Embedding of the vertices of the Auslander–Reiten quiver of an iterated tilted algebra of Dynkin type \( \Delta \) in \( \mathbb{Z}\Delta \)

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Abstract

Let \( \Delta \) be a Dynkin diagram and \( k \) an algebraically closed field. Let \( A \) be an iterated tilted finite-dimensional \( k \)-algebra of type \( \Delta \) and denote by \( \hat{A} \) its repetitive algebra. We approach the problem of finding a combinatorial algorithm giving the embedding of the vertices of the Auslander–Reiten quiver \( \Gamma_A \) of \( A \) in the Auslander–Reiten quiver \( \Gamma(\text{mod}(\hat{A})) \cong \mathbb{Z}\Delta \) of the stable category \( \text{mod}(\hat{A}) \).

Let \( T \) be a trivial extension of finite representation type and Cartan class \( \Delta \). Assume that we know the vertices of \( \mathbb{Z}\Delta \) corresponding to the radicals of the indecomposable projective \( T \)-modules. We determine the embedding of \( \Gamma_A \) in \( \mathbb{Z}\Delta \) for any algebra \( A \) such that \( T(A) \cong T \).

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Introduction

The algebras to be considered in this paper are basic finite-dimensional algebras over an algebraically closed field \( k \). Any such algebra \( A \) can be written as a bound quiver algebra \( kQ_A/I \), where \( I \) is an admissible ideal of the path algebra \( kQ_A \) and \( Q_A \) is the quiver associated to \( A \).

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For a quiver $Q$, let $Q_0$ denote the set of vertices of $Q$ and $Q_1$ the set of arrows of $Q$. An arrow $a$ of $Q_1$ starts at the vertex $o(a)$ and ends at $e(a)$.

Let $k$ be an algebraically closed field, $\Delta$ a Dynkin diagram and let $A$ be an iterated tilted algebra of type $\Delta$ [1]. Let $T(A) = A \ltimes D_A(A)$ be the trivial extension of $A$ by its minimal injective cogenerator $D_A(A) = \text{Hom}_k(A, k)$. The algebra $T(A)$ is known to be of finite representation type [3] and there exists an embedding of $\text{mod}\ A$ in the stable category $\text{mod}\ T(A)$. Then the set of vertices $(\Gamma_A)_0$ of the AR-quiver $\Gamma_A$ of $A$ can be embedded in the stable part $S\Gamma_{T(A)}$ of the AR-quiver $\Gamma_{T(A)}$ of $T(A)$. Moreover, $T(A)$ admits universal Galois covering $\hat{A} \rightarrow T(A)$, where $\hat{A}$ is the repetitive algebra of $A$, $S\Gamma_{\hat{A}} \simeq \mathbb{Z}\Delta$ and thus $\Gamma_A$ can be embedded in $\mathbb{Z}\Delta$ [1,7,8,11]. This is, the vertices of the AR-quiver $\Gamma_A$ of any iterated tilted algebra $A$ of type $\Delta$ can be embedded in $\mathbb{Z}\Delta$, and in such way that knowing which vertices of $\mathbb{Z}\Delta$ correspond to $A$-modules we can obtain the arrows of $\Gamma_A$ in a canonical way, so that we get the AR-quiver $\Gamma_A$ of $A$. Taking this into account and for simplicity we will just say that the AR-quiver $\Gamma_A$ embeds in $\mathbb{Z}\Delta$ to mean that there is an injective map $\varphi : (\Gamma_A)_0 \rightarrow (\mathbb{Z}\Delta)_0$. Our main objective is to describe this embedding explicitly. We recall that the trivial extensions of finite representation type and Cartan class $\Delta$ are precisely the trivial extensions of iterated tilted algebras of Dynkin type $\Delta$ [3]. We divided the problem in two parts.

Let $T$ be a trivial extension of finite representation type and Cartan class $\Delta$.

1. Assume that we know the vertices of $\mathbb{Z}\Delta$ corresponding to the radicals of the indecomposable projective $T$-modules. Determine the embedding of $\Gamma_A$ in $\mathbb{Z}\Delta$ for any algebra $A$ such that $T(A) \simeq T$.

2. Describe an algorithm to determine which subsets of vertices in $\mathbb{Z}\Delta$ represent the radicals of the indecomposable projective modules over the trivial extension $T$.

In this paper we solve the first part. The second is studied in the first author’s Ph.D. thesis [15] where an algorithm is given for $\Delta = A_n$ and $\Delta = D_n$, and will be published in a forthcoming paper.

We describe the embedding more explicitly. Let $A$ be an iterated tilted algebra of type $\Delta$ and let $T(A) = A \ltimes D_A(A)$ be the trivial extension of $A$ by $D_A(A) = \text{Hom}_k(A, k)$. The canonical epimorphism $p : T(A) \twoheadrightarrow A$ given by $p(a, \varphi) = a$ induces a full and faithful functor

$$F_p : \text{mod}\ A \hookrightarrow \text{mod}\ T(A),$$

which identifies $\text{mod}\ A$ with the full subcategory of $\text{mod}\ T(A)$ whose objects are the $T(A)$-modules annihilated by $D_A(A)$. Moreover, the composition of $F_p$ with the canonical functor $\theta : \text{mod}\ T(A) \hookrightarrow \text{mod}\ T(A)$ is also a full and faithful functor

$$\theta F_p : \text{mod}\ A \hookrightarrow \text{mod}\ T(A).$$

Therefore the AR-quiver $\Gamma_A$ of $A$ can be embedded in the AR-quiver $\Gamma_{T(A)}$ of $T(A)$ and in the stable AR-quiver $S\Gamma_{T(A)}$ making the following diagram commutative.
It is known (see 2.6 in [8]) that there exists a translation quiver morphism \( \pi : \hat{S}_\Gamma \rightarrow S_\Gamma T(A) \), which is the universal covering of \( S_\Gamma T(A) \), and that \( S_\Gamma \hat{A} \simeq \mathbb{Z} \Delta \).

Then we can consider a connected lifting \( S_\Gamma T(A)[0] \) of the quiver \( S_\Gamma T(A) \) to \( \mathbb{Z} \Delta \) (see Section 3). Since the quiver \( \Gamma \hat{A} \) is embedded in \( S_\Gamma T(A) \) the above lifting induces a subquiver \( \Gamma \hat{A}[0] \) of \( S_\Gamma T(A)[0] \) in such way that the following diagram is commutative

\[
\begin{array}{ccc}
\Gamma \hat{A}[0] & \xrightarrow{\pi} & \hat{S}_\Gamma \\
\downarrow{\pi} & & \downarrow{\pi} \\
\Gamma \hat{A} & \xrightarrow{\pi} & S_\Gamma T(A)
\end{array}
\]

We get an embedding of \( \Gamma \hat{A} \) in \( \mathbb{Z} \Delta \) and we are looking for the vertices of \( \mathbb{Z} \Delta \) corresponding to indecomposable \( A \)-modules under such embedding.

