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Pricing commodities[☆]

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ABSTRACT

How should a seller price her goods in a market where each buyer prefers a single good among his desired goods, and will buy the cheapest such good, as long as it is within his budget? We provide efficient algorithms that compute near-optimal prices for this problem, focusing on a commodity market, where the range of buyer budgets is small. We also show that our LP rounding based technique easily extends to a different scenario, in which the buyers want to buy all the desired goods, as long as they are within budget.
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1. Introduction

Pricing goods to maximize revenue is a critical yet difficult task in almost any market. We study the case of a monopolistic seller (only one seller in the market), a restricted scenario that is already quite challenging. One difficulty is to estimate the demand curves (amount of demand at different prices), but even complete knowledge of the demand curves is sufficient only in rather simple cases, e.g. if the monopolist sells only a single type of good, or if the different goods she sells cater to different markets. In such cases, the revenue-maximizing prices can be determined for each good separately, directly from that good's demand curve.

But what if goods of different types are sold all in the same market? Now, *the seller's own goods could be competing against each other* for the attention of the same buyer. This is generally true of a seller who wants to tap into multiple market segments. For example, Dell sells many models of laptops with varying features catering to varying needs of its consumers and it must price the different models carefully so that they do not eat into each other's revenue. As an example on a smaller scale, consider the pricing of movie shows. Different shows are priced differently (for example, matinee vs. evening shows) to attract different audience sections. Again, the pricing is critical—a very cheap matinee show might eat into the evening show revenue and decrease the overall revenue.

On the other hand, multiple goods might lead to higher prices by complementing each other. A very visible example is the marketing of Apple's iPod and various accessories. The strategy there is not to sell the iPod in isolation but to offer various accessories. These accessories vary from items that are expensive (for example, a charger) to items that are inexpensive (for example, songs from iTunes). Pricing for revenue maximization becomes computationally complex precisely because of this interaction between different goods. Indeed, Aggarwal, Feder, Motwani and Zhu [1] and also Guruswami, Hartline, Karlin, Kempe, Kenyon and McSherry [9] studied the computational aspects of these pricing problems, showing that in various such settings, computing the optimal prices is NP-hard.

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In one setting, referred to as *unit-demand consumers* in [9], each buyer wants to buy one good out of his desired set, as follows: There are m buyers, each of whom has an arbitrary set of desirable goods and a spending budget. The (single) seller knows the buyers' types (i.e. desired set and budget) and needs to set a price for each of the n goods. Once prices are set, every buyer buys the (single) cheapest good in his set, provided it is within his budget (breaking ties arbitrarily). Another setting, referred to as *single-minded consumers* in [9], differs from the first setting in that now each buyer only wants to buy his desired set as a bundle. That is, once prices are set, every buyer buys the entire set of his desired goods, provided its total cost is within his budget (if not, he buys nothing). We will also refer to buyers as bidders throughout this paper.

Throughout, we shall assume that the desired set of every buyer has size at most k. As we shall soon see, even the case of small k is nontrivial and interesting. In addition, we shall assume that the goods are available in *unlimited supply*, that is, the seller can sell any number of copies of the item without paying any marginal cost of production.

Several results are known about computing prices that maximize revenue in these two settings. In [1], it is shown that the problem of maximizing revenue in the unit-demand case is not only NP-hard, but APX-hard.[3] For this problem, they also give an $O(\log m)$-approximation algorithm (which uses the best single price). In [9], similar results are shown independently, and it is shown in addition that for the single-minded setting, maximizing revenue is APX-hard and that there is $\log(nm)$-approximation algorithm (which again uses the best single price). Demaine, Feige, Hajiaghayi, and Salavatipour [8] show that the above results are more or less optimal in the general single-minded bidder problem—under some complexity assumptions, there is a fixed $\delta > 0$ such that the problem cannot be approximated to within a factor of $\log^{\delta} n$. Balcan and Blum [3] present a 4-approximation algorithm for single-minded bidders and $k = 2$. Their algorithm extends to larger k, with $O(k)$-approximation. As was observed in [7], their arguments apply to the unit-demand case as well.

These results depict a rather grim landscape (at least computationally) for the problem of pricing to maximize revenue. However, many real-life instances are more specialized, and thus, a more practice-oriented approach is to identify restrictions, under which one can beat the aforementioned $O(\log n)$ factor, or better yet, obtain a very small constant-factor approximation. In particular, every percent of improvement counts in practice, requiring us to improve one (small) constant approximation factor to another.

We thus pay special attention to *commoditized markets*, [4] where the range of buyers budgets is restricted to a "small" set $\mathcal{B}$. In one such restriction, $\mathcal{B} = \{1, C\}$ is a doubleton, representing a bimodal market in which buyers are divided into poor and rich. For example, buyers coming from different referring websites such as `lastminutedeals.com` and `hotels.com` might have significantly different budgets for booking a hotel room. As another example, a tourist might be willing to pay for a Broadway show a significantly higher amount than a local. Another motivation for studying such markets could be the low descriptive complexity for the different buyers' budget types, or equivalently a low communication complexity to identify a buyer's budget. In yet another restriction, $\mathcal{B} = [1, C]$ is a small interval, representing a market with little variation, say within 50%, in the valuation of different buyers, and clearly there are numerous examples for such markets. Note that in both cases, the buyers can be completely idiosyncratic regarding their desired goods, as only the budgets are restricted.

1.1. Results and techniques

Unit-demand setting. In Section 2, we consider inputs with $\mathcal{B} = \{1, C\}$ (i.e., bimodal markets) and $k = 2$ (i.e. a desired set is a pair of goods). On the one hand, the APX-hardness results [1,9] mentioned above are actually shown for such restricted instances (in fact, for $C = 2$). On the other hand, obtaining $(2 - \frac{1}{C})$-approximation is rather easy — simply choose the best single price (same for all goods) among $\{1, C\}$ — obviously a very naive solution, but already better than the (more general) 4-approximation that can be derived from [3]. The challenge in this regime is to improve the approximation below 2, and indeed we present an algorithm achieving factor $\frac{3}{2} - \frac{1}{2C}$. Observe that even when C is not too large, this is a significant improvement (e.g. for $C = 2$, from 1.5 to 1.25). This approach easily extends to larger k, in which case the approximation we achieve is $2 - \frac{1}{k} - \frac{k-1}{kC}$.

Our algorithms are based on randomized rounding of a linear programming (LP) relaxation, a powerful paradigm that is often useful for discrete optimization (for example, see the survey of Srinivasan [15]). We "round" the prices suggested by the LP to prices in the discrete ("integral") set $\{1, C\}$. The rationale behind this rounding is that an optimal pricing may always choose prices from the set $\{1, C\}$. However, it is interesting to note that the pricing problem does not require the prices to be "discrete", and thus, the real reason behind our rounding procedure is the following: In contrast with a "standard" randomized rounding algorithm, where the probability (with which we round a variable upwards) depends linearly on the corresponding LP variable, we use a probability that is polynomial in the LP variable. The only other nonlinear randomized LP rounding algorithms that we are aware of are the approximation algorithm of Goemans and Williamson [12] for MAX SAT, and that for finding the densest k-subgraph problem that is attributed to Goemans [11]. The crux is that at every optimal basic feasible solution of our LP relaxation, all the prices are half integral [16, Chap. 14] (modulo a normalization factor), and this fact greatly simplifies the choice of the polynomial—in fact, our rounding procedure raises the variables to a power. Interestingly, the value of the power is a function of C.

