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BIPARTITE STEINHAUS GRAPHS

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The following two theorems are proven in this paper.

Theorem. *A Steinhaus graph is bipartite if and only if it has no triangles.*

Theorem. *If G is a bipartite Steinhaus graph ($G \neq \overline{K_n}$) with partitions X and Y, where $|X| \leq |Y|$, then G has an X-saturated matching.*

1. Introduction

Let $a_1, \ldots, a_{n-1}$ be a sequence of zeros and ones. Using addition mod 2, $a_1 + a_2, a_2 + a_3, \ldots, a_{n-2} + a_{n-1}$ is an $(n-2)$-long sequence of zeroes and ones. From this sequence, a third sequence can be formed of length $n-3$. Continuing this process, we have $n-1$ sequences of zeroes and ones, which we can display in a triangle of side $n-1$, called a *Steinhaus triangle*. In Fig. 1(a), we show the Steinhaus triangle generated by the sequence 001100.

$$\begin{array}{c}
0\ 0\ 1\ 1\ 0\ 0\\
0\ 1\ 0\ 1\ 0\\
1\ 1\ 1\ 1\\
0\ 0\ 0\\
0\ 0\\
0
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
0 & 1 & 2 & \cdot & \cdot & \cdot & n-1\\
0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}$$

(a) (b)

Fig. 1

To create graphs from these triangles, we form an $n \times n$ symmetric matrix with a diagonal of zeroes and with a Steinhaus triangle as the upper triangular part of the matrix. The matrix generated by 001100 is shown in Fig. 1(b), with the triangle outlined, and the columns numbered from 0 to $n-1$. If $A = (a_{i,j})$ is such a *Steinhaus matrix* then it has the *Steinhaus property*,

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j} \pmod 2 \text{ for } 0 < i < j < n.$$

Of course a symmetric matrix of zeroes and ones is the adjacency matrix of a graph. In Fig. 2, we have drawn in two different ways the Steinhaus graph on $n = 7$ vertices generated by the 6-long sequence 001100.

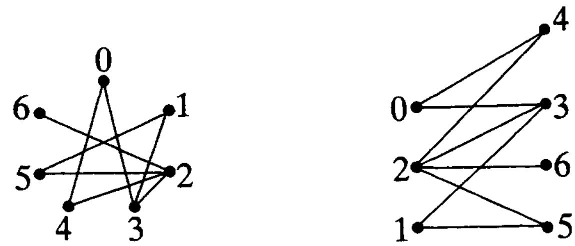

Fig. 2

Steinhaus triangles were first studied by Harborth [5] and later by Chang [1]. Molluzzo [6] made graphs from these triangles although his graphs are the complements of the Steinhaus graphs in this paper. The complements of Steinhaus graphs were further studied in [2] and [4]. Theorem 2 of this paper was proven in [2].

We use n to denote the number of vertices of a Steinhaus graph G. Note that there are 2^{n-1} Steinhaus graphs on n vertices. If we consider the $(n-1)$-long sequence which generates G as a binary number then we use k to denote the decimal representation of that binary number. Usually we label the vertices of a Steinhaus graph from 0 to $n-1$ (as in Fig. 2) and we denote the degree of vertex i by d_i. If v is a vertex of a graph G then $G/\{v\}$ is the graph G except that vertex v and all edges of G incident to v are deleted from G. Note that if G is a Steinhaus graph, then both $G/\{0\}$ and $G/\{n-1\}$ are Steinhaus graphs with $n-1$ vertices, a situation ideal for induction. Also, we use the notation $j = a(b)[c]$ to mean that the index variable j has the values $a, a+b, a+2b, \ldots, c$.

The following three theorems about Steinhaus graphs and their proofs may be found in [3] along with information on eulerian Steinhaus graphs.

Theorem A. *All Steinhaus graphs are connected except for* $\overline{K_n}$ *which is generated by the all zero sequence.*

Theorem B. *For* $n \geq 5$, *the only Steinhaus trees are the star* $K_{1,n-1}$, *generated by* $k = 1$ *and* $k = 2^{n-1} - 1$, *and the path* P_n *generated by* $k = 2^{n-2}$.

Theorem C. *Let* $A(G)$ *be the adjacency matrix of a Steinhaus graph* G. *Then the Steinhaus graph generated by the sequence* $a_{n-2,n-1}, a_{n-3,n-1}, \ldots, a_{0,n-1}$ *is isomorphic to* G *and the correspondence is vertex* j *to vertex* $n-1-j$, *for* $j = 0(1)[(n-1)/2]$, *where* $[x]$ *is the greatest integer in* x.