We start by studying the embedding \( \Gamma \hat{A} \hookrightarrow \Gamma_\hat{T}(A) \) induced by the canonical epimorphism \( p : T(A) \rightarrow A \). Thus, we have to determine which vertices of \( \Gamma_\hat{T}(A) \) correspond to indecomposable \( A \)-modules. We know that \( A \simeq T(A)/D_A(A) \), and that a \( T(A) \)-module \( M \) is an \( A \)-module if and only if \( D_A(A)M = 0 \). Therefore we have to know what the condition \( D_A(A)M = 0 \) means in the Auslander–Reiten quiver \( \Gamma_\hat{T}(A) \). Let \( A = kQ_A/I \), in [9,10] the quiver of \( QT(A) \) is obtained from \( QA \) by adding some arrows. Moreover, the ideal \( D_A(A) \) of \( T(A) \) is generated precisely by these added arrows [9]. On the other hand, given a trivial extension \( T \) of finite representation type a method is given in [9] to obtain the iterated tilted algebras \( B \) such that \( T(B) \simeq T \). In fact, such algebras are obtained by deleting exactly one arrow in each nonzero oriented cycle of \( QT \) and considering the induced relations. Thus \( B \) is the factor of \( T \) by an ideal generated by arrows.

First we will study when an ideal generated by arrows annihilates a module \( M \). In Section 2 we give a characterization of modules \( M \) over a quotient \( k \)-algebra \( A/J \) where \( J \) is an ideal of \( A \) generated by arrows of \( QA \). In particular, when \( A = T(A) \) and \( J = D(A) \) we describe the vertices of \( \Gamma_\hat{T}(A) \) corresponding to \( A \simeq T(A)/J \)-modules.

More precisely, suppose that \( J \) is generated by some arrows \( a_1, a_2, \ldots, a_t \) of \( QT(A) \). We consider the subquiver \( P_{a_1a_2\ldots a_t} \) of \( \Gamma_\hat{T}(A) \) induced by the nonzero paths in \( \Gamma_\hat{T}(A) \) starting at the projective \( P_{\alpha(a_i)} \) and ending at the projective \( P_{\beta(a_i)} \) for some \( i = 1, 2, \ldots, t \). We
prove that the vertices of $\Gamma_A$ are exactly the vertices of $\Gamma_{T(A)}$ which are not contained in $\mathcal{P}_{\alpha_1, \alpha_2, \ldots, \alpha_t}$. A similar description is given in Section 3 for the embedding of $\Gamma_A$ in $\hat{\Gamma}_A$.

To do that, we define an appropriate lifting of $\Gamma_{T(A)}$ to $\hat{\Gamma}_A$, and we study how nonzero paths between projective modules in $\Gamma_{T(A)}$ lift to $\hat{\Gamma}_A$. In this way we obtain the embedding $\Gamma_A \hookrightarrow \hat{\Gamma}_A$, and then the desired embedding $\Gamma_A \hookrightarrow \mathbb{Z}\Delta \simeq S\hat{\Gamma}_A$.

1. Preliminaries

Let $Q$ be a quiver, which may be infinite. A path $\gamma$ in the quiver $Q$ is either an oriented sequence of arrows $\alpha_n \cdots \alpha_1$ with $o(\alpha_i) = o(\alpha_{i+1})$ for $1 \leq i < n$, or the symbol $e_i$ for $i \in Q_0$. The length $\ell(\gamma)$ of $\gamma$ is $n$ in the first case, and $\ell(e_i) = 0$. We call the paths $e_i$ trivial paths and we define $o(e_i) = e(e_i)$. Let $I$ be an ideal of the path algebra $kQ$. We consider $\Lambda = kQ/I$ as a $k$-category whose objects are the vertices $Q_0$ of $Q$ and the morphism space $\Lambda(i,j)$ from $i$ to $j$ is $e_j\Lambda e_i$, where $e_i = e_i + I$ (see [5]).

Let $A$ be a $k$-algebra. For a given vertex $j$ of $Q_A$ we denote by $S_j$ the simple $A$-module corresponding to $j$, by $P_j$ the projective cover of $S_j$, and by $I_j$ the injective envelope of $S_j$. We will use freely properties of the module category $\text{mod} A$ of finitely generated left $A$-modules, the stable category $\text{mod} A_{\text{module projectives}}$, the Auslander–Reiten quiver $\Gamma_A$ and the Auslander–Reiten translations $\tau = D\text{Tr}$ and $\tau^{-1} = \text{Tr}D$, as can be found in [4]. We denote by $\text{ind} A$ (respectively by $\text{ind} A_{\text{stable}}$) the full subcategory of $\text{mod} A$ (resp. $\text{mod} A_{\text{module projectives}}$) formed by chosen representatives of the indecomposable modules. Moreover, we will frequently identify the objects of $\text{ind} A$ with the vertices of the AR-quiver $\Gamma_A$ representing such objects.

We will freely use the notions of locally finite $k$-category, translation quiver, covering functor, well behaved functor and related notions. We refer the reader to [4,5,11,17,18] for definitions and basic properties of these objects.

Let $\Delta$ be an oriented tree. Following Chr. Riedtmann [17] (see also [4]) we will consider the translation quiver $\mathbb{Z}\Delta$, defined as follows:

$$ (\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0, \quad (\mathbb{Z}\Delta)_1 = [-1, 1] \times \mathbb{Z} \times \Delta_1. $$

For an arrow $x \xrightarrow{\alpha} y$ of $\Delta$ we define the arrows $(-1, n, \alpha)$ and $(1, n, \alpha)$ as

$$(n - 1, y) \xrightarrow{(-1, n, \alpha)} (n, x) \quad \text{and} \quad (n, x) \xrightarrow{(1, n, \alpha)} (n, y).$$

Finally, the translation $\tau$ is $\tau(n, y) = (n - 1, y)$.

2. Modules over quotients of quasi-schurian weakly symmetric algebras

We start this section by giving a characterization of modules $M$ over a quotient $k$-algebra $A/J$ where $J$ is an ideal of $A$ generated by arrows of $Q_A$. Then we go on to study the case when $A$ is quasi-schurian and weakly symmetric. Finally, we give an application to trivial extensions of finite representation type.
We recall from [14] that an algebra \( \Lambda \) is quasi-schurian if it satisfies:

(a) \( \dim_k \text{Hom}_\Lambda(P, Q) \leq 1 \) if \( P \) and \( Q \) are non isomorphic indecomposable projective \( \Lambda \)-modules and

(b) \( \dim_k \text{End}_\Lambda(P) = 2 \) for any indecomposable projective \( \Lambda \)-module \( P \).

Let \( A = kQ_A/I \) be a schurian (that is, \( \dim_k \text{Hom}_A(P_i, P_j) \leq 1 \) for any vertices \( i \) and \( j \) of \( Q_A \)) and triangular (that is, \( Q_A \) has non oriented cycles) \( k \)-algebra, with \( I \) admissible ideal. Then the trivial extension \( T(A) \) of \( A \) is a quasi-schurian algebra.

As a consequence we get that the trivial extensions of finite representation type are quasi-schurian. This follows from the fact, proved by K. Yamagata in [20], that the trivial extension of a non triangular algebra is of infinite representation type.

Since we want to describe the \( \Lambda \)-modules \( M \) annihilated by a finite number of arrows of \( Q_\Lambda \), we start by studying when \( \alpha M \neq 0 \) for a given arrow \( \alpha \).

