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Polar syzygies in characteristic zero: The monomial case[☆]

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ABSTRACT

Given a set of forms $\mathbf{f} = \{f_1, \ldots, f_m\} \subset R = k[x_1, \ldots, x_n]$, where k is a field of characteristic zero, we focus on the first syzygy module $\mathcal{Z}$ of the transposed Jacobian module $\mathcal{D}(\mathbf{f})$, whose elements are called *differential syzygies* of $\mathbf{f}$. There is a distinct submodule $\mathcal{P} \subset \mathcal{Z}$ coming from the polynomial relations of $\mathbf{f}$ through its transposed Jacobian matrix, the elements of which are called *polar syzygies* of $\mathbf{f}$. We say that $\mathbf{f}$ is *polarizable* if equality $\mathcal{P} = \mathcal{Z}$ holds. This paper is concerned with the situation where $\mathbf{f}$ are monomials of degree 2, in which case one can naturally associate to them a graph $\mathcal{G}(\mathbf{f})$ with loops and translate the problem into a combinatorial one. The main result is a complete combinatorial characterization of polarizability in terms of special configurations in this graph. As a consequence, we show that polarizability implies normality of the subalgebra $k[\mathbf{f}] \subset R$ and that the converse holds provided the graph $\mathcal{G}(\mathbf{f})$ is free of certain degenerate configurations. One main combinatorial class of polarizability is the class of polymatroidal sets. We also prove that if the edge graph of $\mathcal{G}(\mathbf{f})$ has diameter at most 2 then $\mathbf{f}$ is polarizable. We establish a curious connection with birationality of rational maps defined by monomial quadrics.

1. Introduction

Let k be a field of characteristic zero. Given a set of forms of the same degree, $\mathbf{f} = \{f_1, \ldots, f_m\} \subset R = k[x_1, \ldots, x_n]$, one can consider both the ideal $I = (\mathbf{f}) \subset R$ and the k-subalgebra $A = k[\mathbf{f}] = k[f_1, \ldots, f_m] \subset R$. Looking at the intertwining properties of the subalgebra A and the ideal I was of course Hilbert's original idea to understand the finite generation of certain rings of invariants. As such it became natural to look at the syzygies of the polynomial relations of I. About 25 years before Hilbert's wrap-up of these questions, P. Gordan and M. Noether in their celebrated work [6] about the Hesse problem had this approach sort of turned around by looking instead at an individual polynomial relation $F \in k[\mathbf{T}] = k[T_1, \ldots, T_m]$ of $\mathbf{f}$ in the special case where $n = m$ and $f_1, \ldots, f_m$ were the partial derivatives of a homogeneous polynomial $f \in R$. They posed (and solved) the question of finding all polynomial solutions $\Phi(\mathbf{x}) \in R$ of the partial differential equation

$$\sum_{j=1}^{m} \frac{\partial \Phi}{\partial x_j} F_{T_j}(\mathbf{f}) = 0, \tag{1}$$

where a subscripted variable indicates partial derivative with respect to this variable. In other words, among all syzygies of the ideal $(F_{T_1}(\mathbf{f}), \ldots, F_{T_m}(\mathbf{f}))$ they were looking for the polynomially integrable ones! Particular solutions are of course the very partial derivatives $\Phi_i = f_i$, one for each $i = 1, \ldots, m$ — a consequence of the rule of derivatives for composite functions.

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Now, one can think about the relations

$$\sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial F}{\partial T_j}(\mathbf{f}), \quad i = 1, \dots, m,$$

for each polynomial relation $F \in k[\mathbf{T}]$ of $\mathbf{f}$, as syzygies of the Hessian matrix of the form f. Going back to the more general setting where $\mathbf{f} = \{f_1, \dots, f_m\}$ is a set of m forms of the same degree in $R = k[x_1, \dots, x_n]$, one could ask for the syzygies of the transposed Jacobian matrix of $\mathbf{f}$. This was the original goal in [10] where the syzygies corresponding to the relations

$$\sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i} \frac{\partial F}{\partial T_j}(\mathbf{f}), \quad i = 1, \dots, m,$$

one for each polynomial relation $F \in k[\mathbf{T}]$ of $\mathbf{f}$, have been dubbed *polar syzygies* and it was shown that in a certain special context the whole module of syzygies of the transposed Jacobian matrix of $\mathbf{f}$ is generated by the polar syzygies.