This graph is called the *partner of* G. For example, the partner of the graph generated by 001100 is the graph generated by 000100.

2. The number of bipartite Steinhaus graphs

Since all trees and forests are bipartite, $\overline{K_n}$, $K_{1,n-1}$, and P_n are bipartite Steinhaus graphs. We now show that there are other bipartite Steinhaus graphs.

Theorem 1. *There are at least $n+1$ sequences of length $n-1$ that generate bipartite Steinhaus graphs on n vertices. Of these, $[n/2]+1$ are not isomorphic.*

Proof. First we exhibit the adjacency matrix for a bipartite Steinhaus graph, G, with $\{0, 1, \ldots, r-1\}$ as the vertices in one partition, for $1<r<n-1$, and $\{r, r+1, \ldots, n-1\}$ in the other partition. To do this we set

$$a_{i,r} = 1, \quad \text{for } i = 0(1)(r-1), \quad \text{and}$$
$$a_{r-1,j} = 1, \quad \text{for } j = 0(1)(r-1)$$

in the adjacency matrix $A(G) = (a_{i,j})$ of G. In Fig. 3(a) this is shown for $r = 4$, $n = 8$. (The diagonal elements of $A(G)$ are bold.) Using the Steinhaus property ($a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$) it is easily seen that the columns to the left of column r and the rows below row $r-1$ are all zeroes. Hence, G must be bipartite with partitions as described above. This gives $n-2$ sequences of which the ones for $r = s$ and $r = n-1-s$ are isomorphic. The others are not isomorphic because the number of vertices in the partitions differ. The other three sequences are $k = 0$, 1, $2^{n-1}-1$, yielding $\overline{K_n}$ and $K_{1,n-1}$.

```
             0   r   n-1               0   r   n-1
             0···1···                  00001000
              0··1   ·                  0001100
               0·1   ·                   001010
row r-1         01111       row r-1       01111
                 0                         0000
                  0                         000
                   0                         00
                    0                         0
                   (a)                       (b)
```

Fig. 3

These sequences are not the only ones that generate bipartite Steinhaus graphs but there seems to be no easy way to characterize the others. An example is for $n = 4m$ and $k = 2^{n-5}$. One partition contains all vertices r for which $r = 0, 1, 2, 3 \pmod 8$.

The number $b(n)$ of sequences that generate bipartite Steinhaus graphs on n vertices for $n = 3(1)(24)$ is given in Table 1. It seems from this limited data that $b(n)$ is approximately $2n$. A crude upper bound can be obtained for $b(n)$ by

Table 1

n	$b(n)$	n	$b(n)$	n	$b(n)$
3	4	11	21	19	39
4	6	12	23	20	40
5	9	13	27	21	43
6	10	14	28	22	44
7	13	15	31	23	47
8	15	16	34	24	50
9	19	17	39		
10	19	18	38		

noting the following two facts. First, if G is a bipartite Steinhaus graph on $n+1$ vertices, then $G/\{n\}$ is a bipartite Steinhaus graph on n vertices. Second, if G is a bipartite Steinhaus graph with vertices $\{0, 1, \ldots, n-1\}$ then it can be extended to a Steinhaus graph (not necessarily bipartite) with $n+1$ vertices $\{0, 1, \ldots, n-1, n\}$ in only two ways, depending on whether 0 and n are or are not adjacent. Therefore, $b(n+1) \leq 2 \times b(n)$. Hence $b(n) \leq 50 \times 2^{n-24}$ for $n > 23$.

3. Triangle free implies bipartite

It is known that a graph is bipartite if and only if it has no cycles of odd length. A non-bipartite graph with no triangles is C_5. But for Steinhaus graphs we have

Theorem 2. *If G is a Steinhaus graph with no triangles then G is bipartite.*

Proof. First, if G is bipartite and connected, its partitions are unique. Let n be the least integer such that there is a non-bipartite triangle free Steinhaus graph on n vertices. Obviously $n \geq 5$.

Let G be a Steinhaus graph on n vertices with no triangles. The graphs $G_A = G/A$, $A \subseteq \{0, n-1\}$, $A \neq \emptyset$ are triangle free Steinhaus graphs on $n-1$ or $n-2$ vertices. Hence, these are all bipartite. If G_A is not connected then G is either $\overline{K_n}$, $K_{2,n-2}$, or $K_{1,n-1}$ and hence bipartite. Consequently, if G is to be a counterexample to the theorem, G_A must be connected.

