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Partial orderings with the weak Freese–Nation property

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Abstract

A partial ordering P is said to have the weak Freese–Nation property (WFN) if there is a mapping $f : P \to [P]^{\leqslant \aleph_0}$ such that, for any $a, b \in P$, if $a \leqslant b$ then there exists $c \in f(a) \cap f(b)$ such that $a \leqslant c \leqslant b$. In this note, we study the WFN and some of its generalizations.

Some features of the class of Boolean algebras with the WFN seem to be quite sensitive to additional axioms of set theory: e.g. under CH, every ccc complete Boolean algebra has this property while, under $\boldsymbol{b} \geqslant \aleph_2$, there exists no complete Boolean algebra with the WFN (Theorem 6.2).

1. Introduction

In [10], a Boolean algebra A is said to have the Freese–Nation property (FN, for short) if there exists an FN-mapping on A, i.e. a function $f : A \to [A]^{<\aleph_0}$ such that

(*) if $a, b \in A$ satisfy $a \leqslant b$, then $a \leqslant c \leqslant b$ for some $c \in f(a) \cap f(b)$.

This property is closely connected to the notion of freeness because of the following facts: (a) every free Boolean algebra A has the FN; to see this, fix a subset U of A generating A freely; then for $b \in A$, let $u(b)$ be a finite subset of U generating b

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and $f(b)$ the finite subalgebra of A generated by $u(b)$. The Interpolation Theorem of propositional logic then tells us that $(*)$ holds for this f. Moreover, we have: (b) if A has the FN, then so has every retract of A (see Lemma 2.7 below for a more general statement). From (a) and (b), it follows that: (c) every projective Boolean algebra has the FN.

Historically, the FN was first considered by Freese and Nation in their paper [4] which gives a characterization of projective lattices. In particular, they proved that every projective lattice has the FN.

The FN alone, however, is not equivalent to projectiveness, since, as Heindorf proved in [10], a Boolean algebra A has the FN if and only if A is openly generated in the terminology given below (which is also used in [6]; in [10] these Boolean algebras are called "rc-filtered"). The notion of open generatedness was introduced originally in a topological setting by Ščepin [15]. In the language of Boolean algebras, a Boolean algebra A is said to be openly generated if there exists a closed unbounded subset $\mathscr{C}$ of $[A]^{\aleph_0}$ such that every $C \in \mathscr{C}$ is a relatively complete subalgebra of A. Ščepin found examples of openly generated Boolean algebras which are not projective.

In this paper, we continue the study of Boolean algebras with the following weakening of the Freese–Nation property, begun in [10] or, to some extent, already in [15]: a Boolean algebra A is said to have the *weak Freese–Nation property* (*WFN*, for short) if there is a *WFN mapping on* A, that is, a mapping $f : A \to [A]^{\leqslant\aleph_0}$ satisfying the condition $(*)$ above. We solve some open problems from [10] in Sections 4 and 5.

Clearly the WFN makes perfect sense for arbitrary partial orderings and can be also generalized to any uncountable cardinal κ: we say that a structure A with a distinguished partial ordering $\leqslant$ (we shall call such A a *partially ordered structure*) has the *κ-FN* if there is a *κ-FN mapping on* A, that is, a mapping $f : A \to [A]^{<\kappa}$ satisfying the condition $(*)$. In particular, the FN is the $\aleph_0$-FN and the WFN is the $\aleph_1$-FN. This generalization is also considered in the following sections.

The paper is organized as follows. In Section 2, we collect some basic facts on the κ-FN and its connection to the κ-embedding relation $A \leqslant_\kappa B$ of partially ordered structures. In Section 3, we give some conditions equivalent to the κ-FN which are formulated in terms of elementary submodels, and existence of winning strategies in certain infinitary games respectively. The behavior of Boolean algebras with the κ-FN with respect to the cardinal functions of independence, length and cellularity is studied in Section 4. In Sections 5–7, we deal with the question which members of the following classes of Boolean algebras have the WFN: interval algebras, power set algebras, complete Boolean algebras and $L_{\infty\kappa}$-free Boolean algebras.

Our notation is standard. For unexplained notation and definitions on Boolean algebras, the reader may consult [13] and [14]. Some set theoretic notions and basic facts used here can be found in [11] and/or [12].

The authors would like to thank L. Heindorf for drawing their attention to the weak Freese–Nation property.

2. κ-Freese–Nation property and κ-embedding of partially ordered structures

In this section, we shall look at some basic properties of partially ordered structures with the κ-FN. In the following, A, B, C etc. are always partially ordered structures for an arbitrary (but fixed) signature. Note that this setting includes the cases that A, B, C etc. are (a) Boolean algebras (with their canonical ordering) or (b) bare partially ordered sets without any additional structure.

By the theorem of Heindorf mentioned above, every openly generated Boolean algebra has the WFN. But the class of Boolean algebras with the WFN contains many more Boolean algebras. This can be seen already in the following:

Lemma 2.1. *If $|A| \leqslant \kappa$ then A has the κ-FN.*

Proof. Let $A = \{ b_\alpha : \alpha < \kappa \}$. The mapping $f : A \to [A]^{<\kappa}$ defined by $f(b_\alpha) = \{ b_\beta : \beta \leqslant \alpha \}$ for $\alpha < \kappa$, is a κ-FN mapping on A. □

For A, B such that $A \leqslant B$ (i.e. A is a substructure of B) and $b \in B$ we write:

$$A \restriction b = \{ a \in A : a \leqslant b \}, \quad A \uparrow b = \{ a \in A : a \geqslant b \}.$$

A is a *κ-substructure of B* (or *κ-subalgebra of B* in case of Boolean algebras; notation: $A \leqslant_\kappa B$) if $A \leqslant B$ and, for every $b \in B$, there is a cofinal subset U of $A \restriction b$ and a coinitial subset V of $A \uparrow b$ both of cardinality less than κ. For $\kappa = \aleph_1$ we say also that A is a *σ-substructure/subalgebra* of B and denote it by $A \leqslant_\sigma B$. For $\kappa = \aleph_0$, a κ-substructure/subalgebra A of B is also called a *relatively complete substructure/subalgebra*) of B and this is denoted also by $A \leqslant_{\mathrm{rc}} B$. Note that, if $\leqslant$ is lattice order on A, then $A \leqslant_{\mathrm{rc}} B$ holds if and only if, for all $b \in B$, $A \restriction b$ has a cofinal subset U and $A \uparrow b$ has a coinitial subset V consisting of a single element respectively. In this case these elements are called the *lower* and the *upper projection of b on A* and denoted by $p^B_A(b)$ and $q^B_A(b)$ respectively. Note also that, for Boolean algebras, to show that $A \leqslant_\kappa B$ holds, it is enough to check that $A \restriction b$ is $< \kappa$-generated for every $b \in B$, by duality.

The following lemma can be proved easily:

Lemma 2.2. (a) *If $\lambda \leqslant \kappa$ and $A \leqslant_\lambda B$ then $A \leqslant_\kappa B$. In particular, if $A \leqslant_{\mathrm{rc}} B$ then $A \leqslant_\sigma B$.*

(b) *If $A \leqslant_\kappa C$ and $A \leqslant B \leqslant C$ then $A \leqslant_\kappa B$.*

(c) *For a regular cardinal κ, if $A \leqslant_\kappa B$ and $B \leqslant_\kappa C$ then $A \leqslant_\kappa C$.*

Lemma 2.3. (a) *For a regular cardinal κ, if B has the κ-FN and $A \leqslant_\kappa B$ then A also has the* κ-FN.

(b) *For a regular cardinal κ, if $A \leqslant_\kappa B$, B has the κ-FN and f is a κ-FN mapping on A, then there is a κ-FN mapping $\tilde{g}$ on B extending f.*

(c) *If g is a κ-FN mapping on B and $C \leqslant B$ is closed with respect to g (i.e. $g(c) \subseteq C$ holds for all $c \in C$), then $C \leqslant_\kappa B$.*

Proof. For (a) and (b), let $g : B \to [B]^{<\kappa}$ be a κ-FN mapping on B and, for each $b \in B$, let $U(b)$ and $V(b)$ be such that $U(b)$ is a cofinal subset of $A \restriction b$, $V(b)$ is a coinitial subset of $A \uparrow b$ and $|U(b)|, |V(b)| < \kappa$.

(a): Let f be the mapping on A defined by

$$f(a) = \bigcup\{\, U(b) : b \in g(a)\,\}.$$

Since κ is regular we have $f(a) \in [A]^{<\kappa}$ for every $a \in A$. f is a κ-FN mapping on A. To see this let $a, a' \in A$ be such that $a \leqslant a'$. Then there is $b \in g(a) \cap g(a')$ such that $a \leqslant b \leqslant a'$. Since $U(b)$ is cofinal in $A \restriction b$, there is $c \in U(b)$ ($\subseteq f(a) \cap f(a')$) such that $a \leqslant c$. Since $c \leqslant b$ we also have $c \leqslant a'$. Note that in this proof we only needed that one of $U(b)$ and $V(b)$ is of cardinality less than κ for every $b \in B$.

