) \int_0^t (1+s)^{-2} \|V_{xx}(s)\|^2\, ds, \tag{3.1.42}$$

which implies that

$$\int (V_{xt}^2 + V_{xx}^2)\, dx + \int_0^t \int (V_{xt}^2 + V_{xx}^2)\, dx\, dt \leqslant O(1)(\|V_0\|_2^2 + \|U_0\|_1^2 + 1). \tag{3.1.43}$$

On the other hand, since

$$V_{tt} = \frac{1}{\alpha}\, p(\bar{v})_{xt} - \alpha V_t - [\, p(V_x + \bar{v} + \hat{v}) - p(\bar{v})]_x,$$

we can get from (3.1.43) that

$$\int V_{tt}^2\, dx + \int_0^t \int V_{tt}^2\, dx\, dt \leqslant O(1)(\|V_0\|_2^2 + \|U_0\|_1^2 + 1), \tag{3.1.44}$$

and (3.1.22) follows from (3.1.43) and (3.1.44). This completes the proof of Lemma 3.3 and so does Theorem 3.2. Consequently, Theorem 3.1 follows also.

3.2. *The Second Step*: *Decay Estimates* (1.21) *and* (1.22)

This subsection is devoted to the proof of the decay estimates (1.21) and (1.22). To this end, we first give the following energy estimates

LEMMA 3.4 (Some energy estimates). *Under the assumptions of Theorem* 1.3, *we have*

$$\frac{d}{dt} \int \left\{ \frac{1}{2} V_t^2 - \left[\int_{\bar{v}}^{V_x + \bar{v} + \hat{v}} p(s)\, ds - p(\bar{v})\, V_x \right] \right\} dx + \frac{\alpha}{2} \int V_t^2\, dx$$
$$\leqslant O(1)(1+t)^{-5/2} + O(1)(1+t)^{-1} \int V_x^2\, dx; \tag{3.2.1}$$

$$\frac{d}{dt}\int \{V_{xt}^2 - p'(V_x+\bar{v}+\hat{v})\,V_{xx}^2\}\,dx + \alpha\int V_{xt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-7/2} + O(1)(1+t)^{-2}\int V_x^2\,dx$$
$$+\,O(1)(\|V_{xt}\|_{L^\infty} + (1+t)^{-1})\int V_{xx}^2\,dx, \qquad (3.2.2)_1$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_x^2 + V_xV_{xt}\right\}dx + \frac{C_2}{2}\int V_{xx}^2\,dx$$
$$\leqslant O(1)(1+t)^{-5/2} + O(1)(1+t)^{-2}\int V_x^2\,dx + \int V_{xt}^2\,dx, \qquad (3.2.2)_2$$

$$\frac{d}{dt}\int \{V_{tt}^2 - p'(V_x+\bar{v}+\hat{v})\,V_{xt}^2\}\,dx + \alpha\int V_{tt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-9/2} + O(1)(1+t)^{-3}\int V_x^2\,dx + O(1)(1+t)^{-2}\int V_{xx}^2\,dx$$
$$+\,O(1)(\|V_{xt}\|_{L^\infty} + (1+t)^{-1})\int V_{xt}^2\,dx, \qquad (3.2.2)_3$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_t^2 + V_tV_{tt}\right\}dx + \frac{C_2}{2}\int V_{xt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-7/2} + O(1)(1+t)^{-2}\int V_x^2\,dx + \int V_{tt}^2\,dx; \qquad (3.2.2)_4$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_{xx}^2 + V_{xx}V_{xxt}\right\}dx + \frac{C_2}{2}\int V_{xxx}^2\,dx$$
$$\leqslant O(1)(1+t)^{-7/2} + O(1)(1+t)^{-2}\int V_x^2\,dx + \int V_{xxt}^2\,dx$$
$$+\,O(1)(\|V_{xx}\|_{L^\infty}^2 + (1+t)^{-1})\int V_{xx}^2\,dx, \qquad (3.2.3)_1$$