The motivation for the terminology stems from the tradition of having the rational map induced by the partials of f called the *polar map* of the hypersurface defined by f.

Let us explain the setup of our work in a more systematic way. Let $\Omega_{A/k}$ denote the module of Kähler k-differentials of A and let $A \simeq k[\mathbf{T}]/P$ be a presentation of A over a polynomial ring $k[\mathbf{T}]$. Consider the well-known conormal exact sequence

$$P/P^2 \xrightarrow{\delta} \sum_{j=1}^{m} A\,\mathrm{d}T_j \longrightarrow \Omega_{A/k} \to 0, \tag{2}$$

where δ is induced by the transposed Jacobian matrix over $k[\mathbf{T}]$ of a generating set of P. Let $\mathcal{P} \subset \sum_{j=1}^{m} R\,\mathrm{d}T_j$ denote the R-submodule generated by $\delta(P/P^2)$ – the elements of which are called *polar syzygies* of $\mathbf{f}$. This module is actually a submodule of the first syzygy module $\mathcal{Z}$ of the transposed Jacobian module $\mathcal{D}(\mathbf{f})$ when the latter is viewed in its natural embedding in $\sum_{i=1}^{n} R\,\mathrm{d}x_i$ – the elements of $\mathcal{Z}$ could be called *differential syzygies* of $\mathbf{f}$. We say that $\mathbf{f}$ (or the embedding $A \subset R$) is *polarizable* if $\mathcal{P} = \mathcal{Z}$.

One basic principle will tell us that, on a far more general setting, the two modules always have the same rank and allow for a comparison (Lemma 2.3).

When $\mathbf{f}$ are monomials of degree 2, a special case of the presently envisaged problem had been taken up earlier in [10], where A was, up to degree normalization, the homogeneous coordinate ring of a coordinate projection of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$. The main result was that the k-subalgebra generated by a subset of the monomials

$$\{y_i z_j \mid 0 \le i \le r, 0 \le j \le s\} \subset k[y_0, \dots, y_r; z_0, \dots, z_s]$$

is polarizable.

In this work we vastly enlarge the picture, obtaining a full combinatorial characterization of polarizability. The combinatorial gadget that plays a main role is a graph with loops – this is allegedly a nontrivial work over the usual simple graphs, where no loops are present. In the more general context of admitting loops, the given monomial generators $\mathbf{f}$ of A over k still correspond to (traditional) edges and loops and the corresponding graph is denoted $\mathcal{G}(\mathbf{f})$. Even in this generalized setting we will stick to the terminology that has A called the *edge-algebra* associated to $\mathcal{G}(\mathbf{f})$.

For the purpose of establishing edge-algebra polarizability, we dwell on the fine points of the structure of both $\mathcal{P}$ and $\mathcal{Z}$, by describing their sets of natural minimal generators in terms of combinatorial substructures of the corresponding graph $\mathcal{G}(\mathbf{f})$. We were thus led to isolate two special configurations of $\mathcal{G}(\mathbf{f})$, called *cycle arrangements* and *molecules*, respectively. These configurations are natural supports of closed walks of $\mathcal{G}(\mathbf{f})$ and, provided these closed walks are *even*, give rise to natural sets of both differential and polar syzygies. In order to detect minimal generators among these we further impose certain restrictions and arrive to the notion of *non-split* and *indecomposable* even closed walks. A consequence of these methods is a complete characterization of polarizability in terms of the above configurations.

Besides throwing light into polarizability, it is to expect that these configurations yield some new numerical invariants of the graph that may have some curious reflection into the structure of the corresponding algebra.

An almost immediate consequence is a new proof of the result that the edge-algebra of a connected bipartite graph is polarizable – this is precisely the main theorem in [10, Theorem 2.3] for the projections of the Segre embedding.