(b): Let $\tilde{g}$ be the mapping on B defined by

$$\tilde{g}(b) = \begin{cases} f(b); & \text{if } b \in A, \\ g(b) \cup \bigcup\{\, f(c) : c \in U(b) \cup V(b)\,\}; & \text{otherwise.} \end{cases}$$

Clearly $f \subseteq \tilde{g}$. $\tilde{g}$ is a κ-FN mapping: since κ is regular, we have $\tilde{g}(b) \in [B]^{<\kappa}$ for every $b \in B$. Let $b, b' \in B$ be such that $b \leqslant b'$. We want to show that there is $c \in \tilde{g}(b) \cap \tilde{g}(b')$ such that $b \leqslant c \leqslant b'$. If b, $b' \in A$ or b, $b' \in B \setminus A$, this follows immediately from the definition of $\tilde{g}$. Suppose that $b \in A$ and $b' \in B \setminus A$. Then there is $d \in U(b')$ such that $b \leqslant d$. Hence there is $c \in f(b) \cap f(d)$ such that $b \leqslant c \leqslant d$ holds. Since $f(b) = \tilde{g}(b)$ and $f(d) \subseteq g(b')$ by $d \in U(b')$, it follows that $c \in \tilde{g}(b) \cap \tilde{g}(b')$ and $b \leqslant c \leqslant b'$. The case, $b \in B \setminus A$ and $b' \in A$, can be treated similarly.

(c): Let $C \leqslant B$ be closed with respect to g. For $b \in B$, let $U = g(b) \cap (C \restriction b)$ and $V = g(b) \cap (C \uparrow b)$. Then clearly U and V are of cardinality $< \kappa$. We show that U is cofinal in $C \restriction b$: if $c \leqslant b$ for some $c \in C$ then there is $e \in g(c) \cap g(b)$ such that $c \leqslant e \leqslant b$ holds. Since $g(c) \subseteq C$, we have $e \in C \restriction b$. Hence $e \in U$. Similarly we can also show that V is coinitial in $C \uparrow b$. □

As already mentioned in the introduction, the Boolean algebras with the FN property are exactly the openly generated Boolean algebras [10]. Hence it follows from the next lemma that, if $(B_\alpha)_{\alpha<\delta}$ is a continuously increasing chain of openly generated Boolean algebras such that $B_\alpha \leqslant_\sigma B_{\alpha+1}$ for every $\alpha < \delta$, then $\bigcup_{\alpha<\delta} B_\alpha$ is also openly generated. The original proof of this fact in [15] employed very complicated combinatorial arguments, while our proof below and also the proof of the characterization of openly generated Boolean algebras as those with the FN property is quite elementary.

Lemma 2.4. *Suppose that κ is a regular cardinal, δ a limit ordinal and $(B_\alpha)_{\alpha\leqslant\delta}$ a continuously increasing chain such that $B_\alpha \leqslant_\kappa B_{\alpha+1}$ for all $\alpha < \delta$. Then*

(a) $B_\alpha \leqslant_\kappa B_\beta$ *for every* $\alpha \leqslant \beta \leqslant \delta$.
(b) *If* B_α *has the* κ-*FN for every* $\alpha < \delta$, *then* B_δ *also has the* κ-*FN.*

Proof. (a): By induction on β, using Lemma 2.2(c) for successor steps.
(b): By Lemma 2.3(b), we can construct a continuously increasing sequence $(f_\alpha)_{\alpha<\delta}$ such that for each $\alpha < \delta$, f_α is a κ-FN mapping on B_α. $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$ is then a κ-FN mapping on B_δ. □

Lemma 2.5. *Suppose that* $\mu < \kappa$, $\mathrm{cf}(\mu) < \mathrm{cf}(\kappa)$ *and* $(B_\alpha)_{\alpha\in\mu}$ *is an increasing sequence of* κ-*substructures of* B. *Then* $\bigcup_{\alpha\in\mu} B_\alpha$ *is also a* κ-*substructure of* B.

Proof. Without loss of generality we may assume that $\mu = \mathrm{cf}(\mu)$ holds. For $b \in B$ let $U_\alpha(b)$ be a cofinal subset of $B_\alpha \restriction b$ and $V_\alpha(b)$ a coinitial subset of $B \uparrow b$ both of cardinality less than κ. Then $U(b) = \bigcup_{\alpha\in\mu} U_\alpha(b)$ is a cofinal subset of $(\bigcup_{\alpha\in\mu} B_\alpha) \restriction b$ and $V(b) = \bigcup_{\alpha\in\mu} V_\alpha(b)$ is a coinitial subset of $(\bigcup_{\alpha\in\mu} B_\alpha) \uparrow b$. Since $\mu < \mathrm{cf}(\kappa)$, we have $|U(b)|, |V(b)| < \kappa$. □

Lemma 2.6. *Suppose that* κ *is a regular cardinal,* δ *a limit ordinal and* $(A_\alpha)_{\alpha<\delta}$ *an increasing chain such that* $A_\alpha \leqslant_\kappa A_{\alpha+1}$ *for all* $\alpha < \delta$ *and* $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ *for all limit* $\gamma < \delta$ *with* $\mathrm{cf}(\gamma) \geqslant \kappa$. *Let* $A = \bigcup_{\alpha<\delta} A_\alpha$. *If* A_α *has the* κ-*FN for every* $\alpha < \delta$, *then* A *also has the* κ-*FN.*

Proof. Let $(B_\alpha)_{\alpha\leqslant\delta}$ be defined by

$$B_\alpha = \begin{cases} A_\alpha; & \text{if } \alpha \text{ is a successor or of cofinality } \geqslant \kappa, \\ \bigcup_{\beta<\alpha} A_\beta; & \text{otherwise.} \end{cases}$$

Then $(B_\alpha)_{\alpha\leqslant\delta}$ is continuously increasing, $B_\delta = A$ and $B_\alpha \leqslant_\kappa B_{\alpha+1}$ for all $\alpha < \delta$: for a limit $\alpha < \delta$ with $\mathrm{cf}(\alpha) < \kappa$, this follows from Lemma 2.5.

Hence, using Lemma 2.4(b), we can show by induction that B_α has the κ-FN for every $\alpha \leqslant \delta$. □

Lemma 2.7. *Suppose that there are order preserving mappings* $i : A \to B$ *and* $j : B \to A$ *such that* $j \circ i = id_A$. *If* B *has the* κ-*FN, then* A *also has the* κ-*FN. In particular, for Boolean algebras* A, B, *if* A *is a retract of* B *and* B *has the* κ-*FN, then* A *also has the* κ-*FN.*

Proof. Let $g : B \to [B]^{<\kappa}$ be a κ-FN mapping on B and f be the mapping on A defined by $f(a) = j[g(i(a))]$. We show that f is a κ-FN mapping on A. Clearly $f(a) \in [A]^{<\kappa}$ for every $a \in A$. Suppose that $a, a' \in A$ are such that $a \leqslant a'$. Then we have $i(a) \leqslant i(a')$. Hence there is $b \in g(i(a)) \cap g(i(a'))$ such that $i(a) \leqslant b \leqslant i(a')$. It follows that

$$a = j \circ i(a) \leqslant j(b) \leqslant j \circ i(a') = a'$$

and $j(b) \in f(a) \cap f(a')$. □

3. Characterizations of partially ordered structures with the weak Freese–Nation property

For a partially ordered structure B, let us say that a regular cardinal χ is sufficiently large if the nth power of B, $\mathscr{P}^n(B)$, is in $\mathscr{H}_\chi$ for every $n \in \omega$, where $\mathscr{H}_\chi$ is the set of every sets of hereditary of cardinality less than χ.

Proposition 3.1. *For a regular κ and a partially ordered structure B, the following are equivalent:*

(1) *B has the κ-FN;*

(2) *For some, or equivalently, any sufficiently large χ, if $M \prec \mathscr{H}_\chi = (\mathscr{H}_\chi, \in)$ is such that $B \in M$, $\kappa \subseteq M$ and $|M| = \kappa$ then $B \cap M \leqslant_\kappa B$ holds;*

(3) *$\{ C \in [B]^\kappa : C \leqslant_\kappa B \}$ contains a club set;*

(4) *There exists a partial ordering $I = (I, \leqslant)$ and an indexed family $(B_i)_{i\in I}$ of substructures of B of cardinality κ such that*

(i) *$\{ B_i : i \in I \}$ is cofinal in $([B]^\kappa, \subseteq)$,*

(ii) *I is directed and for any $i, j \in I$, if $i \leqslant j$ then $B_i \leqslant B_j$,*

(iii) *for every well-ordered $I' \subseteq I$ of cofinality $\leqslant \kappa$, $i' = \sup I'$ exists and $B_{i'} = \bigcup_{i\in I'} B_i$ holds, and*

(iv) *$B_i \leqslant_\kappa B$ holds for every $i \in I$.*

Proof. (1)$\Rightarrow$(2): Let f be a κ-FN mapping on B. Since χ is sufficiently large for B, we have B, $f \in \mathscr{H}_\chi$. Let $M \prec H_\chi$ be such that $B \in M$, $\kappa \subseteq M$ and $|M| = \kappa$. Then there is a κ-FN mapping f' on B in M. Clearly $B \cap M$ is closed with respect to f'. Hence it follows by Lemma 2.3 that $B \cap M \leqslant_\kappa B$.