$$\frac{d}{dt}\int \{V_{xxt}^2 - p'(V_x+\bar{v}+\hat{v})\,V_{xxx}^2\}\,dx + \alpha\int V_{xxt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-9/2} + O(1)(1+t)^{-3}\int V_x^2\,dx$$
$$+\,O(1)((1+t)^{-2} + (1+t)^{-1}\,\|V_{xx}\|_{L^\infty}^2)\int V_{xx}^2\,dx$$
$$+\,O(1)(\|V_{xt}\|_{L^\infty} + \|V_{xx}\|_{L^\infty}^2 + (1+t)^{-1})\int V_{xxx}^2\,dx, \qquad (3.2.3)_2$$

$$\frac{d}{dt}\int\{V_{ttt}^2-p'(V_x+\bar{v}+\hat{v})\,V_{xtt}^2\}\,dx+\alpha\int V_{ttt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-13/2}+O(1)(1+t)^{-5}\int V_x^2\,dx$$
$$+O(1)(1+t)^{-4}\int V_{xx}^2\,dx$$
$$+O(1)(1+t)^{-2}\int V_{xxt}^2\,dx$$
$$+O(1)((1+t)^{-1}+\|V_{xt}\|_{L^\infty})\int V_{xtt}^2\,dx$$
$$+O(1)(\|V_{xt}\|_{L^\infty}^2+(1+t)^{-2})(\|V_{xx}\|_{L^\infty}^2+(1+t)^{-1})\int V_{xt}^2\,dx. \tag{3.2.3$_3$}$$

Proof. Lemma 3.4 follows essentially the same arguments as those used in the proofs of Lemma 3.2 and Lemma 3.3. In fact, performing $(3.1.4)_1\times V_t$, $((3.1.4)_1)_x\times V_{xt}$, $(3.1.4)_1\times V_{xx}$, $((3.1.4)_1)_t\times V_{tt}$, $((3.1.4)_1)_t\times V_t$, $((3.1.4)_1)_x\times V_{xxx}$, $((3.1.4)_1)_{xx}\times V_{xxt}$, $((3.1.4)_1)_{tt}\times V_{ttt}$, and integrating the resulting equations with respect to x over R respectively, we can get (3.2.1), $(3.2.2)_1$–$(3.2.2)_4$, $(3.2.3)_1$–$(3.2.3)_3$ by employing the method of integrations by parts, Cauchy–Schwarz's inequality, Lemma 2.1, and Lemma 2.2. The details are omitted. This completes the proof of Lemma 3.4.

Having obtained Lemma 3.4, we find that to employ the techniques developed by Nishihara in [16, 17] to get the decay estimates (1.21) and (1.22), one need only to get the following estimates

$$\begin{aligned}\|V_{xx}\|_{L^\infty}&\leqslant O(1)(1+t)^{-1/2},\\ \|V_{xt}\|_{L^\infty}&\leqslant O(1)(1+t)^{-1}.\end{aligned} \tag{3.2.4}$$

Notice that in [16, 17], (3.2.4) is a direct consequence of the *a priori* assumption (1.24). But since in our analyses, we do not ask the initial error and the strength of the diffusion waves to be small, the techniques developed by Nishihara in [16, 17] can not be used directly and our main novelty in this paper in deducing the decay estimates (1.21) and (1.22) is to get the estimates (3.2.4) without any smallness assumptions on the initial error and on the strength of the nonlinear diffusion waves. Our main observation is that from estimates (3.1.1) obtained in Theorem 3.1, if we let

$$\delta:=\sup_{t\geqslant T^*}\left\{\|V_{xt}\|_{L^\infty}+\frac{1}{1+t}+\|V_{xx}\|_{L^\infty}^2\right\} \tag{3.2.5}$$

for some fixed positive constant $T^*>0$, then we can choose a fixed $T^*>0$ sufficiently large such that δ can be chosen as small as we wanted. For such a δ, we can also find a suitably small positive constant λ such that

$$\begin{aligned} &\frac{1}{2}>\frac{\lambda}{\alpha}\alpha, \\ &\frac{C_2}{2}\lambda>O(1)\,\delta, \end{aligned} \tag{3.2.6}$$