3 An optimization problem is APX-hard if there exists a constant $\rho > 1$ such that it is NP-hard to approximate the optimum within factor ρ.

4 A commoditized market is one characterized by price-competition with little or no differentiation by brand.

Table 1
Comparison of results: Unit-demand setting.

	Budgets range:	Unlimited	$\{1, C\}$	$[1, C]$
$k = 2$	Here	–	$\frac{3}{2} - \frac{1}{2C}$	$1 + \ln C$
	Previous [3]	4	4	4
	Hardness [1,9]	APX-hard	APX-hard	APX-hard
General k	Here	–	$2 - \frac{1}{k} - \frac{k-1}{kC}$	–
	Previous [1,9]	$O(\ln m)$	–	–
	Hardness [1,9]	APX-hard	APX-hard	APX-hard

Table 2
Comparison of results: Single-minded setting.

	Budget range:	Unlimited	$\{1\}$
$k = 2$	Here	–	$\frac{6+\sqrt{2}}{5+\sqrt{2}} \approx 1.15$
	Previous [3,13]	4	4/3
	Hardness [9]	APX-hard	APX-hard
General k	Here	–	–
	Previous [9]	$\log nm$	–
	Hardness [8,9]	$\log^{\delta} n$	APX-hard

From a technical viewpoint, the case of $k = 2$ is really a quadratically constrained discrete optimization problem. The algorithm of [3] essentially avoids the quadratic constraints completely by eliminating a random half of the variables (setting goods prices to 0). Our LP-based approach linearizes the constraints, and thus does not decouple the two goods in each desired set, and the difficult part is of course to bound the resulting profit loss.

We further show that our algorithm can be derandomized, and that its approximation matches the LP's integrality gap, and thus it is optimal with respect to this LP. In addition, we observe that in the case where budgets come from an interval $\mathcal{B} = [1, C]$, a simple algorithm achieves $(1 + \ln C)$-approximation by computing the best single price, and that this factor matches the integrality gap of an LP relaxation that extends the LP mentioned above for the case $\{1, C\}$. We summarize our results (along with previously known results) for the unit-demand setting in Table 1.

Single-minded setting. Recently, Khandekar, Könemann and Markakis [13] have studied the case of single-minded bidders with desired sets of size at most 2, and the same budget for all the buyers, and gave a 4/3-approximation algorithm. Subsequently (but using independently derived techniques), we found out that our LP rounding approach mentioned above is easy to adapt to this setting as well, achieving $\frac{6+\sqrt{2}}{5+\sqrt{2}} \approx 1.15$ approximation. In Section 3 we briefly present this algorithm, and show a matching integrality gap. Again, this problem is known to be APX-hard because the results of [9] are actually shown for such restricted instances. Further, our algorithm obtains much better approximation than a 3/2-approximation achievable by choosing the best single price in the set $\{1/2, 1\}$, which was already better than the (more general) 4-approximation of [3]. We summarize our result, along with previously known results for the single-minded setting in Table 2.

Online pricing. Finally, we consider in Section 4 inputs with $k = 2$ (and no restriction on the budget). Using a variation of the algorithm designed by Balcan and Blum [3], we design an algorithm that works even in an *online setting*, where goods arrive sequentially (together with the bids of all the buyers interested in that good), and the seller has to determine the price of a good immediately as it arrives. This model may correspond for instance to Comcast cable TV selling video on demand, where new offerings are announced (with prices) on a regular basis. Our algorithm achieves 4-approximation, compared to the best (offline) prices. We note that [3] also give an online pricing algorithm, but in their setting buyers arrive online, and the prices (of a fixed set of goods) need to be updated.

Truthful mechanisms. We assume throughout the paper that the seller knows the budget of each bidder. We may also be interested in settings where the seller does not know such information about the market. Balcan, Blum, Hartline and Mansour [4] show that every approximation algorithm for revenue maximization can be converted into a truth-revealing mechanism. They design a general technique that loses only an additional factor of $1 + \epsilon$ in the approximation, if certain technical conditions (like sufficiently many bidders) are satisfied. Similarly to [3], we note that this technique is applicable in our setting, and thus converts our algorithms to truthful mechanisms, provided that the number of bidders is at least (roughly) Cn/ϵ^2.

1.2. Related work

The notion of revenue-maximizing pricing of goods in unlimited supply was introduced by Goldberg, Hartline, Karlin, Saks and Wright [10]. In their setting, the goods were "independent" and hence the optimization problem was trivial, and they focused on designing *truthful* mechanisms to maximize revenue. There have been numerous followup work, and we only mention here results that are directly related to our work.

Guruswami, Hartline, Karlin, Kempe, Kenyon and McSherry [9] considered the problem of revenue maximization in a variety of settings, including both unit-demand and single-minded bidders, and also envy-free pricing of goods in limited supply. As mentioned earlier, they showed logarithmic upper bounds and APX-hardness for both types of bidders. The results for the unit-demand case were also obtained independently by Aggarwal, Feder, Motwani and Zhu [1]. For single-minded bidders, a polylogarithmic hardness result, which complements the result above, was obtained by Demaine, Feige, Hajiaghayi, and Salavatipour [8]. The problem of the single-minded bidder case, where the size of the demand sets was upper bounded by k, was considered by Briest and Krysta [6] who gave an $O(k^2)$-approximation for the problem, and was improved by Blum and Balcan [3] to $O(k)$. For the special case of $k = 2$, they obtain a 4-approximation algorithm [3].

Another paper that is less directly related but was also a starting point for our work is the work of Bansal, Cheng, Cherniavsky, Rudra, Schieber and Sviridenko [5], which studies a problem of pricing over time, that was proposed in [9]. A special case of their problem gives another interpretation for the unit-demand setting: The seller is selling just one type of good (in unlimited supply), and does so over a period of n days, and can set a different price on each day. Each of the m buyers has a subset of size k of the n days, which represent the days on which he can purchase the item, and will choose to buy a copy of the good at the cheapest price he sees over the k days. The seller's aim is to maximize revenue.

1.3. Problem definitions

Our pricing problems involve one seller and m buyers. The seller has a collection V of n goods (also called items). Each $j \in V$ is a digital good, i.e., the seller has 0 marginal cost of production, or equivalently, the number of copies of j is at least the number of buyers m. Once the seller sets the prices of the goods, each buyer will buy a collection of goods, based on his own utility function. The seller's problem is to determine a price p_j of each good $j \in V$ so as to maximize revenue. Depending on the utility functions of the buyers, we have the following variations of the pricing problem. The first variation is our main focus, but we will also show how the techniques we develop also work for the second variation.