Using the known characterization of the integral closure of the corresponding edge-algebra (see [13, Theorem 1.1], [9, Corollary 2.3]), we are able to show that polarizability implies normality of the algebra and the converse holds provided the graph is free of certain degenerate configurations. Both polarizability and normality involve the existence of the so-called *bow tie* configurations which are special cases of the previous configurations (the terminology itself was introduced in [13] and the notion was based on an earlier construct of M. Hochster).

We further consider the question as to how the problem of polarizability is affected by "variable collapsing" when A is generated by monomials of degree 2. This collapsing can be thought of as a loop-contraction operation on the edges of a graph (its geometric interpretation in terms of Proj (A) is that of projecting down to a one dimension less ambient by cutting with a suitable elementary hyperplane). We show that it preserves the k-algebra A by a k-isomorphism if and only the given graph is bipartite, which can be viewed as yet another characterization of connected bipartite graphs. Conversely, by "resolving" a loop issuing from an odd cycle we improve the chances of the given generators become polarizable.

From a close scrutiny of the data in a long list of computed examples, we are naturally led to guess that there is a strong relationship between the syzygies of the given k-algebra generators $\mathbf{f}$ of A and polarizability. In this vein, we first show that the condition that the module of syzygies of $\mathbf{f}$ is generated by linear relations is equivalent to the edge graph of $\mathcal{G}(\mathbf{f})$ having diameter at most 2, an easy result that gives an algebraic tint to the notion of diameter – one would be tempted to ask whether the exact value of the diameter reflects a numerical algebraic invariant, such as the dimension of the subspace of syzygies spanned in degree 2 (or 4 by considering the usual degree shift). Merging with the aforementioned combinatorial characterization of polarizability we show that linear presentation implies polarizability.

A curious consequence of the theory is that the rational map $\mathbb{P}^{n-1} - - \rightarrow \mathbb{P}^{m-1}$ defined by a polarizable set $\mathbf{f}$ of monomials of degree 2, such that $\dim k[\mathbf{f}] = n$, maps $\mathbb{P}^{n-1}$ birationally onto its image. This includes rational maps defined by polymatroidal sets of monomials of degree 2 of maximal rank – a subclass of which are the so-called algebras of Veronese type. This result recovers a couple of theorems proved in [15] with a different approach.

As a final note, the reason to tackle solely monomials of degree 2 – and not more general toric algebras as would be the case – is due to an as yet not completely understood phenomenon by which such monomial k-subalgebras generated in degree higher than 2 easily fail to be polarizable.

2. Statement of the problem

Let $A = k[\mathbf{f}] = k[f_1, \ldots, f_m] \subset R = k[x_1, \ldots, x_n]$. Consider a presentation $A \simeq S/P$ via $S = k[T_1, \ldots, T_m] \twoheadrightarrow A$ by mapping $T_j \mapsto f_j$. We assume throughout that $\mathrm{char}(k) = 0$.

Recall the well-known conormal sequence

$$P/P^2 \xrightarrow{\delta} \sum_{j=1}^{m} A\, \mathrm{d}T_j \longrightarrow \Omega_{A/k} \to 0, \tag{3}$$

where δ is induced by the transposed Jacobian matrix over S of a generating set of P, namely

$$\delta : F \,(\bmod P^2) \mapsto \sum_j \frac{\partial F}{\partial T_j} \,(\bmod P)\, \mathrm{d}T_j.$$

The embedding $A \subset R$ induces an embedding $\sum_{j=1}^{m} A\, \mathrm{d}T_j \subset \sum_{i=1}^{m} R\, \mathrm{d}T_j$.

Throughout, we set $\mathcal{P} = \delta(P/P^2)\, R \subset \sum_{j=1}^{m} R\, \mathrm{d}T_j$, the R-submodule generated by the image of δ. Then $\mathcal{P}$ is generated by the vectors $\sum_j \frac{\partial F}{\partial T_j}(\mathbf{f})\, \mathrm{d}T_j$, where F runs through a set of generators of P. On the other hand, by the usual rules of composite derivatives, if $F \in P$ then $\sum_{j=1}^{m} \frac{\partial F}{\partial T_j}(\mathbf{f})\, \mathrm{d}f_j = 0$. This means that $\mathcal{P} \subset \mathcal{Z}$, where $\mathcal{Z}$ is the first syzygy module of the differentials $\mathrm{d}\mathbf{f}$.