Let X and Y be the unique partitions of $G/\{0, n-1\}$, and let X_0, Y_0 be the unique partitions of $G_0 = G/\{0\}$. The vertex $n-1$ is either in X_0 or Y_0. Since $G/\{0, n-1\} = G_0/\{n-1\}$ then either $X = X_0$ and $Y = Y_0/\{n-1\}$, or $X = X_0/\{n-1\}$ and $Y = Y_0$. Similarly either $X \cup \{0\}$ and Y, or X and $Y \cup \{0\}$ are the partitions of $G/\{n-1\}$. Without loss of generality, we can assume either $\{0, n-1\} \subseteq X$, or 0 is in X and $n-1$ is in Y. In the latter case G is bipartite. In the former case G is bipartite unless 0 and $n-1$ are adjacent. Therefore, for G to be a counterexample to the theorem, vertices 0 and $n-1$ are adjacent and $\{0, n-1\} \subseteq X$.

If $d_0 = 1$ or $d_{n-1} = 1$, then G is the star $K_{1,n-1}$ and hence bipartite. So let r be the smallest vertex adjacent to vertex 0. Then $A(G)$ would be as in Fig. 4.

$$
\begin{array}{rl}
 & 0 \;\; r \quad\; n-1 \\
 & 0001 \cdots 1 \\
 & \;\; 0 \cdot\cdot \quad \cdot \\
 & \;\;\; 0 \cdot \quad \cdot \\
\text{row } r & \;\;\;\; 0 \cdot\cdot\cdot\cdot
\end{array}
$$

Fig. 4

Since 0 is in X, then r is in Y and so for vertex j, $0 \leqslant j < r$, we have that j is in X. Hence vertex j, $0 < j < r$, is not adjacent to vertex $n-1$ because $n-1$ is also in X. Thus in $A(G)$, $a_{i,n-1} = 1$, $i = 1(1)(r-1)$. Also vertex r is not adjacent to vertex $n-1$ for then $(0, r, n-1)$ is a triangle. Hence, $A(G)$ is as depicted in Fig. 5.

$$
\begin{array}{rl}
 & 0 \;\; r \quad\; n-1 \\
 & 0001 \cdots 1 \\
 & \;\; 001 \quad 0 \\
 & \;\;\; 01 \quad 0 \\
\text{row } r & \;\;\;\; 0 \cdots 0
\end{array}
$$

Fig. 5

Using the Steinhaus property, it is evident that vertex 0 is adjacent to vertices $n-2-j$, $j = 0(1)(r-1)$. Again, since G has no triangles, vertex r is not adjacent to vertices $n-2-j$, $j = 0(1)(r-1)$. (See Fig. 6(a).) Thus vertex 0 is adjacent to vertices $n-2-r-j$, $j = 0(1)(r-1)$. (See Fig. 6(b).) Continuing this process we find that $r = 1$ and $G = K_{1,n-1}$. If this is not the case, $a_{r-1,r} = 1$, $a_{r-1,r+1} = 0$ and $a_{r,r+1} = 0$, which violates the Steinhaus property of $A(G)$. Hence we cannot find a counterexample on n vertices. Therefore, if G has no triangles then G is bipartite.

$$
\begin{array}{rl}
 & 0 \;\; r \qquad\quad n-1 \\
 & 0001 \cdots 1111 \\
 & \;\; 001 \quad\;\; \cdot 000 \\
 & \;\;\; 01 \quad\;\; \cdot\cdot 00 \\
\text{row } r & \;\;\;\; 0 \cdots 0000 \\
 & \qquad \text{(a)}
\end{array}
\qquad
\begin{array}{rl}
 & 0 \;\; r \qquad\quad n-1 \\
 & 00011111111 \\
 & \;\; 001 \cdot 000000 \\
 & \;\;\; 01 \cdot\cdot 00000 \\
\text{row } r & \;\;\;\; 0 \cdots 0000 \\
 & \qquad \text{(b)}
\end{array}
$$

Fig. 6

4. Matchings in bipartite Steinhaus graphs

Let G be a bipartite graph with partitions X and Y where $|X| \leqslant |Y|$. A set of edges in G with no common endpoints is called *independent*. A set of pairwise nonadjacent vertices is also called independent. A *matching* in G is a set of independent edges, and an X-*saturated matching* is a matching that involves all the vertices in X. A *perfect matching* of G is a matching that includes all the vertices of G, and hence $|X| = |Y|$. Finally, a *vertex cover* of G is a set S of vertices of G for which any vertex in G/S is adjacent to a vertex in S.