1. Unit-demand bidders:. We let $UD_k(\mathcal{B})$ denote the problem of item pricing for unit-demand bidders with sets of size at most k, and bids from the set $\mathcal{B}$, as follows. Buyer i has a budget of $u_i \in \mathcal{B}$ and a subset S_i of desirable goods, with $|S_i| \leq k$. He is interested in buying *exactly one* good from S_i, and given prices on the goods, he will buy the cheapest good in S_i, provided that its price is at most u_i. For a price vector $\mathbf{p} = (p_1, \ldots, p_m)$, let $\pi_i(\mathbf{p})$ be the revenue that the seller obtains from buyer i if the prices are set to $\mathbf{p}$. Thus

$$\pi_i(\mathbf{p}) = \begin{cases} \min\{p_j : j \in S_i\} & \text{if } \min\{p_j : j \in S_i\} \leq u_i \\ 0 & \text{otherwise.} \end{cases}$$

Thus the seller's problem is: Find $\mathbf{p}$ so as to maximize $\sum_{i=1}^{n} \pi_i(\mathbf{p})$. We are interested in the following special cases of this problem, defined by different values of k and $\mathcal{B}$: (1) $UD_k([1, C])$ for $C > 1$; and (2) $UD_k(\{1, C\})$ for $C > 1$.

2. Single-minded bidders:. We let $SM_k(\mathcal{B})$ denote the problem of item pricing for single-minded bidders, who have sets of size k and bids from the set $\mathcal{B}$, as follows. Buyer i has a budget of $u_i \in \mathcal{B}$ and a subset S_i of V of desirable goods with $|S_i| \leq k$. He is interested in buying *all* the goods in the set S_i. For a price vector $\mathbf{p}$, let π_i be the revenue that the seller obtains from buyer i, if the prices are set to $\mathbf{p}$. Thus

$$\pi_i(\mathbf{p}) = \begin{cases} \sum_{j \in S_i} p_j & \text{if } \sum_{j \in S_i} p_j \leq u_i \\ 0 & \text{otherwise.} \end{cases}$$

Again, the seller's problem is: Find $\mathbf{p}$ so as to maximize $\sum_{i=1}^{n} \pi_i(\mathbf{p})$. We will show how our techniques for $UD_2(\{1, C\})$ extend to give an algorithm for the case $SM_2(\{1\})$. Note that when $|\mathcal{B}| = 1$, then all budgets can be scaled to 1, and we may take $\mathcal{B} = \{1\}$ without loss of generality.

The case of $k = 2$*: Pricing on a graph*

Following [3], for $k = 2$, $UD_2(\mathcal{B})$ becomes a problem of pricing the vertices of a graph, with the buyers' desired sets corresponding to the edges of the graph. This will be our main focus in demonstrating our techniques and analysis. We study two settings of budget ranges: $\mathcal{B} = \{1, C\}$ and $\mathcal{B} = [1, C]$, for $C > 1$.

Given a graph $G = (V, E)$ (possibly with self-loops and parallel edges), along with edge weights $c_{ij} \in \mathcal{B}$ for every edge $(i, j) \in E$, the goal is to set prices p_i on every vertex i so as to maximize the total revenue, where the revenue from an edge $(i, j) \in E$ is:

$$\pi_{ij} = \begin{cases} \min(p_i, p_j) & \text{if } \min(p_i, p_j) \leq c_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

The case of $SM_2(\mathcal{B})$, studied in [3], is defined as a pricing problem on a graph analogously.

$$\max \sum_{(i,j)\in E} \pi_{ij} \qquad \text{subject to:}$$

$$\begin{array}{llr}
\forall (i,j)\in E, c_{ij}=C & \pi_{ij} \le 1+p_i & (1)\\
\forall (i,j)\in E, c_{ij}=C & \pi_{ij} \le 1+p_j & (2)\\
\forall (i,j)\in E, c_{ij}=1 & \pi_{ij} \le 1 & (3)\\
\forall (i,j)\in E, i\neq j, c_{ij}=1 & \pi_{ij} \le 2-\dfrac{p_i+p_j}{C-1} & (4)\\
\forall (i,i)\in E, c_{ii}=1 & \pi_{ii} \le 1-\dfrac{p_i}{C-1} & (5)\\
\forall i\in V & 0\le p_i \le C-1 & (6)\\
\forall (i,j)\in E & \pi_{ij}\ge 0 & (7)
\end{array}$$

Fig. 1. LP relaxation for the unit-demand setting.

1.4. *LP terminology*

We shall use standard LP terminology, with a slight abuse notation due to the fact that all our LPs are bounded. Recall that a solution to an LP is called *feasible* if it satisfies all the constraints. A feasible solution is called an *extreme point* if it cannot be written as the convex combination of two distinct feasible solutions. For every bounded LP, the optimum can be attained at an extreme point, and furthermore such an optimum solution can be computed in polynomial time. In the sequel, we shall freely exchange the notion of an extreme point with the (perhaps more common) notion of a basic feasible solution.

2. Unit-demand buyers in commoditized markets

In this section, we look at pricing for unit-demand bidders with restricted valuations. We start with valuations restricted to the set $\{1, C\}$ for some $C > 1$. In other words, we are interested in pricing schemes for the $UD_k(\{1, C\})$ model. Our main result is a pricing scheme that generates a revenue within a factor $\frac{(2k-1)C-k+1}{kC}$ of the optimal revenue (Theorem 2.4). For ease of exposition, we present the proofs for the $k = 2$ case.

Our pricing scheme rounds an LP relaxation for the problem. Theorem 2.2 shows that our rounding algorithm (for the case $k = 2$) has an approximation factor of $\frac{3C-1}{2C}$. We show in Section 2.4 that the integrality gap of our LP relaxation is at least $\frac{3C-1}{2C}$ demonstrating that our rounding procedure is tight (optimal).

It is not difficult to verify that the LP in Fig. 1 is a relaxation for our pricing problem $UD_2(\{1, C\})$; note that $p_i \in [0, C-1]$ and that the price set to vertex i is actually $p_i + 1$.

2.1. *On the optimal LP solutions*

We first observe that an optimal basic feasible solution to the LP relaxation is half integral, in the sense that all the p_i variables are from the set $\{0, \frac{C-1}{2}, C-1\}$. Recall that the price of vertex i is actually $p_i + 1$.

Proposition 2.1. *Every optimal basic feasible solution to the LP in Fig. 1 is half integral. More precisely, if $(\{p_i^*\}_{i\in V}, \{\pi_{ij}^*\}_{(i,j)\in E})$ is such a solution then $p_i^* \in \{0, \frac{C-1}{2}, C-1\}$ for all i.*

Proof. Let $(\{p_i^*\}_{i\in V}, \{\pi_{ij}^*\}_{(i,j)\in E})$ be an optimal solution, and suppose that the $p_{(\cdot)}$ are not half integral. We will show that such an assignment is not an extreme point by exhibiting two feasible solutions (p^-, π^-) and (p^+, π^+) such that for all $i \in V$, $p_i^* = \frac{1}{2}(p_i^- + p_i^+)$ and for every $(i,j) \in E$, $\pi_{ij}^* = \frac{1}{2}(\pi_{ij}^+ + \pi_{ij}^-)$.