(2)$\Rightarrow$(3): Clear.

(3)$\Rightarrow$(4): Let $I \subseteq \{ C \in [B]^\kappa : C \leqslant_\kappa B \}$ be a club subset of $[B]^\kappa$ with the substructure relation. For $A \in I$, let $B_A = A$. Then $(I, \leqslant)$ and $(B_A)_{A\in I}$ satisfy the conditions in (4).

(4)$\Rightarrow$(1): We prove this in the following two claims. Let I and $(B_i)_{i\in I}$ be as in (4). For a directed $I' \subseteq I$, let $B_{I'} = \bigcup_{i\in I'} B_i$.

Claim 3.1.1. *If $I' \subseteq I$ is directed, then $B_{I'} \leqslant_\kappa B$.*

Proof. Otherwise there is $b \in B$ such that either $B_{I'} \restriction b$ does not have any cofinal subset of cardinality less than κ or $B_{I'} \uparrow b$ does not have any coinitial subset of cardinality less than κ. For simplicity, let us assume the first case. Then there exists an increasing sequence $(I_\alpha)_{\alpha<\kappa}$ of directed subsets of I' of cardinality less than κ such that $B_{I_\alpha} \restriction b$ is not cofinal in $B_{I_{\alpha+1}} \restriction b$. By (iii), $i_\alpha = \sup I_\alpha$ exists and $B_{i_\alpha} = B_{I_\alpha}$ holds for every $\alpha < \kappa$. $(i_\alpha)_{\alpha<\kappa}$ is an increasing sequence in I. Hence, again by (iii), there exists $i^* = \sup_{\alpha<\kappa} i_\alpha$ and $B_{i^*} = \bigcup_{\alpha<\kappa} B_{i_\alpha}$. By (iv), $B_{i^*} \leqslant_\kappa B$. But by the construction, $B_{i^*} \restriction b$ cannot have any cofinal subset of cardinality less than κ. This is a contradiction. This proves the claim. $\square$

Claim 3.1.2. *If $I' \subseteq I$ is directed, then $B_{I'}$ has the κ-FN.*

Proof. We prove the claim by induction on $|I'|$. If $|I'| \leqslant \kappa$, we have $|B_{I'}| = \kappa$. Hence, by Lemma 2.1, $B_{I'}$ has the κ-FN. Assume that $|I'| = \lambda > \kappa$ and that we have proved the claim for every directed $I'' \subseteq I$ with $|I''| < \lambda$. Take a continuously increasing sequence $(I_\alpha)_{\alpha<\mathrm{cf}(\lambda)}$ of directed subsets of I' such that $|I_\alpha| < \lambda$ for every $\alpha < \mathrm{cf}(\lambda)$ and $I' = \bigcup_{\alpha<\mathrm{cf}(\lambda)} I_\alpha$. $(B_{I_\alpha})_{\alpha<\mathrm{cf}(\lambda)}$ is then a continuously increasing sequence of substructures of $B_{I'}$ and $B_{I'} = \bigcup_{\alpha<\mathrm{cf}(\lambda)} B_{I_\alpha}$. By the induction hypothesis, B_{I_α} has the κ-FN and, by Claim 3.1.1, we have $B_{I_\alpha} \leqslant_\kappa B$. Hence, by Lemma 2.4(b), $B_{I'}$ has also the κ-FN. This proves the claim. □

Now by applying Claim 3.1.2 to $I' = I$, we can conclude that $B = B_I$ has the $\kappa - FN$. □

Now we give yet another characterization of partially ordered structures with the κ-FN by means of a game. This characterization will be used later in the proof of Propositions 4.1, 7.3, etc. For a partially ordered structure B, let $\mathscr{G}^\kappa(B)$ be the following game played by Players I and II: in a play in $\mathscr{G}^\kappa(B)$, Players I and II choose subsets X_α and Y_α of B of cardinality less than κ alternately for $\alpha < \kappa$ such that

$$X_0 \subseteq Y_0 \subseteq X_1 \subseteq Y_1 \subseteq \cdots \subseteq X_\alpha \subseteq Y_\alpha \subseteq \cdots \subseteq X_\beta \subseteq Y_\beta \subseteq \cdots$$

for $\alpha \leqslant \beta < \kappa$. So a play in $\mathscr{G}^\kappa(B)$ looks like

$$\begin{array}{ll} \text{Player I}: & X_0, X_1, \ldots, X_\alpha, \ldots \\ \text{Player II}: & Y_0, Y_1, \ldots, Y_\alpha, \ldots \end{array}$$

where $\alpha < \kappa$. Player II wins the play if $\bigcup_{\alpha<\kappa} X_\alpha = \bigcup_{\alpha<\kappa} Y_\alpha$ is a κ-substructure of B. Let us call a strategy τ for Player II simple if, in τ, each Y_α is decided from the information of the set $X_\alpha \subseteq B$ alone (i.e. also independent of α).

For a sufficiently large χ (with respect to B), an elementary submodel M of $\mathscr{H}_\chi$ is said to be V_κ-like if, either $\kappa = \aleph_0$ and M is countable, or there is an increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of M of cardinality less than κ such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$ and $M = \bigcup_{\alpha<\kappa} M_\alpha$. If M is V_κ-like, we say that a sequence $(M_\alpha)_{\alpha<\kappa}$ as above witnesses the V_κ-likeness of M. The notion of V_κ-like elementary submodels of $\mathscr{H}_\chi$ is a weakening of internally approachable elementary submodels introduced in [3]. An elementary submodel M of $\mathscr{H}_\chi$ is said to be internally approachable if M is the union of continuously increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of smaller elementary submodels such that $(M_\beta)_{\beta\leqslant\alpha} \in M_{\alpha+1}$ for every $\alpha < \kappa$. The main reason of the use of V_κ-like elementary submodels here instead of internally approachable ones is the following Lemma 3.2(b) which seems to be false in general for internally approachable elementary submodels.

Lemma 3.2. (a) *If M is a V_κ-like elementary submodel of $\mathscr{H}_\chi$ such that $\kappa \in M$, then $\kappa \subseteq M$ holds. Hence, if x is of cardinality less or equal to κ and $x \in M$ then we have $x \subseteq M$.*

(b) *If* $(N_\alpha)_{\alpha<\kappa}$ *is an increasing sequence of* V_κ*-like elementary submodels of* $\mathscr{H}_\chi$, *then* $M = \bigcup_{\alpha<\kappa} N_\alpha$ *is also a* V_κ*-like elementary submodel of* $\mathscr{H}_\chi$.

Proof. (a): Let $(M_\alpha)_{\alpha<\kappa}$ witness the V_κ-likeness of M. Assume that $\kappa \not\subseteq M$. Let

$$\alpha_0 = \min\{\, \alpha \in \kappa : \alpha \notin M \,\}.$$

Then we have $\alpha_0 \subseteq M$. Let

$$\alpha_1 = \min\{\, \alpha \leqslant \kappa : \alpha_0 \leqslant \alpha,\, \alpha \in M \,\}.$$

Since α_0 is of cardinality less than κ, there exists $\alpha < \kappa$ such that $\alpha_0 \subseteq M_\alpha$. Then we have $\alpha_1 \in M_{\alpha+1}$. Since $M_\alpha \in M_{\alpha+1}$, $\alpha_0 = \{\, \beta \in M_\alpha : \beta < \alpha_1 \,\}$ is an element of $M_{\alpha+1} \subseteq M$. This is a contradiction. Hence we have $\kappa \subseteq M$.

If x is of cardinality less or equal to κ and $x \in M$, then there is a surjection $f : \kappa \to x$ in M. Since $\kappa \subseteq M$, it follows that $x = f[\kappa] \subseteq M$.

