REGULAR ELEMENTS IN SANDWICH SEMIGROUPS OF BINARY RELATIONS

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This note gives a counterexample to a published characterization of regular elements in sandwich semigroups of binary relations. The method of that paper is used to characterize those elements having a right identity. A characterization of regular elements is obtained following the approach of Schein.

0. Introduction

Let $R$ be a fixed binary relation on a set $X$. For any two relations $A$ and $B$ in $\mathcal{B}_X$, the set of all binary relations on $X$, define $A \ast B$ as $ARB$ where juxtaposition is the usual composition of relations. Denote by $\mathcal{B}_X(R)$ the semigroup defined on $\mathcal{B}_X$ by the $\ast$ operation. $\mathcal{B}_X(R)$ is called the sandwich semigroup of binary relations with sandwich relation $R$. These semigroups were studied by K. Chase in [1-3] with some motivation towards applications to automata theory. Others (e.g. [8]) have used sandwich semigroups in other contexts.

The main result of [3] characterizes regular elements of $\mathcal{B}_X(R)$ in terms of two functions from $\mathcal{B}_X(R)$. Unfortunately the conditions as stated are sufficient but not necessary. The methods used by Chase can be modified to characterize those elements of $\mathcal{B}_X(R)$ which have a right identity and for finding an algorithm to determine the maximal right identity.

In the final section of this paper a single necessary and sufficient condition for the regularity of an element in $\mathcal{B}_X(R)$ is found. The method here is a slight modification of that due to Schein [10] and produces the maximal inverse for each regular element. Many of the results here apply with minor changes to the semigroups of closed relations considered in [5-7]. The author thanks the referees for helpful comments.

1. Preliminaries

Binary Boolean matrices can be used to represent binary relations on $X$. For $A \in \mathcal{B}_X(R)$ the matrix corresponding to $A$ has a one in the $(x, y)$ position iff...
Then the composition of relations is the same as binary Boolean matrix multiplication. The converse of $\Lambda$,
\[ A^{-1} = \{(y, x) \mid (x, y) \in A\} \]
as a matrix is the transpose. For any $x \in X$,
\[ xA = \{y \mid (x, y) \in A\} \]
is called the $x$th row of $A$. The $x$th column is similarly defined as
\[ Ax = \{y \mid (y, x) \in A\} = xA^{-1}. \]
For any subset $S \subseteq X$,
\[ [S]A = \bigcup \{sA \mid s \in S\} \quad \text{and} \quad A[S] = \bigcup \{As \mid s \in S\} = [S]A^{-1}. \]
Then the range of $A$ is $[X]A$ and the domain is $A[X]$. The collection $\{[S]A \mid S \subseteq X\}$ is called the row space of $A$ and forms a lattice $\mathcal{Y}(A)$ under inclusion. The column space lattice is $\mathcal{Y}(A^{-1}) = \{A[S] \mid S \subseteq X\}$ under inclusion. Equality and isomorphisms of these lattices determine the Green's relations in $\mathcal{B}_X$ but not in $\mathcal{B}_X(R)$, (see [2, 9, 11]).

For each $A \in \mathcal{B}_X(R)$ two other relations may be defined as follows:
\[ W(A) = \{(x, y) \mid \text{for some } v \in X, x \in ARv \text{ and } ARv \subseteq Ay\}, \]
\[ B(A) = \{(x, y) \mid ARx \subseteq Ay\}. \]
It is easy to check that $W(A)$ has the property that for each $x$, $xW(A)$ is $W_{xAR}$ as defined in [3, Proposition 2.3]. The definition of $B(A)$ here is close to how $B$ was defined in [3, Proposition 2.3]. Note that in contrast to how $B$ was defined it is not required that $xB(A) = \emptyset$ when $xA = \emptyset$. If this were required, then
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
gives a counterexample to $A * B = A$ which is claimed in the proof of [3, Proposition 2.3]. Because of [3, Example 2.6] it appears that requirement was not really intended.