In the sequel, we may use values given to $\{p_i\}_{i\in V}$ to define, for each $(i,j) \in E$,

$$\pi_{ij} = \begin{cases} 1+\min(p_i, p_j) & \text{if } c_{ij} = C\\ \min\left(1, 2-\frac{p_i+p_j}{C-1}\right) & \text{if } i\neq j \text{ and } c_{ij}=1\\ 1-\frac{p_i}{C-1} & \text{if } i=j \text{ and } c_{ij}=1. \end{cases} \tag{8}$$

Note that if $0 \le p_i \le C-1$ for every $i \in V$ then the resulting solution (p, π) is feasible. Furthermore, the above assignment to π_{ij} maximizes the objective function. In particular, since (p^*, π^*) is an optimal solution, we may assume that $\{\pi_{ij}^*\}$ were defined from $\{p_i^*\}$ according to (8).

We proceed to exhibit the two aforementioned solutions (p^+, π^+) and (p^-, π^-). In fact, we shall only define explicitly p^+ and p^-; the corresponding π^+ and π^- are defined according to (8). Define the following two subsets of vertices:

$V^+ = \{i \mid \frac{C-1}{2} < p_i^* < C-1\}$ and $V^- = \{i \mid 0 < p_i < \frac{C-1}{2}\}$. By the assumption that p^* is not half integral, $V^- \cup V^+ \neq \emptyset$. Let $\epsilon > 0$ be a small enough number (to be defined later). We define the two related "price" assignments.

$$p_i^+ = \begin{cases} p_i^* + \epsilon & \text{if } i \in V^+ \\ p_i^* - \epsilon & \text{if } i \in V^- \\ p_i^* & \text{otherwise} \end{cases} \qquad p_i^- = \begin{cases} p_i^* - \epsilon & \text{if } i \in V^+ \\ p_i^* + \epsilon & \text{if } i \in V^- \\ p_i^* & \text{otherwise.} \end{cases}$$

We set $\epsilon = \frac{1}{4}\min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, C-1\}$ where:

$$\begin{aligned} \epsilon_1 &= \min\{|p_i^* - p_j^*| : p_i^* \neq p_j^*, (i,j) \in E \text{ and } c_{ij} = C\}, \\ \epsilon_2 &= \min\left\{\left|1 - \frac{p_i^* + p_j^*}{C-1}\right| : p_i^* + p_j^* \neq C-1, (i,j) \in E, \text{ and } c_{ij} = 1\right\} \\ \epsilon_3 &= \min\{p_i^* : i \in V^-\} \\ \epsilon_4 &= \min\{C - 1 - p_i^* : i \in V^+\}. \end{aligned}$$

First note that $\epsilon > 0$, which implies that $p^+ \neq p^*$ and $p^- \neq p^*$. Further, by the choice of ϵ_3, ϵ_4 and the fact that $\epsilon < \min(\epsilon_3, \epsilon_4, (C-1)/2)$, we have that for every $i \in V$, $0 \le p_i^+ \le C-1$ and $0 \le p_i^- \le C-1$. As observed earlier, this implies that (p^+, π^+) and (p^-, π^-) are feasible solutions to the LP.

Obviously, for all $i \in V$, $p_i^* = \frac{1}{2}\left(p_i^+ + p_i^-\right)$. To complete the proof, we will show that for every $(i,j) \in E$,

$$\pi_{ij}^* = (\pi_{ij}^+ + \pi_{ij}^-)/2. \tag{9}$$

Let $E_1 = \{(i,j) \in E \mid i,j \notin V^+ \cup V^-\}$. Clearly, for $(i,j) \in E_1$, we have $\pi_{ij}^* = \pi_{ij}^+ = \pi_{ij}^-$ and hence (9) holds. It thus remains to consider edges $(i,j) \in E \setminus E_1$, i.e., edges with at least one endpoint in $V^+ \cup V^-$.

First consider the case where $i \neq j$ and $c_{ij} = C$. In this case, $\pi_{ij}^* = 1 + \min(p_i^*, p_j^*)$, $\pi_{ij}^+ = 1 + \min(p_i^+, p_j^+)$ and $\pi_{ij}^- = 1 + \min(p_i^-, p_j^-)$. If $p_i^* = p_j^*$ (both equal to p^*, say), then $\pi_{ij}^* = 1 + p^*$. Further, either both $i, j \in V^+$ or both $i, j \in V^-$. In the former case, $\pi_{ij}^+ = 1 + p^* + \epsilon$ and $\pi_{ij}^- = 1 + p^* - \epsilon$, while in the latter case $\pi_{ij}^+ = 1 + p^* - \epsilon$ and $\pi_{ij}^- = 1 + p^* + \epsilon$. In either case, (9) is satisfied. If $p_i^* \neq p_j^*$, then by the definition of ϵ if p_i^* (without loss of generality) is the minimum price for (i,j), then so are p_i^+ and p_i^-; this is because by the choice of ϵ_1 and the fact that $\epsilon \le \epsilon_1/4$, we have $p_i^* + \epsilon < p_j^* - \epsilon$. Again by the definitions of p^+ and p^-, (9) is satisfied.

Now consider the case where $i \neq j$ and $c_{ij} = 1$. We now consider three subcases. First if $p_i^* + p_j^* < C-1$, then as $\epsilon < \epsilon_2/2$, we have $p_i^+ + p_j^+ < C-1$ and $p_i^- + p_j^- < C-1$. Thus, we have $\pi_{ij}^* = \pi_{ij}^+ = \pi_{ij}^- = 1$, which implies that (9) is satisfied. Second if $p_i^* + p_j^* > C-1$, then again as $\epsilon < \epsilon_2/2$, we have $p_i^+ + p_j^+ > C-1$ and $p_i^- + p_j^- > C-1$. This implies that $\pi_{ij}^* = 2 - (p_i^* + p_j^*)/(C-1)$, $\pi_{ij}^+ = 2 - (p_i^+ + p_j^+)/(C-1)$ and $\pi_{ij}^- = 2 - (p_i^- + p_j^-)/(C-1)$, which implies (9). Finally, if $p_i^* + p_j^* = C-1$ then exactly one of p_i^*, p_j^* is in V^- and the other is in V^+, thus $\pi_{ij}^* = \pi_{ij}^+ = \pi_{ij}^-$, which implies (9).

If $i = j$, then π_{ii} depends linearly on p_i. As $p_i^* = \frac{1}{2}(p_i^+ + p_i^-)$, in this case (9) is also satisfied. ■

2.2. A rounding algorithm

Consider the following randomized algorithm, where $\tau > 0$ is a parameter (to be chosen later).

Algorithm **Algo**(τ):

1. Solve the LP in Fig. 1 and obtain an optimal basic feasible solution with prices variables $\{p_i\}_{i \in V}$.
2. For every $i \in V$, independently assign a price of C with probability $\left(\frac{p_i}{C-1}\right)^\tau$ and a price of 1 with probability $1 - \left(\frac{p_i}{C-1}\right)^\tau$.

We now analyze the performance of the rounding algorithm above.