**Lemma 2.1.** Let \( \Lambda = kQ_A/I \) be a \( k \)-algebra with \( I \) an admissible ideal. Let \( \alpha : i \to j \) be an arrow in \( Q_\Lambda \) and \( M \in \text{mod} \Lambda \).

The following conditions are equivalent:

(a) \( \alpha M \neq 0 \).

(b) \( \text{Hom}_\Lambda(\rho_\alpha, M) : \text{Hom}_\Lambda(P_i, M) \to \text{Hom}_\Lambda(P_j, M) \) is nonzero, where \( \rho_\alpha : P_j \to P_i \) is the right multiplication by \( \alpha \).

**Proof.** The proof is straightforward. \( \square \)

**Lemma 2.2.** Let \( \Lambda = kQ_A/I \) be a \( k \)-algebra with \( I \) an admissible ideal. Let \( \alpha : i \to j \) be an arrow in \( Q_\Lambda \) and \( M \in \text{mod} \Lambda \). Then

(a) If \( \alpha M \neq 0 \) then there are morphisms \( f : P_i \to M, g : M \to I_j \) such that \( gf \neq 0 \).

(b) Assume that \( \text{Hom}_\Lambda(\rho_\alpha, I_j) : \text{Hom}_\Lambda(P_i, I_j) \to \text{Hom}_\Lambda(P_j, I_j) \) is a monomorphism, where \( \rho_\alpha : P_j \to P_i \) is the right multiplication by \( \alpha \). If there are morphisms \( f : P_i \to M, g : M \to I_j \) with \( gf \neq 0 \), then \( \alpha M \neq 0 \).

**Proof.** (a) From Lemma 2.1 we know that there is a nonzero morphism \( f : P_i \to M \) such that \( f \rho_\alpha : P_j \to M \) is nonzero. Then there is \( g : M \to I_j \) such that \( gf \rho_\alpha \neq 0 \), and consequently \( gf \neq 0 \).

(b) Assume that \( \text{Hom}_\Lambda(\rho_\alpha, I_j) \) is a monomorphism and let \( f : P_i \to M, g : M \to I_j \) such that \( gf \neq 0 \). Then \( 0 \neq \text{Hom}_\Lambda(\rho_\alpha, I_j)(g) = (gf)\rho_\alpha = g(f \rho_\alpha) \), proving that \( f \rho_\alpha \neq 0 \). Thus \( \text{Hom}_\Lambda(\rho_\alpha, M)(f) \neq 0 \) and by Lemma 2.1 we get that \( \alpha M \neq 0 \). \( \square \)

In case \( \Lambda \) is a quasi-schurian weakly symmetric algebra we obtain the following theorem.
Theorem 2.3. Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric $k$-algebra with $I$ an admissible ideal. Let $\alpha : i \to j$ be an arrow in $Q_\Lambda$. Then the following conditions are equivalent for an indecomposable $\Lambda$-module $M$:

(a) $\alpha M \neq 0$.
(b) There are morphisms $P_i \xrightarrow{f} M, M \xrightarrow{g} P_j$ with $gf \neq 0$.

Proof. (a) $\Rightarrow$ (b) Since $\Lambda$ is weakly-symmetric then $P_j = I_j$ for any vertex $j \in Q_\Lambda$. So Lemma 2.2(a) proves the result in this case.

(b) $\Rightarrow$ (a) Assume that $i \neq j$. Using Lemma 2.2(b) we only need to prove that 

$$\text{Hom}_\Lambda(\rho_\alpha, P_j) : \text{Hom}_\Lambda(P_i, P_j) \to \text{Hom}_\Lambda(P_j, P_j)$$

is nonzero. Since $\Lambda$ is quasi-schurian and weakly-symmetric it is not hard to prove that there exists a path $\delta$ starting at $j$, ending at $i$ and such that $\delta \alpha$ is nonzero (see in [14, 2.2 and 3]). In particular, from [14, Theorem 3, IV] we obtain that $\alpha \delta$ is nonzero. Thus $\text{Hom}_\Lambda(\rho_\alpha, P_j)$ is nonzero.

If $i = j$ then $\alpha$ is a loop. Now, the only (up to isomorphisms) indecomposable quasi-schurian and weakly-symmetric $k$-algebra with loops is $\Lambda \simeq k[x]/(x^2)$ (see [14, Lemma 14]). Assume that $\varepsilon(\alpha) = o(\alpha) = 1$. Then the projective $P_1$ and the simple $S_1$ are the unique (up to isomorphism) indecomposable $\Lambda$-modules.

Suppose that $M = P_1$. Then $\mathfrak{S} P_1 \neq 0$ and the morphisms $f = \rho_\alpha$ and $g = 1_{P_1}$ satisfy (b).

Let $M = S_1$, then $\mathfrak{S} S_1 = 0$. On the other hand, since $\text{rad}^2(P_1, P_1) = 0$ we get that $gf = 0$ for any $f : P_1 \to S_1$ and $g : S_1 \to P_1$.

Corollary 2.4. Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric $k$-algebra with $I$ an admissible ideal. Let $\alpha_i : a_i \to b_i$ be arrows in $Q_\Lambda$ for $i = 1, 2, \ldots, t$. Then the following conditions are equivalent for an indecomposable $\Lambda$-module $M$:

(a) $M$ is a $A/(\mathfrak{a}_1, \ldots, \mathfrak{a}_t)$-module.
(b) If $f : P_{a_i} \to M, g : M \to P_{b_i}$ are morphisms in $\text{mod} \Lambda$, then their composition $gf$ is zero for all $i = 1, 2, \ldots, t$.

Proof. Follows easily from the preceding theorem.

Corollary 2.5. Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric $k$-algebra of finite representation type, with $I$ an admissible ideal. Let $\alpha_i : a_i \to b_i$ be arrows in $Q_\Lambda$ for $i = 1, 2, \ldots, t$. Then the following conditions are equivalent for an indecomposable $\Lambda$-module $M$:
(a) $M$ is a $\Lambda/\langle \alpha_1, \ldots, \alpha_t \rangle$-module.
(b) Any chain of irreducible maps in $\text{ind } \Lambda$

$$X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_j = M \xrightarrow{f_{j+1}} X_{j+1} \rightarrow \cdots \xrightarrow{f_r} X_r$$

with $X_0 = P_{a_i}$, $X_r = P_{b_i}$ has zero composition for all $i = 1, 2, \ldots, t$.

**Proof.** Follows from the above corollary using that if $\Lambda$ is of finite representation type, then each nonzero morphism between indecomposable modules can be written as a sum of compositions of irreducible morphisms between indecomposable modules [4].

Let $\Lambda$ be a $k$-algebra as in the preceding corollary, and let $A = \Lambda/\mathcal{J}$ where $\mathcal{J}$ is the ideal of $\Lambda$ generated by some arrows $\alpha_1, \alpha_2, \ldots, \alpha_t$ of $Q_\Lambda$. We denote by $P_{\alpha_1, \alpha_2, \ldots, \alpha_t}$ the subquiver of $\Gamma_\Lambda$ induced by the nonzero paths in $k(\Gamma_\Lambda)$ starting at the projective $P_{\alpha_i}$ and ending at the projective $P_{\alpha_i}$. Then by Corollary 2.5 we have that the vertices of $\Gamma_A$ can be identified with the vertices of $\Gamma_\Lambda$ which are not in $P_{\alpha_1, \alpha_2, \ldots, \alpha_t}$. That is, $(\Gamma_A)_0 = (\Gamma_\Lambda)_0 \setminus (P_{\alpha_1, \alpha_2, \ldots, \alpha_t})_0$.