In Fig. 7 we show a bipartite graph without a perfect matching. Note that $S = \{0, 1, 2, 7, 8, 9\}$ is an independent set of vertices and $|S| > |Y|$. Also, $T = \{3, 4, 5, 6\}$ is a vertex cover of G with $|T| < |X|$. Such graphs cannot be Steinhaus graphs because of Theorem 3.

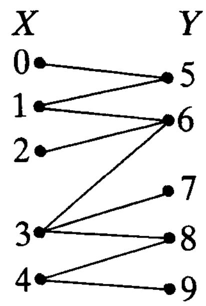

Fig. 7

Theorem 3. *If G is a bipartite Steinhaus graph ($G \neq \overline{K_n}$) with partitions X and Y, where $|X| \leqslant |Y|$, then G has an X-saturated matching.*

From this theorem there are two immediate corollaries.

Corollary 4. *A connected bipartite Steinhaus graph with partitions X and Y has a perfect matching if and only if $|X| = |Y|$.*

Corollary 5. *Let X and Y, $|X| \leqslant |Y|$, be the partitions in a connected bipartite Steinhaus graph G. Then X is a minimal vertex cover of the edges of G and Y is a maximal independent set of the vertices of G.*

Proof of Theorem 3. Let n be the least integer for which a graph on n vertices can be found as a counterexample to the statement of Theorem 3. By inspection, the theorem is true for all Steinhaus graphs with less than 11 vertices and so $n \geqslant 11$.

Let G be a connected bipartite Steinhaus graph on n vertices with partitions X and Y, $|X| \leqslant |Y|$. Let $a \in \{0, n-1\}$. If $|X| < |Y|$ and a is in Y, then $G/\{a\}$ is bipartite and is either connected or $\overline{K_{n-1}}$. In the first case, $G/\{a\}$ has an X-saturated matching and hence G also does. In the latter case, $G = K_{1,n-1}$. Thus G is not a counterexample to the theorem, unless $|X| = |Y|$ or $\{0, n-1\} \subseteq X$.

If $|X| = |Y|$ we always choose X to be the partition containing vertex 0. Note that if vertices 0 and $n-1$ are in different partitions, then $|X| = |Y|$.

We denote by r (s) the smallest (largest) vertex adjacent to vertex 0. Since 0 is in X, then r is in Y and, as can be seen in Fig. 8, vertex j is in X for $j = 0(1)(r-1)$. Hence if $r > [n/2]$, $|X| > |Y|$. Consequently, for G to be a counterexample, we can assume that $r \leqslant [n/2]$. There are two cases: $r > n - s$ and $r \leqslant n - s$.

```
         0   r     n − 1
         0001 · · · 1
          001       ·
           01       ·
row r       0 · · · ·
```

Fig. 8

Case 1. $r > n - s$.

First, $r \neq s$ (i.e., $d_0 \neq 1$) for if so, $r > n/2$. By the Steinhaus property, see Fig. 9, $a_{i,r} = 1$ and $a_{i,s+i} = 1$ for $i = 0(1)(r-1)$. Since $n - s < r$, then $n - 1 - s < r - 1$. Therefore, vertex $n - 1 - s$ is adjacent to vertex r. Also vertex $n - 1 - s$ is adjacent to vertex $n - 1$ and so $\{r, n-1\} \subseteq Y$. As previously noted, 0 is in X. Since $n - 1$ is in Y we have that $|X| = |Y|$.

Consider $H = G/\{0, 1, \ldots, n-1-s, s, s+1, \ldots, n-1\}$. (In Fig. 9, H is outlined.) Now $(r-1, r)$ is an edge in H and $n - s$ vertices were deleted from both X and Y to give $X/\{0, \ldots, n-1-s\}$ and $Y/\{s, s+1, \ldots, n-1\}$ as the partitions of H. So H is a connected bipartite Steinhaus graph on less than n vertices. Therefore, H has a perfect matching. This partial matching of G can be completed to a perfect matching for G by adding the edges $(i, s+i)$ for $i = 0(1)(n-1-s)$.

```
        0       r     s   n − 1
        00000001 · · · 10000
         0     01       1000
          0    01        100
           0   01         10
            0  01     ·    1
            ____________
             001       |·
              01     H |·
row r          0       |
                0      |    ·
                 0     |    ·
                  0    |    ·
                   0   |
row s               0
                     0
                      0
                       0
                        0
```

Fig. 9

Case 2. $r \leqslant n - s$.