Theorem 2.2. *For every $C > 1$, there is $\tau > 0$ such that* **Algo**(τ) *is a $(3C-1)/(2C)$-approximation for the pricing problem with unit-demand bidders, $k = 2$, and budgets from $\mathcal{B} = \{1, C\}$. That is, the expected revenue of* **Algo**(τ) *is at least $\frac{2C}{3C-1}$ fraction of the optimum for $UD_2(\{1, C\})$.*

Proof. Set $\tau = \frac{1}{2}\log\left(\frac{3C-1}{C-1}\right)$. For notational convenience, we will denote **Algo**(τ) by **Algo**. Let the optimal (extreme point) solution of the LP assign prices p_i^* to every vertex i and obtain a revenue of π_{ij}^* from every edge (i,j). We will show that for every edge $(i,j) \in E$, the expected revenue of **Algo** from that edge is at least $\frac{2C}{3C-1} \cdot \pi_{ij}^*$; the theorem then follows by linearity of expectation. For the rest of the proof, it will be convenient to define, for every $i \in V$, $q_i = \frac{p_i^*}{C-1}$. By Proposition 2.1, we have $q_i \in \{0, \frac{1}{2}, 1\}$. The main idea is to analyze the different edge types and chose τ so as to balance the revenues obtained in the different cases.

Let us first consider the case when $i \neq j$. We have two subcases.

Case 1a: $c_{ij} = 1$. In this case $\pi^*_{ij} \leq \min(1, 2-q_i-q_j)$, while **Algo** obtains an expected revenue of $0 \cdot (q_i^\tau q_j^\tau) + 1 \cdot (1 - q_i^\tau q_j^\tau) = 1 - (q_i q_j)^\tau$. When $q_i = q_j = 1$ then both the LP and **Algo** obtain a revenue of 0. When $q_i + q_j = \frac{3}{2}$ then the ratio of the revenue obtained by **Algo** to π^*_{ij} (which is 1/2) is $2(1 - \frac{1}{2^\tau}) > 1 - \frac{1}{2^{2\tau}}$.[5] Finally, when $q_i + q_j \leq 1$, then $\pi^*_{ij} = 1$, while **Algo** obtains the least revenue when $q_i = q_j = \frac{1}{2}$, which implies a ratio of at least $1 - \frac{1}{2^{2\tau}} = \frac{2C}{3C-1}$ in all the possibilities.

Case 1b: $c_{ij} = C$. In this case $\pi^*_{ij} \leq 1 + (C-1)\min(q_i, q_j)$, while **Algo** obtains an expected revenue of $C \cdot (q_i^\tau q_j^\tau) + 1 \cdot (1 - q_i^\tau q_j^\tau) = 1 + (C-1)(q_i q_j)^\tau$. W.l.o.g. assume that $q_j \geq q_i$. Thus, the ratio of the revenue obtained by **Algo** and π^*_{ij} is at least:

$$\begin{aligned}
\min_{q_i, q_j \in \{0, \frac{1}{2}, 1\}, q_j \geq q_i} \frac{1 + (C-1)(q_i q_j)^\tau}{1 + (C-1)\min(q_i, q_j)} &\geq \min_{q_i \in \{0, \frac{1}{2}, 1\}} \frac{1 + (C-1)q_i^{2\tau}}{1 + (C-1)q_i} \\
&= \frac{1 + \frac{C-1}{2^{2\tau}}}{1 + \frac{C-1}{2}} \\
&= \frac{2C}{3C-1}. \qquad (10)
\end{aligned}$$

We now consider the case $i = j$. Again we have two subcases.

Case 2a: $c_{ii} = 1$. In this case $\pi^*_{ii} \leq \min(1, 1 - q_i) = 1 - q_i$, while **Algo** gets a revenue of $0 \cdot q_i^\tau + 1 \cdot (1 - q_i^\tau) = 1 - q_i^\tau$. Thus, the ratio of the revenue of **Algo** to π^*_{ii} is at least

$$\min_{q_i \in \{0, \frac{1}{2}, 1\}} \frac{1 - q_i^\tau}{1 - q_i} = \min(1, 2 - 2^{1-\tau}) \geq 1 - \frac{1}{2^{2\tau}} = \frac{2C}{3C-1}.$$

Case 2b: $c_{ii} = C$. In this case $\pi_{ii} \leq 1 + (C-1)q_i$. The expected revenue for **Algo** is $1 \cdot (1 - q_i^\tau) + C \cdot q_i^\tau = 1 + (C-1)q_i^\tau \geq 1 + (C-1)q_i^{2\tau}$. Thus, from (10), the ratio is at least $\frac{2C}{3C-1}$.

Thus, in all cases for every edge $(i, j) \in E$, **Algo** obtains an expected revenue of at least $\frac{2C}{3C-1} \cdot \pi^*_{ij}$, as desired. ■

2.3. Derandomization

Algorithm **Algo**(τ) can be derandomized in a straightforward way using standard techniques. In particular, observe that the analysis of the randomized rounding step only required pairwise independence among the random choices. One can use a small family of pairwise independent random variables (see the survey [14] for such constructions) and exhaustively try all the possibilities in this space.

Alternatively, one can employ the method of conditional expectation [2,15], since the expected revenue after randomized rounding is an easy formula to calculate (given the probabilities).

2.4. A tight integrality gap

Next, we show that Theorem 2.2 is the best one can hope from any algorithm that rounds the LP. Formally, we prove the following.

Proposition 2.3. *There exist an instance of $UD_2(\{1, C\})$ for which the integrality gap of the LP in Fig. 1 is at least $\frac{3C-1}{2C}$.*

Proof. Consider the graph with two vertices and C parallel edges—one of which has a cost of C and the rest have a cost of 1. (This assumes that C is integral; if however C is not integral, we need to choose an appropriate number of cost 1 edges and cost C edges such that their ratio is C.) The optimal revenue is C. However, the LP can set a price of $\frac{C+1}{2}$ on both the vertices to get a revenue of $\frac{C+1}{2}$ from the cost C edge and a revenue of 1 from each of the cost 1 edges. Thus, the integrality gap is at least

$$\frac{1 \cdot (\frac{C+1}{2}) + (C-1) \cdot 1}{C} = \frac{3C-1}{2C}. \quad ■$$

2.5. The general case

The results presented for $k = 2$ in the previous sections can be suitably modified to work for the general case. The LP relaxation for general k is the natural one. For example, the constraint (4), the sum $p_i + p_j$ will be replaced by $\sum_{j=1}^{k} p_{i_j}$ for the hyperedge $(p_{i_1}, p_{i_2}, \ldots, p_{i_k})$. The "half integrality" gap result will now say that the prices are in the set

[5] To see why this is true set $a = 2^{-\tau}$ and note that we have to show that $2 - 2a > 1 - a^2$, which is true for $a \neq 1$. The latter is true as $\tau > 0$.