For δ, λ, T^* chosen as above, we have the following result

LEMMA 3.5. *If all the assumptions given Theorem* 1.3 *are satisfied, then there exists a positive constant $\beta>0$ such that for each $t\geqslant T^*$, we have*

$$\frac{d}{dt}\int\left(\frac{1}{2}V_t^2-\left[\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}}p(s)\,ds-p(\bar{v})\,V_x\right]\right)(t)\,dx+\frac{\alpha}{2}\int V_t^2(t)\,dx$$

$$\leqslant O(1)(1+t)^{-5/2}+O(1)(1+t)^{-1}\int V_x^2(t)\,dx; \tag{3.2.7}$$

$$\frac{d}{dt}\int\left(\frac{1}{2}V_{xt}^2-\frac{1}{2}p'(V_x+\bar{v}+\hat{v})\,V_{xx}^2+\frac{\lambda\alpha}{2}V_x^2+\lambda V_xV_{xt}\right)(t)\,dx$$

$$+\beta\int(V_{xt}^2+V_{xx}^2)(t)\,dx$$

$$\leqslant O(1)(1+t)^{-5/2}+O(1)(1+t)^{-2}\int V_x^2(t)\,dx, \tag{3.2.8$_1$}$$

$$\frac{d}{dt}\int\left(\frac{1}{2}V_{tt}^2-\frac{1}{2}p'(V_x+\bar{v}+\hat{v})\,V_{xt}^2+\frac{\lambda\alpha}{2}V_t^2+\lambda V_tV_{tt}\right)(t)\,dx$$

$$+\beta\int(V_{xt}^2+V_{tt}^2)(t)\,dx$$

$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int(V_{xx}^2+V_x^2)(t)\,dx; \tag{3.2.8$_2$}$$

$$\frac{d}{dt}\int\left(\frac{1}{2}V_{xxt}^2-\frac{1}{2}p'(V_x+\bar{v}+\hat{v})\,V_{xxx}^2+\frac{\lambda\alpha}{2}V_{xx}^2+\lambda V_{xx}V_{xxt}\right)(t)\,dx$$

$$+\beta\int(V_{xxx}^2+V_{xxt}^2)(t)\,dx$$

$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int V_x^2(t)\,dx$$

$$+O(1)(\|V_{xx}\|_{L^\infty}^2+(1+t)^{-1})\int V_{xx}^2(t)\,dx. \tag{3.2.9}$$

Here

$$\beta = \min\left\{\frac{\lambda C_2}{2} - O(1)\,\delta, \frac{\alpha}{2} - \lambda\right\}.$$

Proof. We only give the sketch of the proof of $(3.2.8)_1$. The rest can be treated similarly. In fact multiplying $(3.2.2)_2$ by λ, adding the resulting inequality to $(3.2.2)_1$, and noticing (3.2.5), (3.2.6), and (3.2.10), we can immediately get $(3.2.8)_1$. This completes the proof of Lemma 3.5.

Having obtained Lemma 3.5, we now turn to prove the estimates (3.2.4).

First, multiplying (3.2.7) by $1+t$ and integrating the results with respect to t over $[T^*, t]$, we have

$$\begin{aligned}
&(1+t)\int\left(\frac{1}{2}V_t^2 - \left[\int_{\bar v}^{V_x+\bar v+\hat v} p(s)\,ds - p(\bar v)\,V_x\right]\right)(t)\,dx \\
&\qquad + \frac{\alpha}{2}\int_{T^*}^{t}(1+s)\int V_t^2(s)\,dx\,ds \\
&\quad \leqslant (1+T^*)\int\left(\frac{1}{2}V_t^2 - \left[\int_{\bar v}^{V_x+\bar v+\hat v} p(s)\,ds - p(\bar v)\,V_x\right]\right)(T^*)\,dx \\
&\qquad + O(1)\int_{T^*}^{t}(1+s)^{-3/2}\,ds + O(1)\int_{T^*}^{t}\int V_x^2(s)\,dx\,ds \\
&\qquad + \int_{T^*}^{t}\int\left(\frac{1}{2}V_t^2 - \left[\int_{\bar v}^{V_x+\bar v+\hat v} p(s)\,ds - p(\bar v)\,V_x\right]\right)(s)\,dx\,ds.
\end{aligned}$$