(b): It is clear that M is an elementary submodel of $\mathscr{H}_\chi$. To prove that M is V_κ-like, let $M = \{\, m_\xi : \xi < \kappa \,\}$ and, for each $\alpha < \kappa$, let $(N_{\alpha,\beta})_{\beta<\kappa}$ be an increasing sequence of elementary submodels of $\mathscr{H}_\chi$ of cardinality less than κ witnessing the V_κ-likeness of N_α. Since $N_{\alpha,\beta} \in N_\alpha \subseteq M$ and $\bigcup_{\beta<\kappa} N_{\alpha,\beta} = N_\alpha$, we can choose α_ξ, $\beta_\xi < \kappa$ for $\xi < \kappa$ inductively such that

(a) $(N_{\alpha_\xi,\beta_\xi})_{\xi<\kappa}$ is an increasing sequence,

(b) $N_{\alpha_\xi,\beta_\xi} \in N_{\alpha_{\xi+1},\beta_{\xi+1}}$ holds for every $\xi < \kappa$ and

(c) $m_\xi \in N_{\alpha_\xi,\beta_\xi}$ for every $\xi < \kappa$.

Then $(N_{\alpha_\xi,\beta_\xi})_{\xi<\kappa}$ witnesses the V_κ-likeness of M. □

Proposition 3.3. *For regular* κ *and a partially ordered structure* B, *the following are equivalent*:

(1) *B has the* κ*-FN*;

(2) *Player II has a simple winning strategy in* $\mathscr{G}^\kappa(B)$;

(3) *For some, or equivalently any, sufficiently large* χ, *if* $M \prec \mathscr{H}_\chi$ *is* V_κ*-like such that* $B, \kappa \in M$, *then* $B \cap M \leqslant_\kappa B$.

Proof. Assume that κ is uncountable (for $\kappa = \aleph_0$, the proof is easier than the following one and given in [7]).

(1)$\Rightarrow$(2): Let $f : B \to [B]^{<\kappa}$ be a κ-FN mapping on B. Then Player II can win by the following strategy: in the αth move, Player II chooses Y_α so that $X_\alpha \subseteq Y_\alpha$ and Y_α is a substructure of B of cardinality less than κ closed under f. After κ moves, $\bigcup_{\alpha<\kappa} Y_\alpha$ is a substructure of B closed under f. Hence, by Lemma 2.3(c), it is a κ-substructure of B.

(2)$\Rightarrow$(3): Let M be a V_κ-like elementary submodel of $\mathscr{H}_\chi$ such that $B, \kappa \in M$. We have to show that $B \cap M \leqslant_\kappa B$. Let $(M_\alpha)_{\alpha<\kappa}$ witness the V_κ-likeness of M. Without loss of generality we may assume that $B \in M_0$. By $M_0 \prec \mathscr{H}_\chi$, there is a simple winning strategy $\tau \in M_0$ for Player II in $\mathscr{G}^\kappa(B)$ (hence $\tau \in M_\alpha$ for every $\alpha < \kappa$). Let $(X_\alpha, Y_\alpha)_{\alpha<\kappa}$ be the play in $\mathscr{G}^\kappa(B)$ such that at his αth move, Player I took

$B \cap M_{\xi_\alpha}$ for some $\xi_\alpha < \kappa$ and Player II played always according to τ. Such a game is possible since if Player I chooses $B \cap M_{\xi_\alpha}$ at his αth move, then $B \cap M_{\xi_\alpha} \in M_{\xi_\alpha+1}$. Hence Player II's move Y_α taken according to τ is also an element of $M_{\xi_\alpha+1}$. Since Y_α is of cardinality less than κ, we have $Y_\alpha \subseteq M$ by Lemma 3.2(a). Hence there is some $\xi_{\alpha+1} \geqslant \xi_\alpha$ such that $Y_\alpha \subseteq M_{\xi_{\alpha+1}}$. Thus Player I may take $B \cap M_{\xi_{\alpha+1}}$ at his next move.

Now we have $B \cap M = B \cap (\bigcup_{\alpha<\kappa} M_{\xi_\alpha}) \leqslant_\kappa B$ since τ was a winning strategy of Player II.

(3)$\Rightarrow$(1): Let

$$\mathscr{C} = \{M : |M| = \kappa, M \text{ is a union of an increasing sequence of } V_\kappa\text{-like elementary submodels of } \mathscr{H}_\chi \text{ such that } B, \kappa \in M\}.$$

Then it is easy to see that $\mathscr{C}$ is club in $[\mathscr{H}_\chi]^\kappa$. Hence $\mathscr{C}' = \{B \cap M : M \in \mathscr{C}\}$ contains a club subset of $[B]^\kappa$. By Proposition 3.1(3), it follows from the claim below that B has the κ-FN.

Claim 3.3.1. *$B \cap M \leqslant_\kappa B$ holds for every $M \in \mathscr{C}$.*

Proof. If $M \in \mathscr{C}$ is union of a sequence $(M_\alpha)_{\alpha<\rho}$ of V_κ-like elementary submodels of $\mathscr{H}_\chi$ for some $\rho < \kappa$ such that $B, \kappa \in M_0$, then we have $B \cap M_\alpha \leqslant_\kappa B$ for every $\alpha < \rho$ by (3). Hence we have $B \cap M = \bigcup_{\alpha<\rho}(B \cap M_\alpha) \leqslant_\kappa B$ by Lemma 2.5. If $M \in \mathscr{C}$ is the union of a κ-chain of V_κ-like elementary submodels of $\mathscr{H}_\chi$, then it follows by Lemma 3.2(b) that M itself is V_κ-like. Hence we have $B \cap M \leqslant_\kappa B$ by the assumption. This proves the claim and the proposition. □

Under $2^{<\kappa} = \kappa$, Propositions 3.1 and 3.3 can be yet improved. This is because of the following fact:

Lemma 3.4. *Assume that κ is a regular cardinal such that $2^{<\kappa} = \kappa$. Let B be a partially ordered structure, χ be sufficiently large for B and $M \subseteq \mathscr{H}_\chi$. Then the following are equivalent:*

(1) M is a V_κ-like elementary submodel of $\mathscr{H}_\chi$ such that $B, \kappa \in M$;

(2) $M \prec \mathscr{H}_\chi$, $|M| = \kappa$, $B, \kappa \in M$ and $[M]^{<\kappa} \subseteq M$.

Proof. For $\kappa = \aleph_0$, this is clear. Assume that κ is uncountable.

(1)$\Rightarrow$(2): Let M be a V_κ-like elementary submodel of $\mathscr{H}_\chi$ and let $(M_\alpha)_{\alpha<\kappa}$ be an increasing sequence of elementary submodels of $\mathscr{H}_\chi$ witnessing the V_κ-likeness of M. It is enough to show that $[M_\alpha]^{<\kappa} \subseteq M$ holds for every $\alpha < \kappa$. By $M_\alpha \in M$, we have $[M_\alpha]^{<\kappa} \in M$. By $2^{<\kappa} = \kappa$, $[M_\alpha]^{<\kappa}$ has cardinality κ. Hence, by Lemma 3.2(a), it follows that $[M_\alpha]^\kappa \subseteq M$.

(2)$\Rightarrow$(1): Suppose that $M \prec \mathscr{H}_\chi$, $|M| = \kappa$, $B \in M$ and $[M]^{<\kappa} \subseteq M$. Let $M = \{m_\alpha : \alpha < \kappa\}$. Then we can construct inductively an increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of M of cardinality less than κ such that $M_\alpha, m_\alpha \in M_{\alpha+1}$ for every $\alpha < \kappa$. This is possible since at αth step of the inductive construction, we have that M_α is a subset of M of cardinality less than κ. By $[M]^{<\kappa} \subseteq M$, it follows that $M_\alpha \in M$. So by the downward Löwenheim–Skolem Theorem, we can take $M_{\alpha+1} \prec M$ such that $|M_{\alpha+1}| < \kappa$ and $M_\alpha, m_\alpha \in M_{\alpha+1}$. At a limit $\gamma < \kappa$ we take $\bigcup_{\alpha<\gamma} M_\alpha$. Then $(M_\alpha)_{\alpha<\kappa}$ witnesses the V_κ-likeness of M. □

Proposition 3.5. *Assume that κ is a regular cardinal such that $2^{<\kappa} = \kappa$. Then for a partially ordered structure B, the following are equivalent:*

(1) *B has the κ-FN;*

(2) *For sufficiently large χ and for all $M \prec \mathscr{H}_\chi$, if $B \in M$, $|M| = \kappa$ and $[M]^{<\kappa} \subseteq M$, then $B \cap M \leqslant_\kappa B$ holds;*

(3) *Player II has a winning strategy in $\mathscr{G}^\kappa(B)$.*

Proof. (1)$\Leftrightarrow$(2): By Lemma 3.4 and Proposition 3.3.

(1)$\Rightarrow$(3): follows from Proposition 3.3.