Let $A = kQ_A/I$ be an iterated tilted $k$-algebra of Dynkin type, with $I$ an admissible ideal and let $T(A)$ be the trivial extension of $A$. Then $\Lambda = T(A)$ satisfies the hypothesis of Corollary 2.5. This is the case because the trivial extension of an iterated tilted algebra of Dynkin type is of finite representation type (see [3]) and, as we have seen at the beginning of this section, $T(A)$ is quasi-schurian.

**Remark 2.6.** Let $T = kQ_T/I_T$ be a trivial extension of finite representation type and let $A$ be an iterated tilted $k$-algebra of Dynkin type such that $T \simeq T(A)$. As we observed in the introduction, $A$ is obtained by deleting exactly one arrow in each nonzero cycle of $Q_T$, and considering the induced relations. So we have that $A = T/\langle \alpha_1, \ldots, \alpha_t \rangle$ where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are arrows in $Q_T$. Suppose that we know which vertices of the AR-quiver $\Gamma_T$ correspond to the projective $T$-modules $P_j$ associated with each vertex $j$ of $Q_T$. As we observed above, the vertices of $\Gamma_A$ can be identified with the vertices of $\Gamma_T$ which are not in $P_{\alpha_1, \alpha_2, \ldots, \alpha_t}$.

Therefore the embedding $\Gamma_A \hookrightarrow \Gamma_T$ is determined by the position in $\Gamma_T$ of the vertices corresponding to the projective $T$-modules $P_j$ for $j \in (Q_T)_0$.

**Example.** Let $A$ be the iterated tilted algebra of type $D_4$ with ordinary quiver $Q_A$, and with relation $0 = \alpha \delta - \varepsilon \eta$, where
By [10] the ordinary quiver $Q_{T(A)}$ of the trivial extension $T(A)$ of $A$ is

$$
\begin{array}{c}
1 \\
\alpha & \beta \\
\alpha & \beta \\
\end{array}
$$

and the ideal $I$ such that $T(A) = kQ_{T(A)}/I$ is generated by the relations: $\alpha \delta - \varepsilon \eta$, $\delta \beta \varepsilon$, $\eta \beta \alpha$, $\beta \alpha \delta \beta$, $a \delta \beta \alpha$, $\varepsilon \eta \beta \varepsilon$. In this case we have $A = T(A)/(\beta)$. Hence we have to look for the nonzero paths in $\Gamma_{T(A)}$ from $P_{\alpha(\beta)} = P_2$ to $P_{\varepsilon(\beta)} = P_3$. The shaded region of Fig. 1 corresponds to $P_\beta$.

Then we delete from the quiver $\Gamma_{T(A)}$ the modules which are in $\mathcal{P}_\beta$. In Fig. 2 we indicate with $\square$ the vertices of $\Gamma_{T(A)}$ corresponding to $A$-modules.
Then the embedding $\Gamma_A \hookrightarrow S\Gamma_T(A)$ is described in Fig. 3, where we indicate with $\square$ the vertices of $S\Gamma_T(A)$ corresponding to $A$-modules.

The other iterated tilted algebras $B$ such that $T(B) \cong T(A)$ are of the form $T(A)/\langle \alpha, \epsilon \rangle$, $T(A)/\langle \alpha, \eta \rangle$, $T(A)/\langle \delta, \epsilon \rangle$, and $T(A)/\langle \delta, \eta \rangle$. The embedding of $\Gamma_B$ in $S\Gamma_T(A)$ for these algebras $B$ is obtained in the same way.

The embedding $\Gamma_A \hookrightarrow S\Gamma_T(A)$ is reduced to the embedding $\Gamma_A \hookrightarrow \Gamma_T(A)$, since the stable part $S\Gamma_T(A)$ of $\Gamma_T(A)$ is obtained from $\Gamma_T(A)$ by deleting the vertices of $\Gamma_T(A)$ associated to projective modules. In general, we have information about the stable quiver $S\Gamma_T(A)$. Indeed, suppose that the trivial extension $\Lambda = T(A)$ of $A$ is of Cartan class $\Delta$, where $\Delta$ is a Dynkin diagram. Then $S\Gamma_A \cong \mathbb{Z}\Delta/\Pi(S\Gamma_A, x)$ where $\Pi(S\Gamma_A, x)$ is the fundamental group associated to the universal covering $\pi: \mathbb{Z}\Delta \rightarrow S\Gamma_A$ of the stable translation quiver $S\Gamma_A$ (see [17]). Moreover, the group $\Pi(S\Gamma_A, x)$ is generated by $\tau^m_\Delta$, where $m_\Delta$ is the Loewy length of the mesh category $k(\mathbb{Z}\Delta)$ [2,6]. We recall that the values of $m_\Delta$ are: $m_{A_n} = n$, $m_{D_n} = 2n - 3$, $m_{E_6} = 11$, $m_{E_7} = 17$, $m_{E_8} = 29$.

In this way we have information about the structure of the stable quiver $S\Gamma_A$. Our problem now is to recover the structure of $\Gamma_A$ from the knowledge we have about $S\Gamma_A$. To do that, we need to know which vertices of $S\Gamma_A$ correspond to the radicals of the projective modules $P_i$ for $i \in (Q_\Lambda)_0$, since $0 \rightarrow rP_i \rightarrow P_i \sqcup rP_i/\soc P_i \rightarrow P_i/\soc P_i \rightarrow 0$ is an AR-sequence for each vertex $i$ of $Q_\Lambda$. We denote by $\mathcal{C}_A$ the set of vertices of $S\Gamma_A$ representing the radicals of the projective $A$-modules. It is well known that $\mathcal{C}_A$ is a configuration of $S\Gamma_A$, as defined by Chr. Riedtmann in [18]. This is, the elements of $\mathcal{C}_A$ satisfy the following definition.

**Definition 2.7.** [18]. Let $\Gamma$ be a stable translation quiver and $k(\Gamma)$ the mesh-category associated to $\Gamma$. A configuration $\mathcal{C}$ of $\Gamma$ is a set of vertices of $\Gamma$ which satisfies the following conditions:

(a) For any vertex $x \in \Gamma_0$ there exists a vertex $y \in \mathcal{C}$ such that $k(\Gamma)(x, y) \neq 0$,
(b) $k(\Gamma)(x, y) = 0$ if $x$ and $y$ are different elements of $\mathcal{C}$,
(c) $k(\Gamma)(x, x) = k$ for all $x \in \mathcal{C}$.