The above inequality together with Theorem 3.2 implies that

$$(1+t)(\|V_x(t)\|^2 + \|V_t(t)\|^2) + \int_0^t (1+s)\,\|V_t(s)\|^2\,ds \leqslant O(1). \tag{3.2.11}$$

The same process applied to $(3.2.8)_1$ and $(3.2.8)_2$ deduce that

$$\begin{aligned}
&(1+t)(\|V_{xx}(t)\|^2 + \|V_{xt}(t)\|^2 + \|V_x(t)\|^2) \\
&\qquad + \int_0^t (1+s)(\|V_{xt}(s)\|^2 + \|V_{xx}(s)\|^2)\,ds \leqslant O(1),
\end{aligned} \tag{3.2.12}$$

and

$$(1+t)(\|V_{tt}(t)\|^2+\|V_{xt}(t)\|^2+\|V_t(t)\|^2)$$
$$+\int_0^t (1+s)(\|V_{xt}(s)\|^2+\|V_{tt}(s)\|^2)\,ds \leqslant O(1). \tag{3.2.13}$$

Furthermore as a consequence of $(3.2.8)_2$, (3.2.11), (3.2.12), and (3.2.13), we have

$$(1+t)^2\,(\|V_{tt}(t)\|^2+\|V_{xt}(t)\|^2+\|V_t(t)\|^2)$$
$$+\int_0^t (1+s)^2\,(\|V_{xt}(s)\|^2+\|V_{tt}(s)\|^2)\,ds \leqslant O(1). \tag{3.2.14}$$

Due to

$$|V_{xx}(t,x)|^2=2\int_{-\infty}^{x}(V_{xx}V_{xxx})(t,x)\,dx \leqslant \|V_{xx}(t)\|\ \|V_{xxx}(t)\|,$$

we have from (3.2.12) and Theorem 3.2 that

$$\|V_{xx}(t)\|_{L^\infty} \leqslant O(1)(1+t)^{-1/4}. \tag{3.2.15}$$

Substituting (3.2.15) into (3.2.9), we get

$$\frac{d}{dt}\int\left(\frac{1}{2}V_{xxt}^2-\frac{1}{2}p'(V_x+\bar{v}+\hat{v})\,V_{xxx}^2+\frac{\lambda\alpha}{2}V_{xx}^2+\lambda V_{xx}V_{xxt}\right)(t)\,dx$$
$$+\beta\int(V_{xxx}^2+V_{xxt}^2)(t)\,dx$$
$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int V_x^2(t)\,dx$$
$$+O(1)(1+t)^{-1/2}\int V_{xx}^2(t)\,dx. \tag{3.2.16}$$

From (3.2.16), Theorem 3.2, (3.2.6), and (3.2.12), we can deduce that

$$(1+t)(\|V_{xxt}(t)\|^2+\|V_{xxx}(t)\|^2+\|V_{xx}(t)\|^2)$$
$$+\int_0^t (1+s)(\|V_{xxx}(s)\|^2+\|V_{xxt}(s)\|^2)\,ds \leqslant O(1). \tag{3.2.17}$$

With (3.2.17) in hand, we can easily get from (3.2.12) and (3.2.17) that

$$\| V_{xx}(t) \|_{L^\infty} \leqslant O(1)(1+t)^{-1/2}. \tag{3.2.18}$$

This proves $(3.2.4)_1$.