$\{0, (1-1/k)(C-1), C-1\}$. Finally, we can prove the following counterparts of Theorem 2.2 and Proposition 2.3 by straightforward generalizations of their proofs

Theorem 2.4. *For every $C > 1$, there an algorithm that is a $\frac{(2k-1)C-k+1}{kC}$ approximation for the pricing problem with unit-demand bidders with demand size at most k and budgets from $\mathcal{B} = \{1, C\}$.*

Proposition 2.5. *There exist an instance of $UD_k(\{1, C\})$ for which the integrality gap of the LP used above is at least $\frac{(2k-1)C-k+1}{kC}$.*

2.6. Budget range $[1, C]$

Another interesting restriction on the range of buyer's budgets is to an interval $\mathcal{B} = [1, C]$, which clearly generalizes the previous doubleton case $\{1, C\}$. For this case, denoted $UD_2([1, C])$, we obtain the following approximation.

Proposition 2.6. *For every $C > 1$, there is a polynomial time $(1 + \ln C)$-approximation algorithm for the unit-demand pricing problem with $k = 2$ and budgets from $\mathcal{B} = [1, C]$.*

Proof. Consider the best single price (same price for all goods) in the range $[1, C]$. We claim that the revenue from this single price is always within factor $1 + \ln C$ of the sum of budgets of all the buyers (denote this quantity by B), and clearly B is an upper bound on the maximum revenue. Assuming the claim, the proof is complete by observing that the best single price can always be attained by one of the budgets appearing in the input, and thus the algorithm need only try at most n different prices.

We conclude the proof by proving that for every $\epsilon > 0$, there exists a p_ϵ price in $[1, C]$ such that the revenue from p_ϵ is at least $\frac{B}{1+\epsilon+\frac{\epsilon \ln C}{\ln(1+\epsilon)}}$. Note that this implies the existence of a price that attains a revenue of at least $B/(1 + \ln C)$ as the above result holds for every $\epsilon > 0$ (and $\lim_{\epsilon\to 0} \epsilon/\ln(1+\epsilon) \to 1$). Next, we argue the existence of p_ϵ. First round down all the budgets to the largest power of $(1+\epsilon)$. Note that this changes the total budget to $B' \geq B/(1+\epsilon)$. Further, after rounding down there are $m \leq \ln_{1+\epsilon} C$ many distinct budget values greater than 1. For $0 \leq i \leq m$, let there be n_i budgets with values $(1+\epsilon)^i$. Note that $B' = \sum_{i=0}^{m}(1+\epsilon)^i n_i$. Further, for every $0 \leq i \leq m$, let $R_i = (1+\epsilon)^i \sum_{j=i}^{m} n_j$ be the revenue obtained by fixing the single price to $(1+\epsilon)^i$. It is not too hard to verify that

$$B' = R_0 + \frac{\epsilon}{1+\epsilon}\left(\sum_{i=1}^{m} R_i\right).$$

Note that best possible revenue is

$$\max_{0\leq i\leq m} R_i.$$

Thus, the best possible revenue is at least the value of the optimum to the following mathematical program

$$\min\left(\max_{0\leq i\leq m} y_i\right) \text{ subject to:}$$

$$y_0 + \frac{\epsilon}{1+\epsilon}\left(\sum_{i=1}^{m} y_i\right) = B' \tag{11}$$

$$y_i \geq 0 \quad \text{for every } 0 \leq i \leq m.$$

We claim that the optimum above occurs when all the y_i's are equal. (If not, we could change the y_i's and y_j's with the largest and second largest values by small enough amounts so that the constraint (11) is still satisfied but the objective value decreases.)

Thus, the best single price obtains a revenue of at least

$$\frac{B'}{1+\frac{\epsilon m}{1+\epsilon}} \geq \frac{B'}{1+\frac{\epsilon \ln_{1+\epsilon} C}{1+\epsilon}} \geq \frac{B}{(1+\epsilon)\left(1+\frac{\epsilon \ln C}{(1+\epsilon)\ln(1+\epsilon)}\right)} = \frac{B}{1+\epsilon+\frac{\epsilon \ln C}{\ln 1+\epsilon}},$$

as desired. ■

One can try a natural extension of our LP-relaxation technique for $\{1, C\}$ to this more general case $[1, C]$. However, it turns out that the resulting LP has integrality gap $1 + \ln C$, and thus cannot offer improved approximation.

$$\max \sum_{(i,j)\in E} \pi_{ij} \qquad \text{subject to:}$$

$$\begin{aligned}
&\forall (i,i)\in E && \pi_{ii} \le p_i && (12)\\
&\forall (i,j)\in E, i\neq j && \pi_{ij} \le p_i + p_j && (13)\\
&\forall (i,j)\in E, i\neq j && \pi_{ij} \le 2 - p_i - p_j && (14)\\
&\forall i\in V && 0 \le p_i \le 1 && (15)\\
&\forall (i,j)\in E && \pi_{ij} \ge 0 && (16)
\end{aligned}$$

Fig. 2. LP relaxation for the single-minded bidders setting.

3. Single-minded buyers in commoditized markets

We now consider the pricing problem for single-minded bidders when all the bidders have the same budget, which can be assumed w.l.o.g. to be 1. That is, we are interested in pricing schemes for the $SM_2(\{1\})$ model. We extend our techniques from Section 2 to get a pricing algorithm with an approximation factor of $\frac{6+\sqrt{2}}{5+\sqrt{2}} \approx 1.156$ (Theorem 3.2). As in the case of single-minded bidders, our rounding procedure is tight (optimal), as we show that this LP relaxation has a matching integrality gap.

It is not difficult to verify the LP in Fig. 2 is a relaxation for our problem $SM_2(\{1\})$.

As in the $UD_2(\{1, C\})$, we first observe that an optimal basic feasible solution to the LP relaxation is half integral.

Proposition 3.1. *Every optimal basic feasible solution to the LP in Fig. 2 is half integral. More precisely, if $(\{p_i^*\}_{i\in V}, \{\pi_{ij}^*\}_{(i,j)\in E})$ is such a solution then $p_i^* \in \{0, \frac{1}{2}, 1\}$ for all i.*

The proof is very similar to that of Proposition 2.1 and is omitted (we have to analyze fewer because $C = 1$).

We next analyze the following randomized algorithm[6].

Algorithm $\mathbf{Algo}_{SM}$:

1. Solve the LP in Fig. 2 to obtain an optimal basic feasible solution with price variables $\{p_i\}_{i\in V}$.
2. Fix prices according to the three schemes below and pick the one that generates the maximum revenue.
   (a) Assign a price p_i to vertex i.
   (b) If $p_i \neq 1$, assign a price of p_i to vertex i, else assign a price of $1/2$.
   (c) If $p_i \neq 1/2$, assign a price of p_i to vertex i, else assign a price of 0 with probability $1/\sqrt{2}$ and a price of 1 with probability $1 - 1/\sqrt{2}$.