In terms of the relations $W(A)$ and $B(A)$, Theorem 2.5 of [3] becomes: $A$ is regular in $\mathcal{B}_X(R)$ if and only if $A = W(A)$ and for each $x \in X$ there is a $K_x \subseteq X$ with $[K_x]RA = xB(A)$. This last condition is equivalent to $B(A) = C * A$ for some $C$. Because $W(A) = A$ implies $A * B(A) = A$ by Proposition 2.2, this is just that $A$ and $B(A)$ are $L$-related. Let
\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
then

\[ B(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad W(A) = \mathcal{A}, \quad A \ast B(A) = A \]

yet \( B(A) \neq C \ast A \) for any \( C \) in \( \mathcal{B}_x(R) \). If

\[ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

then \( A \ast D \ast A = A \) so \( A \) is regular in \( \mathcal{B}_x(R) \). This contradicts \([3, \text{Theorem 2.5}]\) and shows the condition there is not necessary, although it is sufficient.

2. Right identities in \( \mathcal{B}_x(R) \)

A right identity for \( A \in \mathcal{B}_x(R) \) is an element \( C \) so that \( A \ast C = A \). In this section the relations \( W(A) \) and \( B(A) \) will be used to determine the existence of right identities and to find the largest such identity.

**Lemma 2.1.** For all \( A \in \mathcal{B}_x(R), \ W(A) \subseteq A. \)

**Proof.** If \( (x, y) \in W(A) \), then \( x \in AR_\mathcal{U} \subseteq A y \) for some \( \mathcal{U} \). Hence \( x \in A y \) or \( (x, y) \in A \).

**Proposition 2.2.** \( A \ast B(A) = W(A) \) for all \( A \) in \( \mathcal{B}_xR \).

**Proof.** For \( (x, y) \in W(A) \) choose \( \mathcal{U} \) so that \( x \in A \mathcal{U} R_v \) and \( A \mathcal{U} R_v \subseteq A y \). Thus \( (x, \mathcal{U}) \in AR \). Therefore \( (x, y) \in B(A) \). Hence \( (x, y) \in A \mathcal{R} B(A) = A \ast B(A) \). Conversely, let \( (x, y) \in A \ast B(A) = A \mathcal{R} B(A) \). Then for some \( v, (x, v) \in A R \) and \( (v, y) \in B(A). \) Hence \( x \in A \mathcal{U} R_v \) and \( A \mathcal{U} R_v \subseteq A y \). Therefore \( (x, y) \in W(A). \)

**Lemma 2.3.** Let \( A \) and \( C \) be in \( \mathcal{B}_R(X) \). If \( A \ast C \subseteq A \), then \( C \subseteq B(A). \)

**Proof.** Let \( (x, y) \in C \). If \( A \mathcal{R} x = \emptyset \), then \( (x, y) \in B(A) \). Suppose \( s \in A \mathcal{R} x \), i.e. \( (s, x) \in AR \). Then \( (s, y) \in A RC = A \ast C \subseteq A \). Thus \( (s, y) \in A \) or \( s \in A y \). Hence \( A \mathcal{R} x \subseteq A y \) and \( (x, y) \in B(A) \).

**Corollary 2.4.** \( B(A) = \bigcup \{ C \mid A \ast C \subseteq W(A) \}. \)

**Proof.** \( B(A) \) is included in the union by Proposition 2.2. The opposite inclusion follows from Lemmas 2.1 and 2.3.
Theorem 2.5. An element A has a right identity in $\mathcal{B}_X(R)$ if and only if $W(A) = A$. In this case $B(A)$ is the largest right identity.

Proof. If $C$ is a right identity let $(x, y) \in A = A * C = ARC$. Then choose $v$ so $(x, v) \in AR$ and $(v, y) \in C$. Then $x \in ARv$. Suppose $u \in ARv$, then $(u, v) \in AR$, so $(u, y) \in ARC = A$ or $u \in Ay$. Hence $ARv \subset Ay$, $(x, y) \in W(A)$ and $A \subset W(A)$. By Lemma 2.1, $W(A) = A$. Conversely, if $W(A) = A$, then by Proposition 2.2 $A * B(A) = A$, so A has a right identity. By Lemma 2.3 $B(A)$ contains any right identity.

The following additional facts relating $A$, $W(A)$, and $B(A)$ are fairly easy to derive from the previous results and definitions.