Now substituting (3.2.18) into (3.2.9) again, we have

$$\begin{aligned}
&\frac{d}{dt} \int \left(\frac{1}{2} V_{xxt}^2 - \frac{1}{2} p'(V_x + \bar{v} + \hat{v}) V_{xxx}^2 + \frac{\lambda\alpha}{2} V_{xx}^2 + \lambda V_{xx} V_{xxt} \right)(t)\, dx \\
&\quad + \beta \int (V_{xxx}^2 + V_{xxt}^2)(t)\, dx \\
&\leqslant O(1)(1+t)^{-7/2} + O(1)(1+t)^{-2} \int V_x^2(t)\, dx \\
&\quad + O(1)(1+t)^{-1} \int V_{xx}^2(t)\, dx,
\end{aligned} \tag{3.2.19}$$

from which and (3.2.12), (3.2.17), we can conclude that

$$\begin{aligned}
&(1+t)^2 (\| V_{xxt}(t) \|^2 + \| V_{xxx}(t) \|^2 + \| V_{xx}(t) \|^2) \\
&\quad + \int_0^t (1+s)^2 (\| V_{xxx}(s) \|^2 + \| V_{xxt}(s) \|^2)\, ds \\
&\leqslant O(1) \int_0^t (1+s)(\| V_{xxx}(s) \|^2 + \| V_{xxt}(s) \|^2 + \| V_{xx}(s) \|^2)\, ds \\
&\quad + O(1) \int_0^t (1+s)^{-3/2}\, ds + O(1) \int_0^t \| V_x(s) \|^2\, ds \\
&\leqslant O(1).
\end{aligned} \tag{3.2.20}$$

Thus from (3.2.14) and (3.2.20), we have

$$\begin{aligned}
&\| V_{xt}(t) \| \leqslant O(1)(1+t)^{-1}, \\
&\| V_{xxt}(t) \| \leqslant O(1)(1+t)^{-1},
\end{aligned} \tag{3.2.21}$$

and consequently

$$\| V_{xt}(t) \|_{L^\infty} \leqslant \| V_{xt}(t) \|^{1/2} \| V_{xxt}(t) \|^{1/2} \leqslant O(1)(1+t)^{-1}. \tag{3.2.22}$$

This proves $(3.2.4)_2$.

Combibing (3.2.4) with Lemma 3.4 and Theorem 3.2, we have

COROLLARY 3.1. *If the conditions listed in Theorem* 1.3 *are satisfied, then we can get the following differential-integral inequalities*

$$\frac{d}{dt}\int\left\{\frac{1}{2}V_t^2-\left[\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)\,ds-p(\bar{v})\,V_x\right]\right\}dx+\frac{\alpha}{2}\int V_t^2\,dx$$

$$\leqslant O(1)(1+t)^{-5/2}+O(1)(1+t)^{-1}\int V_x^2\,dx; \tag{3.2.23}$$

$$\frac{d}{dt}\int\{V_{xt}^2-p'(V_x+\bar{v}+\hat{v})\,V_{xx}^2\}\,dx+\alpha\int V_{xt}^2\,dx$$

$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int V_x^2\,dx$$

$$+O(1)(1+t)^{-1}\int V_{xx}^2\,dx, \tag{3.2.24$_1$}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_x^2+V_xV_{xt}\right\}dx+\frac{C_2}{2}\int V_{xx}^2\,dx$$

$$\leqslant O(1)(1+t)^{-5/2}+O(1)(1+t)^{-2}\int V_x^2\,dx+\int V_{xt}^2\,dx, \tag{3.2.24$_2$}$$

$$\frac{d}{dt}\int\{V_{tt}^2-p'(V_x+\bar{v}+\hat{v})\,V_{xt}^2\}\,dx+\alpha\int V_{tt}^2\,dx$$

$$\leqslant O(1)(1+t)^{-9/2}+O(1)(1+t)^{-3}\int V_x^2\,dx+O(1)(1+t)^{-2}\int V_{xx}^2\,dx$$

$$+O(1)(1+t)^{-1}\int V_{xt}^2\,dx, \tag{3.2.24$_3$}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_t^2+V_tV_{tt}\right\}dx+\frac{C_2}{2}\int V_{xt}^2\,dx$$