As before vertex $r-1$ is adjacent to vertex r and since r is in Y, then $r-1$ is in X. Also $r-1$ is adjacent to $s+(r-1)$ and so $s+(r-1)$ is in Y. (If $s+(r-1) > n-1$, then $r > n-s$.) Hence r is not adjacent to $s+(r-1)$. By the Steinhaus property, this implies that $r-1$ is adjacent to $s+(r-2)$, which in turns implies that r is not adjacent to $s+(r-2)$. Continuing this process, we note that vertices $r-1$ and $s+r$ are adjacent to vertices $r+i$ for $i = 0(1)(s-1)$.

Suppose $(n-r)/s \leqslant 2$. In this case X and Y are as in Fig. 10, where $r=3$, $s=12$, $n=18$, $X = \{0, 1, 2, 15, 16, 17\}$ and the matching is $(0,12)$, $(1,13)$, $(2,14)$, $(15,3)$, $(16,4)$, $(17,5)$.

	(a) 0 r s n − 1		(b) 0 r s n − 1
	0001 · · · 100000		0001 · · · 100000
	001 10000		001 10000
	01 1000		011111111000
row r	0 · · · 100	row r	00000000100
			0000000110
			000000101

(a) (b)

Fig. 10

If $2s \leqslant n-1$ then $|X| > |Y|$. Therefore, a graph with such a partition is not a counterexample to the theorem. So $2s > n-1$ and the following X-saturated matching is indicated in Figs. 10 and 11,

$$(a, a+s), \quad 0 \leqslant a \leqslant r-1, \quad \text{and}$$
$$(a, a-s), \quad s+r \leqslant a \leqslant n-1, \text{ for } a \text{ in } X.$$

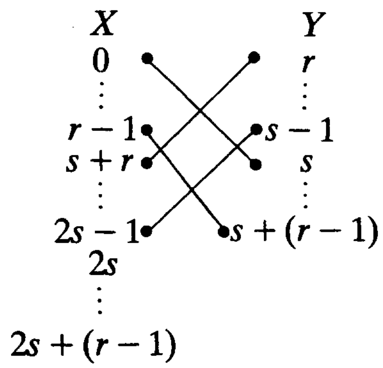

Fig. 11

From this we can now assume $(n-r)/s > 2$. We first show that $s = 2^m$ for some $m \geqslant 1$. Since $n - r > 2s$, vertex $s+r$ is adjacent to vertex $2s+r$. Hence $2s+r$ is in Y.

It is not difficult to see from Fig. 12 that if vertex $s+r$ is adjacent to vertex $s+r+i$, for $i=2(1)(s-2)$ then $(s+r, s+r+i, r+i)$ is a triangle in G. But G has no triangles and so $a_{s+r,s+r+i}=0$, for $i=0(1)(s-1)$. Hence we have by the Steinhaus property that $a_{s+r-1,s+r+i}=1$ for $i=0(1)(s-1)$.

```
0   r      s               2s + r
0001 · · · 100000000000
 001        100            0
  0111111110               0
   000000001               0
     0        011          0
      0       01 1         0
       0      01   1       0
        0     01     1     0
         0    01       1   0
              001        1 0
               01         10
row s + r       0   · · ·  1
```

Fig. 12

In fact, in $A(G)$ rows r to $s+r-1$ and columns $s+r$ to $2s+r-1$ are the first s rows of Pascal's triangle modulo two. Hence, row r of $A(G)$ corresponds to row 0 of Pascal's triangle and row $s+r-1$ of $A(G)$ corresponds to row $s-1$ of Pascal's triangle. But the only rows of Pascal's triangle that are all odd are the rows of binomial coefficients $C(2^m-1, j)$. Therefore $s-1=2^m-1$ or $s=2^m$ for some $m\geqslant 1$. Thus $A(G)$ is mainly copies of the first s rows of Pascal's triangle as shown in Fig. 13. It is easy to see that X and Y are as depicted in Fig. 14.