Let $\Delta$ be a Dynkin diagram, $\Lambda$ a trivial extension of Cartan class $\Delta$, and $\pi: \mathbb{Z}\Delta \rightarrow S\Gamma_A$ the universal covering of $S\Gamma_A$. Since $\mathcal{C}_A$ is a configuration of $S\Gamma_A$, we obtain from [18] that $\tilde{\mathcal{C}}_A = \pi^{-1}(\mathcal{C}_A)$ is a configuration of $\mathbb{Z}\Delta$. We will say that $\tilde{\mathcal{C}}_A$ is the configuration of $\mathbb{Z}\Delta$ associated to $\Lambda$. 

![Fig. 3.](image)
3. The lifting process

Throughout this section $\Delta$ denotes a Dynkin diagram. Let $A$ be an iterated tilted $k$-algebra of type $\Delta$ and let $T(A)$ be the trivial extension of $A$. In the preceding section we described an embedding of $\Gamma_A$ into $\hat{\Gamma}_T(A)$ which we will lift to an embedding of $\hat{\Gamma}_A$ in $\mathbb{Z}\Delta = \hat{\Gamma}_T(A)$. Our purpose now is describing directly this embedding in terms of a section in $\mathbb{Z}\Delta$ and some nonzero paths in $\hat{\Gamma}_A$ between projective modules lift to $\hat{\Gamma}_A$. The lifting process was described in the preceding section for the embedding of $\Gamma_A$ into $\hat{\Gamma}_{T(A)}$. So, we will define a connected lifting $\hat{\Gamma}_{T(A)}[0]$ of $\hat{\Gamma}_{T(A)}$ to $\mathbb{Z}\Delta$ and extend it to a connected lifting $I_{T(A)}[0]$ of $\Gamma_{T(A)}$ to $\hat{\Gamma}_A$. Afterwards we will study how nonzero paths in $\hat{\Gamma}_{T(A)}$ between projective modules lift to $\hat{\Gamma}_A$. Since there are infinitely many $\hat{\Gamma}_A$-projectives and we want to circumscribe to a small part of $\mathbb{Z}\Delta$, we need to study how long the nonzero paths between the projective modules in $\hat{\Gamma}_A$ are. So we start with some preliminaries.

Following [6,12] we denote the Nakayama-permutation on $\mathbb{Z}\Delta$ by $v_\Delta$. This is the bijection $v_\Delta : (\mathbb{Z}\Delta)_0 \rightarrow (\mathbb{Z}\Delta)_0$ which satisfies the following condition: for each vertex $x$ of $\mathbb{Z}\Delta$ there exists a path $w : x \rightarrow v_\Delta(x)$ whose image $\overline{w}$ in the mesh-category $k(\mathbb{Z}\Delta)$ is not zero, and $w$ has longest length among all nonzero paths starting at $x$. The Loewy length $m_\Delta$ of the mesh-category $k(\mathbb{Z}\Delta)$ is the smallest integer $m$ such that $\overline{w} = 0$ in $k(\mathbb{Z}\Delta)$ for all paths $w$ in $\mathbb{Z}\Delta$ whose length is greater than or equal to $m$. Thus $m_\Delta - 1$ is the common length of all nonzero paths from $x$ to $v_\Delta(x)$. Moreover, we have that $\tau - m_\Delta = v_\Delta^{m_\Delta-1}$.

Let $(\Gamma, \tau)$ be a connected stable translation quiver. Following P. Gabriel in [12] we will call slice of $\Gamma$ to a full connected subquiver whose vertices are determined by choosing a unique element in each $\tau$-orbit of $\Gamma_0$. Then for each vertex $x \in \Gamma$ there is a well-determined slice admitting $x$ as its unique source. We call it slice starting at $x$ and denote it by $S_x\rightarrow$. Likewise, the slice ending at $x$ admits $x$ as its unique sink and is denoted by $S_x\leftarrow$.

Let $f : (\mathbb{Z}\Delta)_0 \rightarrow \mathbb{Z}$. We recall that $f$ is additive if it satisfies the equation

$$f(x) + f(\tau(x)) = \sum_{z \in S_x\rightarrow} f(z)$$

for each vertex $x$. It is well known that the additive function $f_\tau$, which has value 1 on $S_x\rightarrow$, determines the support of the functor $k(\mathbb{Z}\Delta)(x, -)$. In fact, $\dim_k k(\mathbb{Z}\Delta)(x, y) = f_\tau(y)$.

**Proposition 3.1.** Let $x$ be a vertex of $\mathbb{Z}\Delta$. Then

(a) $\text{Supp} k(\mathbb{Z}\Delta)(x, -) = \text{Supp} k(\mathbb{Z}\Delta)(-, v_\Delta(x))$.

(b) $\text{Supp} k(\mathbb{Z}\Delta)(x, -) \cap \text{Supp} k(\mathbb{Z}\Delta)(-, v_\Delta^2(x)) = \{v_\Delta(x)\}$.

**Proof.** (a) The proof given by Chr. Riedtmann for the $D_n$ case in [19, page 312] can be adapted to the other Dynkin diagrams.

(b) Follows from (a) and the fact that $\mathbb{Z}\Delta$ has no oriented cycles. □

Let $x$ be a vertex of $\mathbb{Z}\Delta$. Using (a) of the preceding proposition we obtain that the support of the functor $k(\mathbb{Z}\Delta)(x, -)$ is contained in the set of vertices of $\mathbb{Z}\Delta$ laying on or
between the sections $S_{x,y}$ and $S_{x,\nu\Delta(x)}$. Though this inclusion is not in general an equality it is so in the case $\Delta = A_n$.

**Remark 3.2.** Let $\Lambda$ be a trivial extension of Cartan class $\Delta$, and let $F : k(\mathbb{Z}\Delta) \to \text{ind} \Lambda$ be a well-behaved functor induced by the universal covering $\pi : \mathbb{Z}\Delta \to S\Gamma_A$. Since $F$ is a covering functor, then it induces a $k$-vector space isomorphism

$$\bigoplus_{y \in \pi^{-1}(Y)} k(\mathbb{Z}\Delta)(x, y) \sim \text{Hom}_A(\pi(x), Y).$$

Since $\Delta$ is of Dynkin type we can say more: if $\text{Hom}_A(\pi(x), Y) \neq 0$, then the left side has a unique nonzero summand. Dually, if $\text{Hom}_A(X, \pi(y)) \neq 0$ there exists a unique $x \in \pi^{-1}(X)$ such that $k(\mathbb{Z}\Delta)(x, y) \neq 0$.

In fact, we assume that $k(\mathbb{Z}\Delta)(x, y_i) \neq 0$ for $i = 1, 2$ and $\pi(y_1) = \pi(y_2)$. Suppose that $y_1 \neq y_2$. Then $y_1 = \tau^{jm_A} y_2$ for some integer $j$, which we may assume positive. Let $\delta : y_1 \to y_2$ and $\gamma : x \to y_1$ be paths in $\mathbb{Z}\Delta$. Therefore we have a path $\delta \gamma : x \to y_2$ with length $\ell(\delta \gamma) \geq \ell(\delta) = 2jm_A$. Since paths between vertices of $\mathbb{Z}\Delta$ have the same length, we obtain that any path starting at $x$ and ending at $y_2$ has length at least $2jm_A$. This is a contradiction because the longest length of a nonzero path in $k(\mathbb{Z}\Delta)$ is $m_A - 1$. This proves the first statement of the remark. The second statement follows by duality.