(3)$\Rightarrow$(2): Let M be as in (2) and let $B \cap M = \{b_\alpha : \alpha < \kappa\}$. By (3), there is a winning strategy $\tau \in M$ of Player II in $\mathscr{G}^\kappa(B)$. Let $(X_\alpha, Y_\alpha)_{\alpha<\kappa}$ be a play in $\mathscr{G}^\kappa(B)$ such that Player I chooses X_α so that $|X_\alpha| < \kappa$ and $b_\alpha \in X_\alpha$, and Player II played always according to τ. Such a game is possible since, by $[M]^{<\kappa} \subseteq M$, at Player II's αth innings, she has $(X_0, Y_0, \ldots, X_\alpha) \in M$. Hence her move Y_α taken according to τ will be also an element of M. Since $|Y_\alpha| < \kappa$, Y_α is a subset of M. Now we have $\bigcup_{\alpha<\kappa} Y_\alpha = \bigcup_{\alpha<\kappa} X_\alpha = B \cap M$. Since τ was a winning strategy, we also have $B \cap M = \bigcup_{\alpha<\kappa} X_\alpha \leqslant_\kappa B$. □

We can also consider the following variant of the game $\mathscr{G}^\kappa(B)$: for cardinals κ, λ such that $\lambda \leqslant \kappa$ and a partially ordered structure B, $\mathscr{G}^\kappa_\lambda(B)$ is the game just like $\mathscr{G}^\kappa(B)$ except that Player II wins in $\mathscr{G}^\kappa_\lambda(B)$ if and only if $\bigcup_{\alpha<\kappa} X_\alpha \leqslant_\lambda B$. As in Propositions 3.1, 3.3, we can prove the implication (A)$\Rightarrow$(B)$\Rightarrow$(C)$\Rightarrow$(D) for the following assertions for regular κ, λ.

(A) For every sufficiently large χ and $M \prec \mathscr{H}_\chi$ of cardinality κ with $B \in M$, we have $B \cap M \leqslant_\lambda B$.

(B) Player II has a simple winning strategy in $\mathscr{G}^\kappa_\lambda(B)$.

(C) For every sufficiently large χ and $M \prec \mathscr{H}_\chi$, if M is V_κ-like then $B \cap M \leqslant_\lambda B$.

(D) Player II has a winning strategy in $\mathscr{G}^\kappa_\lambda(B)$.

(E) For any sufficiently large χ and $M \prec \mathscr{H}_\chi$ of cardinality κ such that $B \in M$ and $[M^{<\kappa} \subseteq M,\ B \cap M \leqslant_\lambda B]$ holds.

For the implication (A)$\Rightarrow$(B), we fix an expansion of $\mathscr{H}_\chi$ by Skolem functions. For $x \subseteq \mathscr{H}_\chi$, let $\tilde{h}(x)$ be the Skolem hull of x. We may take the Skolem hull operation so that $B \in \tilde{h}(\emptyset)$ holds. Player II then wins if she takes Y_α such that $X_\alpha \subseteq Y_\alpha$ and $\tilde{h}(Y_\alpha) \cap B = Y_\alpha$ hold in each of her αth innings for $\alpha < \kappa$. By the same idea, we can also prove the equivalence of (B) and (C), if we allow Player II to remember her last move in her simple winning strategy in (B). By Lemma 3.4, we have (C)$\Leftrightarrow$(D)$\Leftrightarrow$(E) under $2^{<\kappa} = \kappa$.

4. Cardinal functions on Boolean algebras with the weak Freese–Nation property

In [10] it is shown that, for any openly generated Boolean algebra (i.e. Boolean algebra with the FN), the cardinal functions (those studied in [14], possibly except the topological density d) have the same value as for the free Boolean algebra of the same cardinality, as follows obviously from Lemma 2.1. Later we shall see some more examples of Boolean algebras with the WFN which behave quite differently from free Boolean algebras with respect to cardinal functions. Nevertheless, there are some restrictions on the values of cardinal functions on Boolean algebras with the WFN.

Proposition 4.1. *For every partially ordered structure B, if $\kappa^+ + 1$ or $(\kappa^+ + 1)^*$ is (order isomorphic) embeddable into B then B does not have the κ-FN. In particular, for every Boolean algebra with the κ-FN, we have* $\mathrm{Depth}(B) \leqslant \kappa$.

Proof. Suppose that $i : \kappa^+ + 1 \to B$ is an embedding (the case for $(\kappa^+ + 1)^*$ can be handled similarly). Let $j : B \to \kappa^+ + 1$ be defined by

$$j(b) = \sup\{\, \alpha : i(\alpha) \leqslant b \,\}$$

for $b \in B$. Then j is order preserving and $j \circ i = id_{\kappa^+ + 1}$ holds. Hence, by Lemma 2.7, the following claim proves the proposition:

Claim 4.1.1. $(\kappa^+ + 1, \leqslant)$ *does not have the κ-FN.*

Proof. Player I wins a game in $\mathscr{G}^\kappa(\kappa^+ + 1)$ if he chooses X_α at his α'th move such that $\sup X_\alpha \setminus \{\,\kappa^+\,\} > \sup \bigcup_{\beta<\alpha} Y_\beta \setminus \{\,\kappa^+\,\}$ holds. By Proposition 3.3, it follows that $(\kappa^+ + 1, \leqslant)$ does not have the κ-FN. Note that for the implication $(1) \Rightarrow (2)$ in Proposition 3.3 used here, we do not need the assumption of regularity of κ. This proves the claim and the proposition. □

Theorem 4.2. *For a regular cardinal κ, if a Boolean algebra B has the κ-FN, $\lambda = \lambda^{<\kappa}$ and $X \subseteq B$ is of cardinality $> \lambda$ then there is an independent $Y \subseteq X$ of cardinality $> \lambda$.*

Proof. Essentially the same argument as the following one has been used in [16, Section 4].

Let f be a κ-FN mapping on B. Let $(a_\delta)_{\delta<\lambda^+}$ be a sequence of elements of X such that, letting B_δ be the closure of $\{\, a_\gamma : \gamma < \delta \,\}$ with respect to f and the Boolean operations, $a_\delta \notin B_\delta$ holds for every $\delta < \lambda^+$. By Lemma 2.3, (c), we have $B_\delta \leqslant_\kappa B$ for every $\delta < \lambda^+$. Let $S = \{\, \delta < \lambda^+ : \mathrm{cf}(\delta) \geqslant \kappa \,\}$. For each $\delta \in S$, let I_δ and J_δ be cofinal subsets of $B_\delta \restriction a_\delta$ and $B_\delta \restriction -a_\delta$ respectively, both of cardinality less than κ. Let

$$h(\delta) = \langle\, I_\delta, J_\delta \,\rangle.$$

By Fodor's lemma and $\lambda = \lambda^{<\kappa}$, there is a stationary $T \subseteq S$ such that $h \restriction T$ is constant, say $h(\delta) = \langle I, J \rangle$ for all $\delta \in T$. Let

$$\delta^* = \min\{\, \delta < \lambda^+ : I, J \subseteq B_\delta \,\}.$$

Without loss of generality we may assume that $\delta^* < \delta$ holds for every $\delta \in T$. Let

$$L = \{\, b \in B_{\delta^*} : b \nleqslant i + j \text{ for all } i \in I, j \in J \,\}.$$

Then we have

(1) $1 \in L$ (since, by $a_\delta \notin B_{\delta^*}$ for any $\delta \in T$, $I \bigcup J$ generates a proper ideal of B_{δ^*}). In particular we have $L \neq \emptyset$;

(2) If $b \in L$ and $k \in I \bigcup J$ then $b \cdot -k \in L$.

Now, by (1) above, the following claim shows that $\{\, a_\delta : \delta \in T \,\}$ is independent. Since $|T| = \lambda^+$, this proves the theorem.

Claim 4.2.1. *If $b \in L$ and p is an elementary product over $a_{\delta_0}, \ldots, a_{\delta_{n-1}}$ for $\delta_i \in T$ such that $\delta_0 < \cdots < \delta_{n-1}$ (i.e. p is of the form $(a_{\delta_0})^{\tau_0} \cdot \cdots \cdot (a_{\delta_{n-1}})^{\tau_{n-1}}$ for some $\tau_i \in 2$, $i < n$) then $b \cdot p \neq 0$. Here, for a Boolean algebra B, $b \in B$ and $i \in 2$, we define $(b)^i$ by*

$$(b)^i = \begin{cases} b; & \text{if } i = 1, \\ -b; & \text{if } i = 0. \end{cases}$$

Proof. By induction on n. For $n = 0$, this is trivial since $0 \notin L$. Assume that the claim holds for n. Let $\delta_0, \ldots, \delta_n \in T$ be such that $\delta_0 < \cdots \delta_{n-1} < \delta_n$ and let p be an arbitrary elementary product over $a_{\delta_0}, \ldots, a_{\delta_{n-1}}$. Let $b \in L$. By the induction hypothesis, we have $b \cdot p \neq 0$. We have to show that $b \cdot p \cdot a_{\delta_n} \neq 0$ and $b \cdot p \cdot -a_{\delta_n} \neq 0$. Toward a contradiction, assume that $b \cdot p \cdot a_{\delta_n} = 0$ holds. Then $b \cdot p \leqslant -a_{\delta_n}$. Since $b \cdot p \in B_{\delta_n}$, we can find $j \in J$ such that $b \cdot p \leqslant j$. Hence $(b \cdot -j) \cdot p = 0$. Since $b \cdot -j \in L$ by (2) above, this is a contradiction to the induction hypothesis. Similarly, from $b \cdot p \cdot -a_{\delta_n} = 0$, it follows that $(b \cdot -i) \cdot p = 0$ for some $i \in I$ which again is a contradiction to (2). This proves the claim. □

The next corollary gives a positive answer to a problem by L. Heindorf.