$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int V_x^2\,dx+\int V_{tt}^2\,dx; \tag{3.2.24$_4$}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_{xx}^2+V_{xx}V_{xxt}\right\}dx+\frac{C_2}{2}\int V_{xxx}^2\,dx$$

$$\leqslant O(1)(1+t)^{-7/2}+O(1)(1+t)^{-2}\int V_x^2\,dx+\int V_{xxt}^2\,dx$$

$$+O(1)(1+t)^{-1}\int V_{xx}^2\,dx, \tag{3.2.25$_1$}$$

$$\frac{d}{dt}\int\{V_{xxt}^2-p'(V_x+\bar{v}+\hat{v})\,V_{xxx}^2\}\,dx+\alpha\int V_{xxt}^2\,dx$$

$$\leqslant O(1)(1+t)^{-9/2}+O(1)(1+t)^{-3}\int V_x^2\,dx+O(1)(1+t)^{-2}\int V_{xx}^2\,dx$$

$$+O(1)(1+t)^{-1}\int V_{xxx}^2\,dx, \tag{3.2.25$_2$}$$

$$\frac{d}{dt}\int\{V_{ttt}^2-p'(V_x+\bar{v}+\hat{v})\,V_{xtt}^2\}\,dx+\alpha\int V_{ttt}^2\,dx$$

$$\leqslant O(1)(1+t)^{-13/2}+O(1)(1+t)^{-5}\int V_x^2\,dx+O(1)(1+t)^{-4}\int V_{xx}^2\,dx$$

$$+O(1)(1+t)^{-2}\int V_{xxt}^2\,dx+O(1)(1+t)^{-1}\int V_{xtt}^2\,dx$$

$$+O(1)(1+t)^{-3}\int V_{xt}^2\,dx. \tag{3.2.25$_3$}$$

Having obtained Corollary 3.1, following exactly the arguments developed by Nishihara in [16, 17], we can get the decay estimates (1.21) and (1.22) easily.

Remark 3.1. We must point that the estimates (1.21) and (1.22) also imply that, even in the L^2-setting, the decay estimates obtained in (1.21) and (1.22) can not be optimal. We only use the decay estimate on $V_{xx}(t,x)$ obtained in (1.21) to show this. In fact, from (1.21), we can get

$$\begin{gathered}\|V_{xx}(t)\|\leqslant O(1)(1+t)^{-1},\\ \int_0^t(1+s)\,\|V_{xx}(s)\|^2\,ds\leqslant O(1).\end{gathered} \tag{3.2.26}$$

If the decay estimate $(3.2.26)_1$ is optimal, i.e., there exists a constant $C_3 > 0$ such that

$$\|V_{xx}(t)\| \geqslant C_3(1+t)^{-1}, \tag{3.2.27}$$

then we must have

$$\int_0^t (1+s)\, \|V_{xx}(s)\|^2\, ds \geqslant C_3 \int_0^t \frac{ds}{1+s} = C_3 \ln(1+t), \tag{3.2.28}$$

which can not be bounded by a time-independent constant.

3.3. *The Third Step*: *Optimal* L^p *Decay Estimates* (1.23)

In this final step, we prove the optimal L^p $(2 \leqslant p \leqslant \infty)$ decay estimates (1.23). As pointed out in the intrudction, our analyses are essentially due to [18] with a slight modifications.