As a consequence of the above remark we can see that the information we have about the support of the functor $k(\mathbb{Z}\Delta)(x, -)$ in $\mathbb{Z}\Delta$ can be carried out through the universal covering $\pi : \mathbb{Z}\Delta \to S\Gamma_A$ to determine the support of $\text{Hom}_A(\pi(x), -)$ in $S\Gamma_A$.

**Proposition 3.3.** Let $\Lambda$ be a trivial extension of Cartan class $\Delta$. Then the universal covering $\pi : \mathbb{Z}\Delta \to S\Gamma_A$ induces the following bijections:

(i) $\text{Supp} k(\mathbb{Z}\Delta)(x, -) \sim \text{Supp} \text{Hom}_A(\pi(x), -)$.

(ii) $\text{Supp} k(\mathbb{Z}\Delta)(- , x) \sim \text{Supp} \text{Hom}_A(-, \pi(x))$.

The next result is an interesting application of the preceding corollary.

**Corollary 3.4.** Let $\Lambda$ be a trivial extension of Cartan class $\Delta$ with $\Delta$ a Dynkin diagram. Then for all $X, Y \in \text{ind} \Lambda$ we have

$$\dim_k \text{Hom}_A(X, Y) \leq \begin{cases} 1 & \text{if } \Delta = A_n, \\
2 & \text{if } \Delta = D_n, \\
3 & \text{if } \Delta = E_p \text{ and } p = 6, 7, \\
6 & \text{if } \Delta = E_6. \end{cases}$$

**Proof.** Let $\pi : \mathbb{Z}\Delta \to S\Gamma_A$ be the universal covering of $S\Gamma_A$. To describe $\text{Hom}_A(X, Y)$ we consider a fixed $x \in \pi^{-1}(X)$. We know by Remark 3.2 that there exists a unique $y \in \pi^{-1}(Y)$ such that $\text{Hom}_A(X, Y)$ is isomorphic to $k(\mathbb{Z}\Delta)(x, y)$. On the other hand,
\[ \dim_k k(\mathbb{Z}\Delta)(x, y) = f_x(y) \] where \( f_x \) is the additive function starting at \( x \). We use the work of Gabriel [12, p. 53] where he computes the values of this function for some vertices \( x \) of \( \mathbb{Z}\Delta \), to get the bounds for \( \dim_k \text{Hom}_A(X, Y) = f_x(y) \) above stated. ∎

When \( A \) is an iterated tilted algebra of Cartan class \( \Delta \), there is an embedding \( \text{ind} A \hookrightarrow \text{ind} T(A) \). Thus, the bounds given in the preceding corollary are also bounds for \( \dim_k \text{Hom}_A(X, Y) \) if \( X, Y \in \text{ind} A \).

For a fixed vertex \( x \) of \( \mathbb{Z}\Delta \) we define the partition \( \{ P_x[j] : j \in \mathbb{Z} \} \) of \( \mathbb{Z}\Delta \), where \( P_x[0] \) is the full subquiver of \( \mathbb{Z}\Delta \) with vertices lying on or between the slices \( S_{\tau^{-m}\Delta} \) and \( \tau^{-m\Delta+1}S_{\tau^{-m}\Delta} \), and \( P_x[j] = \tau^{-jm\Delta}P_x[0] \) for any \( j \in \mathbb{Z} \). Let \( z \) be a vertex of \( P_x[0] \), for any integer \( j \) we denote by \( z[j] \) the vertex \( \tau^{-jm\Delta}z \) of \( P_x[j] \).

Let \( \Lambda \) be a trivial extension of Cartan class \( \Delta \), and let \( \pi : \mathbb{Z}\Delta \rightarrow s\Gamma_A \) be the universal covering of \( s\Gamma_A \). Let \( M \in \text{ind} \Lambda \) and let \( M[0] \) be a fixed element of the fibre \( \pi^{-1}(M) \). Then \( \pi|_{\mathbb{P}_M[0]} : \mathbb{P}_M[0] \rightarrow s\Gamma_A \) is a quiver morphism, which is a bijection on the vertices of \( \mathbb{P}_M[0] \), since the quiver \( s\Gamma_A \) is isomorphic to the cylinder \( \mathbb{Z}\Delta/\langle \tau^{m\Delta} \rangle \). The inverse \( \varphi_M : (s\Gamma_A)_0 \rightarrow (\mathbb{Z}\Delta)_0 \) of this bijection defines an embedding of \( s\Gamma_A \) into \( \mathbb{Z}\Delta \). Moreover, the map \( \pi|_{\mathbb{P}_M[0]} \) is injective on the arrows of \( \mathbb{P}_M[0] \) but not surjective. Indeed, the arrows \( X \rightarrow Y \) of \( s\Gamma_A \) with \( X \in S_{\tau^{-M}\Delta} \) and \( Y \in S_{\tau^{M}\Delta} \) are not in the image of \( \pi|_{\mathbb{P}_M[0]} \) (see Fig. 4).

**Definition 3.5.** Let \( \Lambda \) be a trivial extension of Cartan class \( \Delta \) and let \( M \in \text{ind} \Lambda \). We say that the quiver \( s\Gamma_A[0] = \mathbb{P}_M[0] \) is a lifting of \( s\Gamma_A \) to \( \mathbb{Z}\Delta \) at \( M \). Moreover, if we do not want to state precisely the lifting vertex we will say that \( s\Gamma_A[0] \) is a lifting of \( s\Gamma_A \) to \( \mathbb{Z}\Delta \).

For an algebra \( A \) such that \( A \simeq T(A) \) we denote by \( \Gamma_A[0] \) the embedding of \( \Gamma_A \) in \( \mathbb{Z}\Delta \) obtained as the composition of the embeddings \( \Gamma_A \hookrightarrow s\Gamma_{T(A)} \) (given in the preceding section) and \( \varphi_M : s\Gamma_A \hookrightarrow \mathbb{Z}\Delta \).

![Fig. 4](image-url)
Remark 3.6. Let $s\Gamma_A^\ast[0]$ be a lifting of $s\Gamma_A$ to $\mathbb{Z}\Delta$ at $M$, and let $\alpha: X \to Y$ be an arrow of $s\Gamma_A$. For any $j \in \mathbb{Z}$, there exists a unique arrow $\alpha_j: X[j] \to Y_j$ in $\mathbb{Z}\Delta$ such that $\pi(\alpha_j) = \alpha$, where $\pi: \mathbb{Z}\Delta \to s\Gamma_A$ is the universal covering of $s\Gamma_A$. Moreover, we have that $Y_j$ is either equal to $Y[j]$ or to $Y[j + 1]$. The latter case occurs when $Y \in S_{\mathbb{Z}M\to}$. 