Corollary 4.3. *For a regular cardinal κ, if a Boolean algebra B has the κ-FN then $|B| \leqslant \mathrm{Ind}(B)^{<\kappa}$. In particular, for an openly generated Boolean algebra B, $|B| = \mathrm{Ind}(B)$ holds* ([10]). *For a Boolean algebra B with the WFN, we have $|B| \leqslant \mathrm{Ind}(B)^{\aleph_0}$.*

Proof. Assume that B has the κ-FN but $|B| > \mathrm{Ind}(B)^{<\kappa}$ holds. Then by $\mathrm{Ind}(B)^{<\kappa} = (\mathrm{Ind}(B)^{<\kappa})^{<\kappa}$, we have $\mathrm{Ind}(B) > \mathrm{Ind}(B)^{<\kappa}$ by Theorem 4.2. But this is impossible. □

In Corollary 4.3 the equality is attained for every regular κ. For the case of $\kappa = \aleph_1$, the simplest example to see this would be *Intalg* ($\mathbb{R}$) (see Proposition 5.1 below).

The following corollary is an immediate consequence of Theorem 4.2:

Corollary 4.4. ([10] for $\kappa \leqslant \aleph_1$) *Let κ be a regular cardinal. If a Boolean algebra B has the κ-FN, then $c(B) \leqslant 2^{<\kappa}$ and* $\mathrm{Length}(B) \leqslant 2^{<\kappa}$.

5. Interval algebras and power set algebras

Proposition 5.1. (a) *For $\rho \in \mathrm{Ord}$, $Intalg(\rho)$ has the κ-FN if and only if $\rho < \kappa^+$.*

(b) *$Intalg(\mathbb{R})$ has the WFN.*

(c) *For a totally ordered set X, $Intalg(X)$ has the κ-FN if and only if X has the κ-FN.*

(d) *Assume that κ is a regular cardinal. For a linearly ordered set X, if $Intalg(X)$ (hence, by* (c), *also X) has the κ-FN then $|X| \leqslant 2^{<\kappa}$.*

Proof. For $b \in Intalg(X)$, let $ep(b)$ be the set of end points of b, i.e.

$$ep(b) = \{ x_i : i < 2n \}.$$

where $b = \dot{\bigcup}_{j<n}[x_{2j}, x_{2j+1})$ in the standard representation.

(a): For $\rho < \kappa^+$, $Intalg(\rho)$ has cardinality less or equal to κ. Hence, by Lemma 2.1, $Intalg(\rho)$ has the κ-FN. If $\rho \geqslant \kappa^+$, $Intalg(\rho)$ does not have the κ-FN by Proposition 4.1.

(b): For all $b \in Intalg(\mathbb{R})$, the mapping defined by

$$g(b) = \{ c \in Intalg(\mathbb{R}) : ep(c) \subseteq \mathbb{Q} \bigcup ep(b) \}.$$

is a WFN-mapping on $Intalg(\mathbb{R})$.

(c): If $f : X \to [X]^{<\kappa}$ is a κ-FN mapping on X then $g : Intalg(X) \to [Intalg(X)]^{<\kappa}$ defined by

$$g(b) = \{ c \in Intalg(X) : ep(c) \subseteq \bigcup f[ep(b)] \}$$

is a κ-FN mapping on $Intalg(X)$. Conversely, if g is a κ-FN mapping on $Intalg(X)$, then $f : X \to [X]^{<\kappa}$ defined by

$$f(x) = \bigcup \{ ep(b) : b \in g((-\infty, x)) \}$$

is a κ-FN mapping on X.

(d): We have $\mathrm{Ind}(Intalg(X)) = \aleph_0$ (see e.g. [13] Corollary 15.15.) Hence, by Corollary 4.3, $|X| \leqslant |Intalg(X)| \leqslant 2^{<\kappa}$ holds if X (or equivalently $Intalg(X)$) has the $\kappa - FN$. $\square$

Note that Lemma 5.1(c) is not true for tree algebras: e.g. for any cardinal λ such that $\lambda > 2^{\aleph_0}$, the tree $(\kappa, \emptyset)$ has the WFN but $Treealg((\kappa, \emptyset))$ does not by Corollary 4.4.

For any sets x, y we say that x is a subset of y modulo $< \kappa$ (notation: $x \subseteq_{<\kappa} y$) if $|x \setminus y| < \kappa$ holds. The following lemma is well-known:

Lemma 5.2. *Suppose that κ is a regular uncountable cardinal.*

(a) *If $(u_\alpha)_{\alpha<\kappa}$ is a sequence of non-stationary subsets of κ then there exists a non-stationary $u \subseteq \kappa$ such that $u_\alpha \subseteq_{<\kappa} u$ holds for all $\alpha < \kappa$.*

(b) *There exists a strictly $\subseteq_{<\kappa}$-increasing sequence of elements of $\mathscr{P}(\kappa)$ of order type κ^+.*

Proof. (a): For each $\alpha < \kappa$, let $c_\alpha \in \mathscr{P}(\kappa)$ be a club subset of κ such that $u_\alpha \cap c_\alpha = \emptyset$. Let $(\beta_\alpha)_{\alpha<\kappa}$ be a strictly, continuously increasing sequence in κ such that $\beta_\alpha \in \bigcap_{\delta<\alpha} c_\delta$ for every $\alpha < \kappa$. Then $\kappa \setminus \{ \beta_\alpha : \alpha < \kappa \}$ is as desired.

(b): We can construct a $\subseteq_{<\kappa}$-increasing sequence $(u_\alpha)_{\alpha<\kappa}$ of elements of $\mathscr{P}(\kappa)$ inductively so that u_α is non-stationary for every $\alpha < \kappa$: for a successor step let $u_{\alpha+1}$ be the union of u_α and any non-stationary subset of $\kappa \setminus u_\alpha$ of cardinality less than κ. For a limit $\delta < \kappa^+$ with $\mathrm{cf}(\delta) = \lambda < \kappa$, we choose increasing $(\delta_\beta)_{\beta<\lambda}$ such that $\delta = \bigcup_{\beta<\lambda} \delta_\beta$, and let $u_\delta = \bigcup_{\beta<\lambda} u_{\delta_\beta}$. For limit $\delta < \kappa^+$ with $\mathrm{cf}(\delta) = \kappa$, we can take an appropriate u_δ using (a). □

Proposition 5.3. *Suppose that κ is a regular uncountable cardinal. Then*

(a) *$\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$ does not have the κ-FN.*

(b) *$\mathscr{P}(\kappa)$ does not have the κ-FN.*

Proof. (a): By Lemma 5.2(b), $(\kappa^+, \leqslant)$ is embeddable into $\mathscr{P}(\kappa)/[\kappa]^{<\kappa}$. Hence, by Proposition 4.1, $\mathscr{P}(\kappa)/[\kappa]^{\kappa}$ does not have the κ-FN.

(b): Let χ be sufficiently large and let $M \prec \mathscr{H}_\chi$ be V_κ-like such that $\kappa \in M$ (and hence also $\mathscr{P}(\kappa) \in M$). Let $(M_\alpha)_{\alpha<\kappa}$ be an increasing sequence of elementary submodels of M of cardinality less than κ witnessing the V_κ-likeness of M. We construct a sequence $(u_\alpha)_{\alpha<\kappa}$ of non-stationary subsets of κ inductively such that $u_\alpha \in M_{\alpha+1}$, $u \subseteq_{<\kappa} u_\alpha$ and $|u_\alpha \setminus u| = \kappa$ hold for every non-stationary $u \in \mathscr{P}(\kappa) \cap M_\alpha$. This is possible since $M_\alpha \in M_{\alpha+1}$. By Lemma 5.2(a), there is a non-stationary $u^* \in \mathscr{P}(\kappa)$ such that $u_\alpha \subseteq_{<\kappa} u^*$ holds for every $\alpha < \kappa$. Clearly $(\mathscr{P}(\kappa) \cap M) \restriction u^*$ is not generated by any subset of cardinality less than κ. Hence, by Proposition 3.3, it follows that $\mathscr{P}(\kappa)$ does not have the κ-FN. □

By Proposition 5.3, it follows that $\mathscr{P}(\omega_1)$ does not have the WFN. In contrast to this, the statement "$\mathscr{P}(\omega)$ has the WFN" is independent from ZFC or even from ZFC + $\neg$CH. For $x \in \mathscr{P}(\omega)$, let us denote by $[x]$ the equivalence class of x modulo *fin* (the ideal of finite subsets of ω).