As in [18], $V(t, x)$ has the following integral representation

$$\begin{aligned} V(t, x) = &\int G(t, x; 0, y)\, V_0(y)\, dy \\ &+ \frac{1}{\alpha} \int_0^t \int G(t, x; s, y)(F(s, y) - V_{ss}(s, y))\, dy\, ds \\ &+ \int_0^t \int R_G(t, x; s, y)\, V(s, y)\, dy\, ds. \end{aligned} \tag{3.3.1}$$

Here $G(t, x; s, y)$ is an approximate Green function defined by

$$G(t, x; s, y) = \left(\frac{\alpha}{4\pi a(t, x)(t-s)} \right)^{1/2} \exp\left(-\frac{\alpha(x-y)^2}{4A(s, t, y)(t-s)} \right), \tag{3.3.2}$$

with

$$\begin{aligned} a(t, x) &= -p'(\bar{v}(t, x)), \\ A(s, t, y) &= -p'(\varphi(\eta)), \\ \eta &= \begin{cases} \dfrac{y}{\sqrt{1+s}}, & s = \dfrac{t}{2}, \\[2ex] \dfrac{y}{\sqrt{1+\dfrac{t}{2}}}, & s \leqslant \dfrac{t}{2}, \end{cases} \end{aligned} \tag{3.3.3}$$

and

$$R_G(t, x; s, y) = G_s(t, x; s, y) + \frac{1}{\alpha}(a(s, y)\, G_y(t, x; s, y))_y,$$
$$F(t, x) = \frac{1}{\alpha} p(\bar{v}(t, x))_{xt} - [p(V_x + \bar{v} + \hat{v}) - p(\bar{v})]_x (t, x). \tag{3.3.4}$$

Under the above notations, for each $k \leqslant 2$, we can get the following integral representation for $\partial_x^k V_x(t, x)$ and $\partial_x^k U(t, x)$:

$$\begin{aligned}\partial_x^k V_x(t, x) &= \int \partial_x^{1+k} G(t, x; 0, y)\, V_0(y)\, dy \\ &\quad + \frac{1}{\alpha} \int_0^t \int \partial_x^{1+k} G(t, x; s, y)(F(s, y) - V_{ss}(s, y))\, dy\, ds \\ &\quad + \int_0^t \int \partial_x^{1+k} R_G(t, x; s, y)\, V(s, y)\, dy\, ds \\ &:= \sum_{i=1}^{3} I_i,\end{aligned} \tag{3.3.5}$$

and

$$\begin{aligned}\partial_x^k U(t, x) &= \int \partial_t \partial_x^k G(t, x; 0, y)\, V_0(y)\, dy \\ &\quad + \partial_t \left\{ \frac{1}{\alpha} \int_0^t \int \partial_x^k G(t, x; s, y)(F(s, y) - V_{ss}(s, y))\, dy\, ds \right\} \\ &\quad + \partial_t \left\{ \int_0^t \int \partial_x^k R_G(t, x; s, y)\, V(s, y)\, dy\, ds \right\} \\ &:= \sum_{j=1}^{3} J_j.\end{aligned} \tag{3.3.6}$$

By employing the decay estimates (1.21) and (1.22) obtained in the second step, we have by mimicing the arguments developed by Nishihara, Wang, and Yang in [18] that

$$\begin{aligned}|I_i| &\leqslant O(1)(1+t)^{-2k+3/4}, \qquad i = 1, 2, 3, \\ |J_j| &\leqslant O(1)(1+t)^{-2k+5/4}, \qquad j = 1, 2, 3.\end{aligned} \tag{3.3.7}$$

Consequently

$$\begin{aligned} \|\partial_x^k V_x(t)\| &\leqslant O(1)(1+t)^{-2k+3/4}, \\ \|\partial_x^k U(t)\| &\leqslant O(1)(1+t)^{-2k+5/4}, \end{aligned} \tag{3.3.8}$$

and (1.23) follows from (3.3.8) and Sobolev's inequality.

ACKNOWLEDGMENTS

The author thanks Professor Alberto Bressan and Professor Benedetto Piccoli for their kind hospitality. He is also deeply indebted to Professor T. Yang of City University of Hong Kong and Professor C.-J. Zhu of Central China Normal University for their constant encouragement. Particular thanks go to Dr. R.-H. Pan and Dr. S.-Q. Tang for their helpful comments on the draft version of this manuscript.

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