Let $A$ be an iterated tilted algebra of Cartan class $\Delta$, with $\Delta$ a Dynkin diagram. Let $\pi: \mathbb{Z}\Delta \to s\Gamma_{T(A)}$ be the universal covering of $s\Gamma_{T(A)}$, $C_{T(A)} = \{rP_i: i \in (Q_{T(A)})_0\}$ and let $\hat{\mathcal{C}}_{T(A)} = \pi^{-1}(C_{T(A)})$ be the configuration of $\mathbb{Z}\Delta$ associated to $T(A)$. From this data Chr. Riedtmann constructed in [18] the universal covering of $\Gamma_{T(A)}$ by adding to $\mathbb{Z}\Delta$ the “projective vertices”, exactly one for each vertex of the configuration $\mathcal{C}_{T(A)}$, and appropriate arrows. This can be described as follows. Let $s\Gamma_{T(A)}[0]$ be a lifting of $s\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$. Then $[rP_i(j): j \in \mathbb{Z}] = \pi^{-1}(rP_i)$ for any vertex $i$ of $Q_{T(A)}$. We denote by $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$, the translation quiver obtained from $\mathbb{Z}\Delta$ by adding a new vertex $\hat{P}_j[1]$ and arrows $rP_i[j] \to \hat{P}_j[1], \hat{P}_j[1] \to r^{-1}P_i[j]$ for each $rP_i[j] \in \hat{C}_{T(A)}$. The translation of $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ coincides with the translation of $\mathbb{Z}\Delta$ on the common vertices and is not defined on the remaining ones.

The action of $\Pi(s\Gamma_{T(A)}, x) = (\tau^{m\ast})$ on $\mathbb{Z}\Delta$ can be extended to $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ by defining $\tau^{m\ast}(P_i[j]) = P_i[j - 1]$. Moreover, the covering $\pi: \mathbb{Z}\Delta \to s\Gamma_{T(A)}$ admits an extension $\hat{\pi}: \mathbb{Z}\Delta_{\hat{C}_{T(A)}} \to \Gamma_{T(A)}$ by defining $\hat{\pi}(P_i[j]) = P_i$ for any vertex $i$ and $j$. It is not difficult to see that $\hat{\pi}: \mathbb{Z}\Delta_{\hat{C}_{T(A)}} \to \Gamma_{T(A)}$ is the universal covering of $\Gamma_{T(A)}$ and that it induces an isomorphism $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}/\langle \tau^{m\ast}\rangle \cong \Gamma_{T(A)}$.

For any $M \in \text{ind} T(A)$ the embedding $\varphi_M: s\Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta$ can be extended to an embedding $\hat{\varphi}_M: \Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ by defining $\hat{\varphi}_M(P_i) = P_i[0]$ for any vertex $j$ of $Q_{T(A)}$. We denote by $\Gamma_{T(A)}[0]$ the full subquiver of $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ with vertices $\hat{\varphi}_M((\Gamma_{T(A)})_0)$. Then $\hat{\pi}|_{\Gamma_{T(A)}[0]}: \Gamma_{T(A)}[0] \to \Gamma_{T(A)}$ is a quiver morphism, which is a bijection with inverse $\hat{\varphi}_M$ on the vertices of $\Gamma_{T(A)}[0]$. In this way, we have that the lifting $s\Gamma_{T(A)}^\ast[0]$ of $s\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$ extends directly to a lifting $\Gamma_{T(A)}[0]$ of $\Gamma_{T(A)}$ to $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$.

Given a set $X$ of vertices of $\Gamma_{T(A)}[0]$ we denote by $X[j]$ the shifted set $\tau^{-jm\ast}X$.

Proposition 3.7. With the above notation we have that $\Gamma_A^\ast \cong \mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ and the protective vertices $P_i[j]$ of $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$ represent the protective $\hat{A}$-modules. Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}\Delta & \xrightarrow{\pi} & \Gamma_{T(A)}[0] \\
\uparrow{s\Gamma_A^\ast} & & \downarrow{\hat{\pi}} \\
s\Gamma_{T(A)} & \xrightarrow{\gamma} & \Gamma_{T(A)}[0]
\end{array}
$$

Proof. Let $F: k(\mathbb{Z}\Delta_{\hat{C}_{T(A)}}) \to \text{ind} T(A)$ be a well-behaved functor induced by the universal covering $\hat{\pi}: \mathbb{Z}\Delta_{\hat{C}_{T(A)}} \to \Gamma_{T(A)}$. Let $\hat{A}$ be the full subcategory of $k(\mathbb{Z}\Delta_{\hat{C}_{T(A)}})$ whose objects are the protective vertices of $\mathbb{Z}\Delta_{\hat{C}_{T(A)}}$. Then the restriction of the functor $F$ to $\hat{A}$ induces a
The covering functor $F': \tilde{A} \rightarrow T(A)$ (see [11, 2]). This functor is the universal covering since $T(A)$ is standard [13, 3]. On the other hand, it is proven in [16] that the Galois covering $\hat{A} \rightarrow T(A)$ is universal. So $\tilde{A} \simeq \hat{A}$ proving the result. 

Remark 3.8. For any $M \in \text{ind} T(A)$ the embeddings $\varphi_M: sF_{T(A)} \hookrightarrow \mathbb{Z}\Delta_{\tilde{T}(A)}$ and $\tilde{\varphi}_M: \Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta_{\tilde{T}(A)}$ induce embeddings of $\Gamma_A$ in $s\Gamma_A$ and $\hat{\Gamma}_A$, respectively, making the following diagram commutative

\[
\begin{array}{ccc}
s\Gamma_A = \mathbb{Z}\Delta & \hookrightarrow & \mathbb{Z}\Delta_{\tilde{T}(A)} = \hat{\Gamma}_A \\
\pi & \Downarrow & \\
\Gamma_A & \hookrightarrow & \Gamma_{T(A)} \\
\end{array}
\]

Moreover, we have that $\Gamma_A[j] \hookrightarrow s\Gamma_{T(A)}[j] \hookrightarrow \Gamma_{T(A)}[j]$ for any $j \in \mathbb{Z}$.

We know that $A = T(A)/(\alpha_1, \ldots, \alpha_t)$, where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are arrows of $Q_{T(A)}$. In Section 2 we have seen that $(\Gamma_A)_{\mathfrak{0}} = (\Gamma_{T(A)})_{\mathfrak{0}} \setminus (P_{\alpha_1, \alpha_2, \ldots, \alpha_t})_{\mathfrak{0}}$, where $P_{\alpha_1, \alpha_2, \ldots, \alpha_t}$ is the full subquiver of $\Gamma_{T(A)}$ induced by the nonzero paths in $k(\Gamma_{T(A)})$ starting at the projective $P_{\alpha_i}$ and ending at the projective $P_{\alpha_j}$ for some $i = 1, 2, \ldots, t$. Thus, to obtain the embedding $\Gamma_A \hookrightarrow \hat{\Gamma}_A$ and then the desired embedding $\Gamma_A \hookrightarrow \mathbb{Z}\Delta \simeq s\Gamma_A$ we have to lift $P_{\alpha_1, \alpha_2, \ldots, \alpha_t}$ through the universal covering $\tilde{\pi}: \mathbb{Z}\Delta_{\tilde{T}(A)} \rightarrow \Gamma_{T(A)}$.

As we recalled at the beginning of this section, the length of any nonzero path in $k(\mathbb{Z}\Delta)$ is at most $m_{\Delta} - 1$. Though in $\mathbb{Z}\Delta_{\tilde{T}(A)}$, there are longer paths which are nonzero in $k(\mathbb{Z}\Delta_{\tilde{T}(A)})$, we have that the length of these paths is bounded by $2m_{\Delta}$, as follows from the following known result.