Lemma 5.4. *$\mathscr{P}(\omega)$ has the WFN if and only if $\mathscr{P}(\omega)/fin$ in has the WFN.*

Proof. If g is a WFN mapping on $\mathscr{P}(\omega)$, then $g' : \mathscr{P}(\omega)/fin \to [\mathscr{P}(\omega)/fin]^{\leqslant\aleph_0}$ defined by

$$g'([x]) = \bigcup \{ \{ [y] : y \in g(z) \} : z \in \mathscr{P}(\omega), [z] = [x] \}$$

for $x \in \mathscr{P}(\omega)$, is a WFN mapping on $\mathscr{P}(\omega)/fin$.

If f is a WFN mapping on $\mathscr{P}(\omega)/fin$, then $f' : \mathscr{P}(\omega) \to [\mathscr{P}(\omega)]^{\leqslant\aleph_0}$ defined by

$$f'(x) = \{ z : [z] \in f([x]) \}$$

for $x \in \mathscr{P}(\omega)$, is a WFN mapping on $\mathscr{P}(\omega)$. □

Proposition 5.5. (a) (CH) *$\mathscr{P}(\omega)$ has the WFN.*

(b) *In the generic extension of a model of* ZFC + CH *by adding less that $\aleph_\omega$ many Cohen reals (by standard Cohen forcing), $\mathscr{P}(\omega)$ still has the WFN. In particular, the assertion* "MA(*Cohen*) + ¬CH + *$\mathscr{P}(\omega)$ has the WFN*" *is consistent. Here* MA(*Cohen*) *stands for Martin's axiom restricted to partial orderings of the form* $\mathrm{Fn}(\kappa, 2)$.

(c) *If $\boldsymbol{b} \geqslant \aleph_2$, then $\mathscr{P}(\omega)$ does not have the WFN.*

Proof. (a): Since $|\mathscr{P}(\omega)| = \aleph_1$ under CH, the claim follows from Lemma 2.1.

(b): This follows from Theorem 6.3 below.

(c): By Lemma 5.4, it is enough to show that $\mathscr{P}(\omega)/fin$ does not have the WFN. But under $\boldsymbol{b} \geqslant \aleph_2$, $(\omega_2, \leqslant)$ can be embedded into $\mathscr{P}(\omega)/fin$. Hence, by Proposition 4.1, $\mathscr{P}(\omega)/fin$ does not have the WFN. □

Note that, by Proposition 5.5(c), the statement "$\mathscr{P}(\omega)$ has the WFN" is not consistent with MA(σ-*centered*) + ¬CH.

6. Complete Boolean algebras

Lemma 6.1. *For Boolean algebras A, B and regular κ, if $A \leqslant B$, A is complete (but not necessarily a complete subalgebra of B) and B has the κ-FN, then A also has the κ-FN.*

Proof. By Sikorski's Extension Theorem, there is a homomorphism j from B to A such that $j \restriction A = id_A$. Hence the claim of the lemma follows from Lemma 2.7. □

Theorem 6.2. (a) *If $\boldsymbol{b} \geqslant \aleph_2$, then no infinite complete Boolean algebra has the WFN.*

(b) *For regular κ, no complete Boolean algebra without the κ-cc has the κ-FN.*

(c) *If κ is regular and $2^{<\kappa} = \kappa$, then every κ-cc complete Boolean algebra has the κ-FN.*

Proof. (a): If B is complete, then $\mathscr{P}(\omega)$ is embeddable into B. Hence, by Proposition 5.5(c) and Lemma 6.1, B does not have the WFN.

(b): If B is complete and not κ-cc, then $\mathscr{P}(\kappa)$ is embeddable into B. Hence, by Proposition 5.3(b) and Lemma 6.1, B does not have the κ-FN.

(c): Assume that B has the κ-cc. Let χ be sufficiently large and let $M \prec \mathscr{H}_\chi$ be such that $|M| = \kappa$, $B \in M$ and $[M]^{<\kappa} \subseteq M$ hold. Then $B \cap M$ is a κ-complete subalgebra of B. By the κ-cc of B it follows that $B \cap M$ is complete subalgebra of B. Hence $B \cap M \leqslant_{\mathrm{rc}} B$ holds. By Proposition 3.5 it follows that B has the κ-FN. □

By (b) and (c) in Theorem 6.2, under CH, a complete Boolean algebra B has the WFN if and only if B satisfies the ccc. This assertion still holds partially in the model obtained by adding Cohen reals to a model of CH:

Theorem 6.3. *Suppose that $V \models$ "ZFC + CH". Let λ be a cardinal in V such that $V \models$ "$\lambda < \aleph_\omega$". Let $P = \mathrm{Fn}(\lambda, 2)$ and let G be a V-generic filter over P. For $B \in V[G]$ such that $V[G] \models$ "B is* ccc *complete Boolean algebra", if there is a Boolean algebra $A \in V$ such that $B = \overline{A}^{V[G]}$ (where $\overline{A}^{V[G]}$ denotes the completion of A in $V[G]$), then we have: $V[G] \models$ "B has the WFN".*

For the proof of Theorem 6.3 we need the following lemma:

Lemma 6.4. *Let V be a ground model and $P = \mathrm{Fn}(S, 2)$ for some $S \in V$. Let G be V-generic over P and $A \in V$ be such that*

$V \models$ "*A is a* ccc *complete Boolean algebra*".

Then, we have:

(a) *if $\mathscr{S} \in V$ is such that*

$V \models$ "*$\mathscr{S}$ is a σ-directed family of subsets of S and* $\bigcup \mathscr{S} = S$"

then $\overline{A}^{V[G]} = \bigcup\{\overline{A}^{V[G_T]} : T \in \mathscr{S}\}$ where $G_T = G \cap \mathrm{Fn}(T, 2)$;

(b) *For a Boolean algebra B in $V[G]$, if $V[G] \models$ "$A \leqslant B$", $V[G] \models$ "A is a dense subalgebra of B", then $V[G] \models$ "$A \leqslant_\sigma B$".*

Proof. (a): If $b \in \overline{A}^{V[G]}$ then, by the ccc of A, there is a countable $X \subseteq A$ ($X \in V[G]$) such that $b = \Sigma^{\overline{A}^{V[G]}} X$. Hence, by the ccc of $\mathrm{Fn}(S, 2)$ (in V), there is a name $\dot{b}$ of b in which only countably many elements of $\mathrm{Fn}(S, 2)$ appear. Let $T \in \mathscr{S}$ be such that every element of $\mathrm{Fn}(S, 2)$ appearing in $\dot{b}$ is contained in $\mathrm{Fn}(T, 2)$. Then $b \in V[G_T]$, hence $b \in \overline{A}^{V[G_T]}$.

(b): It is enough to show that $V[G] \models$ "$A \leqslant_\sigma \overline{A}^{V[G]}$". Let $b \in \overline{A}^{V[G]}$. Then, as in the proof of (a), there exists a countable $X \subseteq S$ (in V) such that $b \in V[G_X]$ where $G_X = G \cap \mathrm{Fn}(X, 2)$. Let $\dot{b}$ be an $\mathrm{Fn}(X, 2)$ name of b. In $V[G]$, let

$$I = \{\Sigma^A\{c \in A : p \Vdash_{\mathrm{Fn}(X,2)} \text{“}c \leqslant \dot{b}\text{”}\} : p \in G_X\}.$$

Then I is countable and generates $A \restriction b$. □

Proof of Theorem 6.3. It is enough to show the following assertion for all ccc Boolean algebras A in V and for all $n \in \omega$:

$(*)_n$ If H is V – generic over $\mathrm{Fn}(\aleph_n, 2)$ then $\overline{A}^{V[H]}$ has the WFN in $V[H]$.

For $n \leqslant 1$, $V[H]$ still satisfies the CH. Hence, by Theorem 6.2(c), $\overline{A}^{V[H]}$ has the WFN in $V[H]$. Now suppose that $n > 1$ and we have shown $(*)_m$ for all $m < n$. Let H be a V-generic filter over $\mathrm{Fn}(\aleph_n, 2)$. For each $\alpha < \aleph_n$, let $H_\alpha = H \cap \mathrm{Fn}(\alpha, 2)$. Then by the induction hypothesis and Lemma 6.4(a) and (b), the sequence $(\overline{A}^{V[H_\alpha]})_{\alpha<\aleph_n}$ satisfies the conditions of Lemma 2.6. Hence $\overline{A}^{V[H]} = \bigcup_{\alpha<\aleph_n} \overline{A}^{V[H_\alpha]}$ has the WFN in $V[H]$. □

Corollary 6.6. *The assertion "every Cohen algebra has the WFN" is consistent with* $\mathrm{MA}(\mathit{Cohen}) + \neg\mathrm{CH}$.