Theorem 3.2. **Algo**$_{SM}$ *achieves $\frac{6+\sqrt{2}}{5+\sqrt{2}}$ approximation for the pricing problem with single-minded bidders, desired sets of size at most 2, and unit budgets. That is, expected revenue of* **Algo**$_{SM}$ *is at least $\frac{5+\sqrt{2}}{6+\sqrt{2}}$ fraction of the optimum for $SM_2(\{1\})$.*

Proof. Fix an arbitrary edge $(i,j)\in E$. Let π_{ij}^a, π_{ij}^b and π_{ij}^c be the (expected) revenue that the pricing schemes in steps 2(a), 2(b) and 2(c) in **Algo**$_{SM}$ generate for that edge. By linearity of expectation, the claimed result will follow if

$$\max\left(\pi_{ij}^a, \pi_{ij}^b, \pi_{ij}^c\right) \ge \frac{5+\sqrt{2}}{6+\sqrt{2}} \cdot \pi_{ij}. \tag{17}$$

To prove the above, we will show the following inequality.

$$\frac{1}{2}\cdot \pi_{ij}^a + \frac{2}{6+\sqrt{2}}\cdot \pi_{ij}^b + \frac{2+\sqrt{2}}{2(6+\sqrt{2})}\cdot \pi_{ij}^c \ge \frac{5+\sqrt{2}}{6+\sqrt{2}}\cdot \pi_{ij}. \tag{18}$$

By Proposition 3.1, we know that $p_i, p_j \in \{0, \frac{1}{2}, 1\}$. A simple case analysis (for the eight possible values of (p_i, p_j)) proves (18). For example, consider the case when $p_i = 1/2$ and $p_j = 1$. In this case $\pi_{ij} = 1/2$ (due to constraint (14)

[6] A minor technical modification is required: since the probability used in step 2(c) is irrational, we need to approximate it so as to work in polynomial time.

in the LP). It is easy to check that $\pi^a_{ij} = 0, \pi^b_{ij} = 1$ and $\pi_{ij} = 1/\sqrt{2} \cdot 1 + (1 - 1/\sqrt{2}) \cdot 0$. Thus, we have

$$\frac{1}{2} \cdot \pi^a_{ij} + \frac{2}{6+\sqrt{2}} \cdot \pi^b_{ij} + \frac{2+\sqrt{2}}{2(6+\sqrt{2})} \cdot \pi^c_{ij} = \frac{1}{2} \cdot 0 + \frac{2}{6+\sqrt{2}} \cdot 1 + \frac{2+\sqrt{2}}{2(6+\sqrt{2})} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{5+\sqrt{2}}{6+\sqrt{2}} \cdot \frac{1}{2} = \frac{5+\sqrt{2}}{6+\sqrt{2}} \cdot \pi_{ij},$$

as desired. The calculations for the other cases (and the case of self-loops, i.e. $i = j$) are similar and omitted. ■

The rounding procedure above is tight (optimal), as the following proposition shows.

Proposition 3.3. *Let $\epsilon > 0$ be a real number. There exist an instance of $SM_2(\{1\})$ for which the integrality gap of the LP in Fig. 2 is at least $\frac{6+\sqrt{2}}{5+\sqrt{2}} - \epsilon$.*

Proof. Consider the following graph, with all costs being 1. The graph consist of $n + 1$ vertices; n of the vertices form a complete graph, with each edge having a multiplicity of $\frac{a}{n}$, where $a = 1 + \sqrt{2}$.[7] The last vertex, call it w, has $2n$ self-loops and an edge to each of the other n vertices (with multiplicity of 1).

The idea is the following. The LP will set the price of w to be 1 and the rest of the prices are set to $\frac{1}{2}$. Thus, the LP gets a total revenue of

$$\frac{a}{n} \cdot \frac{n(n-1)}{2} + 2n + \frac{n}{2} = \frac{a}{2}(n-1) + \frac{5n}{2} = \frac{a+5}{2}n - \frac{a}{2}.$$

Below we will show that the OPT for this graph is at most (neglecting lower-order terms):

$$\left(\max\left(\frac{a+2}{4}, \frac{a}{2}, \frac{(a+1)^2}{4a}\right) + 2\right) \cdot n \tag{19}$$

Now for $a \geq 2$, $\frac{a}{2} \geq \frac{a+2}{4}$. Further, $\frac{(a+1)^2}{4a} \geq \frac{a}{2}$ if $a^2 - 2a - 1 \leq 0$, which happens if $a \leq 1 + \sqrt{2}$. Finally, $\frac{(a+1)^2}{4a} > \frac{a+2}{4}$. Thus, we have the following:

$$\max\left(\frac{a+2}{4}, \frac{a}{2}, \frac{(a+1)^2}{4a}\right) = \begin{cases} \frac{(a+1)^2}{4a} & \text{if } a \leq 1+\sqrt{2} \\ \frac{a}{2} & \text{if } a \geq 1+\sqrt{2}. \end{cases}$$

The integrality gap is maximized when $a = 1 + \sqrt{2}$, which for large enough n gives a gap of at least

$$\frac{\frac{a+5}{2}}{\frac{a}{2}+2} - \epsilon = \frac{a+5}{a+4} - \epsilon = \frac{6+\sqrt{2}}{5+\sqrt{2}} - \epsilon.$$

Next, we prove (19). It will be convenient to denote $\alpha = \frac{a}{n}$.

First observe that in OPT, there is always a pricing where w is priced at 1,[8] which gives a revenue of $2n$ from the self-loops. Now assume of the remaining n vertices, $x \geq 0$, $y \geq 0$ and $z \geq 0$ nodes have prices 0, 1/2 and 1 respectively. Note that $z + x + y = n$. In this case, the revenue from the non-loop edges is

$$x + \alpha\left(\frac{y(y-1)}{2} + \frac{xy}{2} + xz\right)$$
$$= x + \frac{\alpha y(y-1)}{2} + \frac{\alpha xy}{2} + \alpha nx - \alpha xy - \alpha x^2$$
$$= x(\alpha n + 1 - \alpha x) + \frac{\alpha y(y-x-1)}{2}. \tag{20}$$

Now if $y < x + 1$, then to maximize (20), it is better to set $y = 0$. In this case we want to maximize

$$x(\alpha n + 1 - \alpha x),$$

which happens at $x = \frac{\alpha n+1}{2\alpha}$ giving a value of

$$\frac{\alpha n+1}{2\alpha}\left(\frac{\alpha n+1}{2}\right) = \frac{(\alpha n+1)^2}{4\alpha} = \frac{(a+1)^2}{4a} \cdot n. \tag{21}$$

7 The multiplicity $\frac{a}{n}$ is not an integer or even rational; formally, we approximate it using a rational number to within any desired accuracy, and then increase all the multiplicities by the same large enough factor, which cancels out in the final ratio.

8 If w is priced at 0 then obviously, it is not optimal. If the price is $\frac{1}{2}$, then the total revenue from self-loops is n and another at most n from the edges from w to other nodes. If price is changed to 1, one gets a revenue of $2n$ from the self-loops, which is at least as much as the case when price is set to $\frac{1}{2}$.