Lemma 3.9 [6, 1.2]. Any nonzero path $v: x \rightarrow y$ in $k(\mathbb{Z}\Delta_{\tilde{T}(A)})$ can be extended to a nonzero path $P_{[j]} \xrightarrow{u} x \rightarrow y \xrightarrow{w} P_{[j]} \xrightarrow{\tau^{-m_{\Delta}}} P_{[j]}$ for some $i \in (Q_{T(A)})_{\mathfrak{0}}$ and $j \in \mathbb{Z}$. In particular, the nonzero path $v: x \rightarrow y$ has length $\ell(v) \leq 2m_{\Delta}$.

Remark 3.10. Let $\Lambda$ be a trivial extension of Cartan class $\Delta$, with $\Delta$ a Dynkin diagram. Let $F: k(\mathbb{Z}\Delta_{\tilde{\Lambda}}) \rightarrow \text{ind} \Lambda$ be a well-behaved functor induced by the universal covering $\tilde{\pi}: \mathbb{Z}\Delta_{\tilde{\Lambda}} \rightarrow \hat{\Gamma}_{\Lambda}$. We consider now the isomorphism

$$\bigoplus_{y \in \tilde{\pi}^{-1}(y)} k(\mathbb{Z}\Delta_{\tilde{\Lambda}})(x, y) \cong \text{Hom}_{\Lambda}(\tilde{\pi}(x), Y)$$

induced by the covering functor $F: k(\mathbb{Z}\Delta_{\tilde{\Lambda}}) \rightarrow \text{ind} \Lambda$. In analogy with the result stated in Remark 3.2 for the stable case, we obtain that if $\text{Hom}_{\Lambda}(\tilde{\pi}(x), Y) \neq 0$ then the left side
of (**) has a unique nonzero summand, unless \( \tilde{\pi}(x) \cong Y \). Though this is not true when \( \tilde{\pi}(x) \cong Z \); in this case the left side of (**) has at most two nonzero summands.

In fact, the last claim follows directly from Lemma 3.9. To prove the first, let \( y \in \tilde{\pi}^{-1}(Y) \) be such that \( k(\mathbb{Z}\Delta C_{\tilde{T}}^e)(x, y) \neq 0 \). Using Lemma 3.9 we only need to prove that \( k(\mathbb{Z}\Delta C_{\tilde{T}}^e)(x, \tau^{j\cdot m}y) = 0 \) for \( j = \pm 1 \). Since any path \( w : y \to \tau^{-m}\Delta y \) has length \( 2m \Delta \) and we have a path \( v : x \to y \) with \( x \neq y \), we conclude that any path \( u : x \to \tau^{-m}\Delta y \) has length \( \ell(u) \geq 2m + 1 \). Thus by Lemma 3.9 we obtain that \( k(\mathbb{Z}\Delta C_{\tilde{T}}^e)(x, \tau^{-m}\Delta y) = 0 \). Likewise, we get that also \( k(\mathbb{Z}\Delta C_{\tilde{T}}^e)(x, \tau^{-m}\Delta y) = 0 \), proving the result.

We are now in a position to prove the main result of this section.

**Theorem 3.11.** Let \( A \) be an iterated tilted algebra of Dynkin type \( \Delta \), and let \( A = T(A)/\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \), where \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are arrows of \( QT(A) \). Let \( \Gamma T(A)[0] \) be a lifting of \( \Gamma T(A) \) to \( \mathbb{Z}\Delta \). For any integer \( j \) we denote by \( P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j] \) the full subquiver of \( \mathbb{Z}\Delta C_{\tilde{T}}^e \), induced by the nonzero paths in \( k(\mathbb{Z}\Delta C_{\tilde{T}}^e) \) starting at \( P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j] \) and ending either at \( P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j] \) or \( P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j + 1] \) for some \( i = 1, 2, \ldots, t \). Then the vertices of \( \Gamma A[0] \) are the vertices of \( \tilde{\pi}^{-1}(P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j]) \) which are not in \( \tilde{P}_{\alpha_1, \alpha_2, \ldots, \alpha_n}[-1] \) or \( \tilde{P}_{\alpha_1, \alpha_2, \ldots, \alpha_n}[0] \).

**Proof.** Let \( \tilde{\pi} : \mathbb{Z}\Delta C_{\tilde{T}}^e \to \Gamma T(A) \) be the universal covering of \( \Gamma T(A) \). By Remarks 2.6 and 3.8 we know that \( \Gamma A[0] = \tilde{\pi}^{-1}(P_{\alpha_1, \alpha_2, \ldots, \alpha_n}) \). On the other hand, \( P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j] \cap \tilde{\pi}^{-1}(P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j]) = \emptyset \) for \( j \geq 1 \) and \( j \leq -2 \). Then the desired result follows from the equality

\[
\tilde{\pi}^{-1}(P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j]) = \bigcup_{j \in \mathbb{Z}} P_{\alpha_1, \alpha_2, \ldots, \alpha_n}[j],
\]

which is a consequence of Lemma 3.9 and Remark 3.10.

**Example.** Let \( T \) be the trivial extension of Cartan class \( A_5 \) with ordinary quiver \( QT \) and with the relations \( \alpha_4\alpha_3 = 0, \alpha_1\alpha_6 = 0, \alpha_3\alpha_2\alpha_1 - \alpha_6\alpha_5\alpha_4 = 0, \alpha_2\alpha_1\alpha_3\alpha_2 = 0, \alpha_5\alpha_4\alpha_6 \alpha_5 = 0 \).

![Diagram](image)

Let \( A = T/(\overline{\alpha_2}, \overline{\alpha_3}) \) and \( B = T/(\overline{\alpha_1}, \overline{\alpha_4}) \). Hence \( T(A) = T(B) \) and the embeddings \( \Gamma A[j] \hookrightarrow \Gamma_A, \Gamma B[j] \hookrightarrow \Gamma_B \) for each integer \( j \) are as follows:

1. The shaded regions in Fig. 5 correspond to \( P_{\alpha_2, \alpha_6}[j] \) for \( j \in \mathbb{Z} \). Hence, the vertices of \( \Gamma_A \) which are not in these shaded regions correspond to \( A \)-modules.

2. The shaded regions in Fig. 6 correspond to \( P_{\alpha_3, \alpha_5}[j] \) for \( j \in \mathbb{Z} \). Consequently, the vertices of \( \Gamma_B \) which are not in these regions correspond to \( B \)-modules.
Finally, we can describe $\Gamma_A$ and $\Gamma_B$ from this information. Indeed, the vertices of $\Gamma_A$ can be represented by the vertices of $S\Gamma_{T(A)}[0]$, which are not in the shaded regions. The arrows of $\Gamma_A$ are obtained by studying the paths in $S\Gamma_{T(A)}[-1] \cup S\Gamma_{T(A)}[0] \cup S\Gamma_{T(A)}[1]$, as follows from Remarks 3.2 and 3.6. Then we get the AR-quivers $\Gamma_A$ and $\Gamma_B$.
References