**Proposition 2.6.** Let $A$ and $C$ be any elements in $\mathcal{B}_X(R)$.

(a) $W(A) * B(A) \subset W(A)$.
(b) $B(A) * B(A) \subset B(A) \subset B(B(A))$.
(c) If $R = 1$ equality holds in (b).
(d) $A$ and $C$ $L$-related implies $B(A) = B(C)$ and $W(A)$ is $L$-related to $W(C)$.
(e) If $A = P \times Q$ is a product relation, then $B(A)$ is the relation $X \times Q \cup (X - [Q]R) \times X$.

3. Regular elements in $\mathcal{B}_X(R)$

Schein in [10] has characterized the regular elements of $\mathcal{B}_X$ in a way which allows easy computation of both the regular elements and their largest inverses. His method can be applied to $\mathcal{B}_X(R)$ with slight modifications.

A subinverse of $A \in \mathcal{B}_X(R)$ is any element $C$ with $A * C * A \subset A$. Since the empty relation is a subinverse and the union of subinverses is again a subinverse, a unique greatest subinverse $S(A)$ will always exist. For any relation $C$ in $\mathcal{B}_X(R)$ let $C'$ be the complimentary relation. Thus $C' = \{(x, y) | (x, y) \notin C\}$. Then the greatest subinverse $S(A)$ for $A$ is computable as follows.

**Theorem 3.1.** For any $A \in \mathcal{B}_X(R)$, $S(A) = (R^{-1}A^{-1}A' A'^{-1}R^{-1})'$.

Proof. By the definition of $S(A)$, $(x, y) \in S(A)$ iff $A * (x \times y) * A \subset A$ iff $AR(x \times y)RA \subset A$ iff $(u, v) \in AR(x \times y)RA$ implies $(u, v) \in A$ iff $(u, s) \in A$, $(s, x) \in R$, $(y, t) \in R$ and $(t, v) \in A$ implies $(u, v) \in A$. Thus $(x, y)$ not in $S(A)$ is equivalent to the existence of $u, v, s$, and $t$ so that $(u, s) \in A$, $(s, x) \in R$, $(y, t) \in R$, $(t, v) \in A$ and $(u, v) \notin A$. This is the same as $u, v, s$, and $t$ with $(x, s) \in R^{-1}$, $(s, u) \in A^{-1}$, $(u, v) \in A'$. $(v, t) \in A^{-1}$ and $(t, y) \in R^{-1}$, that is $(x, y) \in R^{-1}A^{-1}A'A'^{-1}R^{-1}$. Thus $(x, y) \in S(A)$ iff $(x, y) \in (R^{-1}A^{-1}A'A^{-1}R^{-1})'$. 

Theorem 2 and its corollary from [10], translated to $B_X(R)$ now following using the same proofs.

**Theorem 3.2.** A relation $A$ in $B_X(R)$ is regular iff $A \subset A \ast S(A) \ast A$.

**Proof.** If $A$ is regular, then $A \ast C \ast A = A$ for some $C$. Thus $C \subset S(A)$. Hence $A = A \ast C \ast A \subset A \ast S(A) \ast A$. Conversely, if $A \subset A \ast S(A) \ast A$, then since $A \ast S(A) \ast A \subset A$, $A = A \ast S(A) \ast A$ and $A$ is regular.

**Corollary 3.3.** If $A \in B_X(R)$ is regular, then its greatest inverse is $S(A) \ast A \ast S(A)$.

**Proof.** If $A$ is regular, then by Theorem 3.2 $S(A) \ast A \ast S(A)$ is an inverse of $A$. If $C'$ is any other inverse of $A$, then $C' \subset S(A)$ and so $C' \prec C \ast A \prec C \subset S(A) \ast A \ast S(A)$.

Zareckii [11] has characterized the regular elements in $B_X$ in terms of the distributivity of the row and column space lattices. Using the methods and equation (3) of [4] the following characterization of regular elements in $B_X(R)$ in terms of lattices of row spaces can be obtained.

**Theorem 3.4.** $A \in B_X(R)$ is regular if and only if for every $U \subset X$, 

$$[U]A = \bigcup \{[Y]RA \mid Y \subset X, [U]AR \notin [Z]A\}.$$ 

Note that when $R$ is the identity the condition of Theorem 3.4 is just that the row space lattice $\mathcal{R}(A)$ is completely distributive.

**References**