Now if $y \geq x + 1$, then for any *fixed* x, it is good to set y as large as possible to maximize (20). That is, set $y = n - x$. Substituting this in (20), we need to maximize

$$\begin{aligned} x(1+\alpha n) - \alpha x^2 + \left(\frac{\alpha n - \alpha x}{2}\right)(n-1-2x) = \\ &= x + \alpha n x - \alpha x^2 + \frac{\alpha n(n-1)}{2} - \alpha n x - \frac{\alpha x(n-1)}{2} + \alpha x^2 \\ &= \frac{\alpha n(n-1)}{2} + x\frac{(2+\alpha-\alpha n)}{2} \\ &\simeq \left(\frac{a}{2} + \frac{x(2-a)}{2n}\right)\cdot n, \end{aligned} \tag{22}$$

where in the last equation, we removed lower-order terms. Now to maximize (22), there are two cases. If $a > 2$, then one should set $x = 0$ and if $a \leq 2$, the x should be set to as large a value as possible, which is $\frac{n}{2}$ (recall that we are in the case that $y \geq x + 1$ and also that $x + y \leq n$). Thus, (22) is at most

$$\max\left(\frac{a}{2}, \frac{a+2}{4}\right)\cdot n. \quad \blacksquare$$

We note that the case of $k = 2$ and $B = \{1\}$ is the simplest nontrivial case for the single-minded buyers model. Our algorithm and proofs crucially use this, and while it may be possible to extend the techniques to more complex scenarios (e.g., for larger k, or $B = \{1, C\}$), we do not see an easy extension.

4. An online 4-approximation

In this section, we consider the following online version of the $UD_2(\cdot)$ and $SM_2(\cdot)$ problems. Buyers are assumed to be "in the system" at the beginning and the goods arrive in an online fashion. When a good arrives, any buyer who is interested submits a bid and the seller has to price this good before the next good arrives. We assume that every buyer is interested in at most two items and the seller knows the identity of each buyer but only finds out about the exact set of elements the buyer is interested in after the buyer has bid for *both* the items he is interested in. The price that a buyer pays follows the same rules as in $UD_2(\cdot)$ and $SM_2(\cdot)$ models respectively. In the graph abstraction of the $UD_2(\cdot)$ and $SM_2(\cdot)$, the online model has the following interpretation. At every step, a vertex in the underlying graph arrives. Once a vertex appears, all the edges incident on it (along with the edge weights) are revealed to the seller. But the only way the seller knows about the other endpoint of an edge is if that vertex had arrived earlier. Under these constraints, the seller has to price every vertex as it arrives, so as to make as much revenue as possible. For the rest of the section, we will only talk about the $UD_2(\cdot)$ model. The discussion holds equally well for the $SM_2(\cdot)$ model (just replace the prices of ∞ by 0).

The (offline) algorithm in [3] can be interpreted in a weaker online model in which when a vertex arrives, the seller has the full information about the edges incident on it. That is, if the other endpoint is in the "future" then the seller also gets to know about this other endpoint. We now restate the algorithm in [3] that works in this full-information scenario. Initially with probability 1/2 decide on "left" or "right". For the ease of exposition, assume that the algorithm chose left. When a vertex (say i) arrives, with probability 1/2 tag it as a left vertex or a right vertex (unless it is already assigned a tag). If i is a right vertex then assign it a price ∞. Otherwise look at the set of neighboring vertices of i (recall that in this weaker model the seller knows everything about the edge incident on i). If some neighbor j has not arrived yet then assign j one of the tags with equal probability. Let $N'(i)$ denote the set of neighbors of i that are tagged right. Now consider all edges between i and $N'(i)$ and set the price of i to be the best fixed price given that the vertices in $N'(i)$ are priced at ∞. By the analysis in [3], this algorithm is 4-competitive.

We now consider the more general (true online) model, where the seller has no information about the vertices that are yet to arrive. To clarify the difference between our online model and the weaker model we used above, consider a graph on four vertices a, b, c, d and two edges $\{a, c\}$ and $\{b, d\}$. After vertices a and b arrive (but before c and d arrive), the seller sees (the identity of) only one endpoint of each edge, and cannot differentiate it from a scenario where the second edge is actually $\{b, c\}$. This is in contrast to the weaker where the identities of both endpoints are revealed and thus the seller can differentiate between these two scenarios.

For our (more general) model, we consider the following *refinement* of the algorithm in [3]. For any vertex i, let p_i^* denote the best fixed price for vertex i, given that all of its neighbors are priced at ∞. Recall that in our online model, once a vertex arrives, the seller knows the weights of all the incident edges. Thus, the seller can calculate the price p_i^*. Given this, the online algorithm is very simple.

Algorithm: When each vertex i arrives,

- Compute its best fixed price p_i^*.
- With probability 1/2 set its price $p_i = \infty$ and with probability 1/2 set $p_i = p_i^*$.

We have the following performance guarantee.

Theorem 4.1. *For the online $UD_2(\cdot)$ model, the algorithm above is 4-competitive in the expected sense.*

Proof. For any vertex i, let $R(i)$ denote the maximum revenue obtainable from i. That is $R(i) = \pi_{ii}(p_i^*) + \sum_{(i,j)\in E} \pi_{ij}(p_i^*, \infty)$, where $\pi_{ij}(p_i, p_j)$ is zero if $\min(p_i, p_j) > c_{ij}$ and $\min(p_i, p_j)$ otherwise. $\pi_{ii}(p_i)$ is zero if $p_i > c_{ii}$ and p_i otherwise[9]. It is not too hard to check that

$$\begin{aligned} OPT &\leq \sum_{i\in V} R(i) \\ &= \sum_{(i,j)\in E} \left(\pi_{ij}(p_i^*, \infty) + \pi_{ij}(\infty, p_j^*)\right) + \sum_{(i,i)\in E} \pi_{ii}(p_i^*), \end{aligned} \tag{23}$$

where OPT is the revenue of the optimal offline pricing. Now fix an edge $(i, j) \in E$, with $i \neq j$. With probability 1/4 each, the algorithm sets the prices to $(p_i, p_j) = (p_i^*, \infty)$ and $(p_i, p_j) = (\infty, p_j^*)$. Thus, the expected revenue that the algorithm generates for the edge (i, j) is at least

$$\frac{1}{4} \cdot \pi_{ij}(p_i^*, \infty) + \frac{1}{4} \cdot \pi_{ij}(\infty, p_j^*). \tag{24}$$

Now consider a self-loop $(i, i) \in E$. With probability of 1/2, the algorithm set the price $p_i = p^*$. Thus, the expected revenue that the algorithm generates from the self-loop (i, i) is at least $\frac{1}{2} \cdot \pi_{ii}$, which along with (23) and (24) and linearity of expectation completes the proof. ■

5. Conclusions

We have shown near-optimal algorithms for computing profit-maximizing prices in various scenarios of restricted bidder types. As argued earlier, these restricted scenarios may be applicable in many situations that do not require solving the most general problem. In practice, every percent of improvement counts, requiring us to improve one (small) constant to another. In fact, these problems fall in a familiar regime—the challenge of designing algorithmic techniques that beat the rather naive approach of a single price for all goods. Indeed, in one example given in Section 1.1, our algorithm can increase the guaranteed profit by 20% and decrease the "potentially lost" profit almost by half.

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[9] For the ease of exposition, we assume that every vertex has at most one self-loop.