7. $L_{\infty\kappa}$-free Boolean algebras

A Boolean algebra B is called *$L_{\infty\kappa}$-free* if $B \equiv_{L_{\infty\kappa}} \mathrm{Fr}\,\kappa$, i.e. if B is elementary equivalent to $\mathrm{Fr}\,\kappa$ in the infinitary logic $L_{\infty\kappa}$. B is *$L_{\infty\kappa}$-projective* if $B \equiv_{L_{\infty\kappa}} C$ for some projective C. It is easily seen that if B is $L_{\infty\kappa}$-projective then $B \oplus \mathrm{Fr}\,\kappa$ is $L_{\infty\kappa}$-free. In [9] it is shown that for every κ, there exists an $L_{\infty\aleph_1}$-free Boolean algebra B which does not satisfy the κ-cc. By Corollary 4.4, it follows that

Proposition 7.1. *For any κ, there exists an $L_{\infty\aleph_1}$-free Boolean algebra which does not have the κ-FN.*

On the other hand, we show in Corollary 7.4 below that every ccc $L_{\infty\aleph_1}$-free Boolean algebra has the WFN. Let us begin with the following lemma:

Lemma 7.2. *If B satisfies the* ccc *and $(A_\alpha)_{\alpha<\kappa}$ is an increasing sequence of relatively complete subalgebras of B with $\mathrm{cf}(\kappa) > \omega$ then $\bigcup_{\alpha<\kappa} A_\alpha \leqslant_{\mathrm{rc}} B$ holds.*

Proof. Suppose not. Then there is some $b \in B$ without projection on $\bigcup_{\alpha<\kappa} A_\alpha$. Then, for $\mu = \mathrm{cf}(\kappa)$, we can construct an increasing sequence of ordinals $(\gamma_\beta)_{\beta<\mu}$ such that $\gamma_\beta < \kappa$ and $p^B_{A_{\gamma_\beta}}(b) < p^B_{A_{\gamma_{\beta+1}}}(b)$ holds for every $\beta < \mu$. But this is impossible since B satisfies the ccc. □

Proposition 7.3. *If a Boolean algebra B satisfies the* ccc *and $\{\, C : C \leqslant_{\mathrm{rc}} B, |C| = \aleph_0 \,\}$ is cofinal in $[B]^{\aleph_0}$ then B has the WFN.*

Proof. By Proposition 3.3, it is enough to show that Player II has a simple winning strategy in $\mathscr{G}^{\omega_1}(B)$. By Lemma 7.2, Player II wins if he takes $Y_\alpha \geqslant X_\alpha$ in his αth move such that $Y_\alpha \leqslant_{\mathrm{rc}} B$ holds (actually this is a simple winning strategy of Player II in $\mathscr{G}^{\omega_1}_{\aleph_0}(B)$). □

Corollary 7.4. *Every* ccc *$L_{\infty\aleph_1}$-projective Boolean algebra B has the WFN.*

Proof. Let B be a ccc $L_{\infty\aleph_1}$-projective Boolean algebra. The statement "$\{\, C : C \leqslant_{\mathrm{rc}} B$, $|C| = \aleph_0 \,\}$ is cofinal in $[B]^{\aleph_0}$" can be formulated in $L_{\infty\aleph_1}$ and is true in any projective

Boolean algebra. Hence it is also true in B. By Proposition 7.3, it follows that B has the WFN. □

Since the ccc is expressible in $L_{\infty\aleph_2}$ and it is true in any projective Boolean algebra, every $L_{\infty\aleph_2}$-projective Boolean algebra satisfies the ccc. Hence, by Corollary 7.4 every $L_{\infty\aleph_2}$-projective Boolean algebra has the WFN. Under Axiom R we can obtain a stronger result. Recall that Axiom R is the following statement:

(Axiom R): *For any $\lambda \geqslant \aleph_2$ with uncountable cofinality, stationary $\mathscr{S} \subseteq [\lambda]^{\aleph_0}$ and a $\mathscr{T} \subseteq [\lambda]^{\aleph_1}$ which is closed under union of increasing chains of order type ω_1, there exists an $X \in \mathscr{T}$ such that $\mathscr{S} \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.*

Axiom R follows from MM ([1]) but its consistency with CH can be also shown under a supercompact cardinal. The following theorem was proved in [6] (see also [7]):

Theorem 7.5. (Axiom R) *Every $L_{\infty\aleph_2}$-projective Boolean algebra B has the FN.*

The theorem above is not provable in ZFC: under $V = L$, we can obtain a counterexample to Theorem 7.5 ([7]). In contrast to Corollary 7.4, we have the following:

Proposition 7.6. *For any cardinal κ, there is a subalgebra of* $\operatorname{Fr}\kappa^+$ *without the κ-FN.*

Proof. The topological dual to the Boolean algebra B below is considered in Engelking [2] to show that there exists a non-projective subalgebra of a free Boolean algebra (in the language of topology, this means that there exists a dyadic space which is not a Dugundji space).

Let X be a set of cardinality κ^+. We shall show that there is a subalgebra of $\operatorname{Fr}X$ without the WFN. Let U_1 and U_2 be the ultrafilters of $\operatorname{Fr}X$ generated by X and $\{-x : x \in X\}$ respectively. Let

$$B = \{b \in \operatorname{Fr}X : b \in U_1 \Leftrightarrow b \in U_2\}.$$

Clearly B is a subalgebra of $\operatorname{Fr}X$. We claim that B does not have the κ-FN. For $Y \subseteq X$, let $B_Y = B \cap \operatorname{Fr}Y$.

Claim 7.6.1. *For every $Y \in [X]^\kappa$, B_Y is not a κ-subalgebra of B.*

Proof. Let $x_0 \in Y$ and let x_1, x_2 be two distinct elements of $X \setminus Y$. Let

$$b = x_0 + x_1 + -x_2.$$

Since $b \in U_1$ and $b \in U_2$, we have $b \in B$. Let $I = B_Y \restriction b$. We show that I is not $< \kappa$-generated: let J be any subset of I of cardinality less than κ. For $c \in J$, we have $c \leqslant x_0$. Since x_0 is not an element of B, c is strictly less than x_0. Let Y' be a subset of Y of cardinality less than κ such that $J \subseteq B_{Y'}$. Let y_1, y_2 be two distinct elements of $Y \setminus Y'$. Let

$$d = x_0 \cdot y_1 \cdot -y_2.$$

Then we have $d \notin U_1$, $d \notin U_2$ and $d \leqslant x_0$. Hence $d \in B_Y \restriction b$. But d is incomparable with every non-zero element of J. This proves the claim. □

Since there are club many $C \in [B]^\kappa$ of the form $C = B_Y$ for $Y \in [X]^\kappa$, it follows that B does not have the κ-FN by Proposition 3.1. Note that the implication $(1) \Rightarrow (3)$ in Proposition 3.1 used here does not require the assumption of regularity of κ. This proves the proposition. □

A subalgebra of an openly generated Boolean algebra B is openly generated (i.e. has the FN) if and only if B has the Bockstein separation property ([10]).

Problem 7.7. *Is there a Boolean algebra with the Bockstein separation property but without the WFN?*

In [7], the following partial answer to the problem is given: *if B is stable and satisfies the ccc and the Bockstein separation property, then B has the WFN.* Here a Boolean algebra B is said to be stable if, for any countable subset X of B, only countably many types over X are realized in B.

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Note added in proof

After this paper was submitted we found that the proof of $(3) \Rightarrow (1)$ in Theorem 3.3 is incorrect. The correct proof of the implication we know at the moment needs the following additional assumptions:

(i) for every $\mu \geqslant \kappa$ of cofinality $\geqslant \kappa, ([\mu]^{<\kappa}, \subseteq)$ has a cofinal subset of cardinality μ;

(ii) for every $\mu \geqslant \kappa$ of cofinality $< \kappa$, $\square_\mu$ holds.

Thus, in Proposition 3.5, Theorem 6.2, (c) and Theorem 6.3, corresponding assumptions should be added. Meanwhile it is known that we do need some additional assumptions for $(3) \Rightarrow (1)$ of Proposition 3.3: under the consistency strength of a huge cardinal, there is a model of ZFC in which we have a counter-example to the implication. More details about these results are to be found in: *S. Fuchino and Lajos Soukup; More set-theory around the weak Freese–Nation property*, which is in preparation.

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