Vertex $n-1$ is either in X or Y. If $n-1$ is in X, then $q=[(n-r-1)/s]$ is odd. There are two possibilities. First, if $n-1\geqslant(q+1)s$, then $|X|>|Y|$ and G is not a counterexample. Second, if $n-1<(q+1)s$, then for a in X,

$$(a, a+s), \quad 2js\leqslant a\leqslant 2js+(r-1), \text{ for } 0\leqslant j\leqslant(q-1)/2,$$

and

$$(a, a-s), \quad (2j-1)s+r\leqslant a<2js, \text{ for } 1\leqslant j\leqslant(q+1)/2$$

is an X-saturated matching for G.

If $n-1$ is in Y, then $q=[(n-r-1)/s]$ is even. Again there are two possibilities. First, if $(qs+2r)>n$, then $|X|>|Y|$ and G cannot be a counterexample. Second, if $(qs+2r)\leqslant n$ then a partial matching for G is, for a in X,

$$(a, a+s), \quad 2js\leqslant a<2js+(r-1), \qquad 0\leqslant j\leqslant(q-2)/2,$$

and

$$(a, a-s), \quad (2j-1)s+r\leqslant a<2js, \quad 1\leqslant j\leqslant q/2.$$

To complete this matching we must find a matching of $\{qs, \ldots, qs+(r-1)\}$ into

```
0  r   s                  2s + r    3s + r    4s + r
00011001100 00000000 00000000 00000000 000 000
 001010101            0        0        0
  011111111           0        0        0
   00000000|1         0        0        0
    0     0|11        0        0        0
     0    0|101       0        0        0
      0   0|1111      0        0        0
       0  0|10001     0        0        0
        0 0|110011    0        0        0
         00|1010101   0        0        0
          0|11111111  0        0        0
row s + r    0        |1       0        0
              0       |11      0        0
               0      |101     0        0
                0     |1111    0        0
                 0    |10001   0        0
                  0   |110011  0        0
                   0  |1010101 0        0
                    0 |11111111 0       0
row 2s + r            0        |1       0
                       0       |11      0
                        0      |101     0
                         0     |1111    0
                          0    |10001   0
                           0   |110011  0
                            0  |1010101 0
                             0 |11111111 0
row 3s + r                     0        |1
                                0       |11
                                 0      |101
                                  0     |111 1
                                   0    |100 01
row 4s                              0   |110|011
                                     0  |101|010
                                      0 |111|111
row 4s + r                               000 000
                                          00 000
                                           0 000
                                             000
                                              00
                                               0
```

Fig. 13. $r = 3$, $s = 8$.

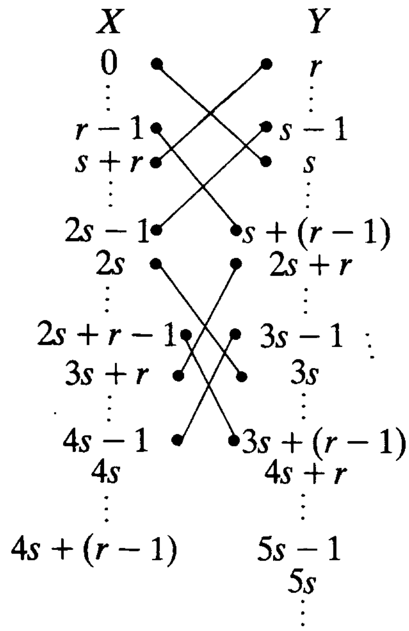

Fig. 14

$\{qs + r, \ldots, n - 1\}$. Columns $qs + r$ to $qs + 2r - 1$ and rows qs to $qs + (r - 1)$, (outlined in Fig. 13), are an inverted form of Pascal's triangle modulo two, i.e. leftmost bottom entry corresponds to $C(0, 0)$ and the row and column of ones are $C(b, 0)$ and $C(b, b)$. From this $r \times r$ square we must pick exactly one '1' from each row and column. This is easy for $r = 3$. For $r > 3$, choose the highest diagonal of ones in the upper part of the square. The remaining rows and columns form another of these squares of smaller order and hence we can choose exactly one '1' from each remaining row and column. (See Fig. 15.)

$$\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

—diagonal of ones

remaining square

Fig. 15

Thus, we have completed the matching for G. Finally, if $r = s$, then $A(G)$ is as depicted in Fig. 13 if the first three rows are ignored. For a in X, $(a, a + s)$, $2js \leqslant a < (2j + 1)s$ in an X-saturated matching where $2j \leqslant [(n - 1)/s]$.

In each case we have shown that all bipartite Steinhaus graphs on n vertices have an X-saturated matching. Hence the theorem is true. □

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