Definition 2.1. As a way of terminology, the elements of $\mathcal{Z}$ (respectively, $\mathcal{P}$) are called *differential syzygies* (respectively, *polar syzygies*). Thus, $\mathcal{Z}$ (respectively, $\mathcal{P}$) will be referred to as the *differential syzygy module* (respectively, the *polar syzygy module*) of $\mathbf{f}$.

The set $\mathbf{f}$ (or, by a slight abuse, the embedding $A \subset R$ defined by these generators) is said to be *polarizable* if $\mathcal{P} = \mathcal{Z}$.

A preliminary fact in this framework is the following result, which seems to be partially folklore (but see [11, Proposition 1.1] for a proof and a feeling of this result and its previous history).

Proposition 2.2. *If* $\mathit{char}(k) = 0$ *then* $\dim k[\mathbf{f}] = \mathit{rank}\, \mathcal{D}(\mathbf{f})$.

It will be used in the proof of the main supporting evidence for the potential equality $\mathcal{P} = \mathcal{Z}$, as given by the following result of general nature.

Lemma 2.3. $\mathit{rank}_R(\mathcal{P}) = \mathit{rank}_R(\mathcal{Z})$ $(= \mathit{height}\ P\,)$.

Proof. Since $P/P^{(2)}$ and $\mathcal{P}$ are generated by the same generating set, computing rank by the familiar determinantal method yields $\mathrm{rank}_A(P/P^{(2)}) = \mathrm{rank}_R(\mathcal{P})$. But $P^{(2)}/P^2$ is a torsion A-module and P is generically a complete intersection on S, hence $\mathrm{rank}_A(P/P^{(2)}) = \mathrm{rank}_A(P/P^2) = \text{height } (P)$. On the other hand, $\mathrm{rank}_R(\mathcal{Z}) = m - \mathrm{rank}_R(\mathcal{D}(\mathbf{f})) = m - \dim A = \text{height } (P)$ using Proposition 2.2. Since $\mathcal{P} \subset \mathcal{Z}$, we are through. $\square$

As it turns the theory in the case of monomials of degree 2 is fairly under grasp; in particular, we will give a complete characterization of when $\mathbf{f}$ is polarizable in terms of its underlying combinatorial nature. For this, we are led to introduce several configurations of that nature drawing largely from the theory of graphs.

3. Related graph substructures

In this section we develop the graph-theoretic material needed to translate the stated problem into combinatorics. The general reference for algebraic graph theory in this section is [16].

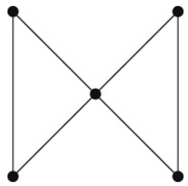


Fig. 1. Path-degenerate bow tie.

3.1. Non-split even closed walks

Recall that, given a set $\mathbf{f} = \{f_1, \ldots, f_m\} \subset R = k[x_1, \ldots, x_n]$ of distinct monomials of degree 2, one associates to it a graph $\mathcal{G}(\mathbf{f})$ with loops whose vertices correspond to the variables, and where, given i, j, $1 \leq i \leq j \leq n$, the vertices x_i and x_j of $\mathcal{G}(\mathbf{f})$ are connected by an edge whenever $x_i x_j \in \mathbf{f}$. The ideal $(\mathbf{f})$ is radical if and only if $\mathcal{G}(\mathbf{f})$ is a simple graph, i.e., has no loops.

The notion of even closed walk on $\mathcal{G}(\mathbf{f})$ is central in this part, so let us recall its main features along with some extra precision needed for the purpose of this paper.

An *even closed walk of length* $2r$ in $\mathcal{G}(\mathbf{f})$ is given by a sequence $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$, where $g_j \in \mathbf{f}$ and $\gcd(g_j, g_{j+1}) \neq 1$ for $1 \leq j \leq 2r$ (with the proviso $g_{2r+1} = g_1$). We call $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ the structural *edge sequence* of the even closed walk.

Often, by abuse, we make no distinction between an even closed walk and its structural edge sequence. Note that for a given j the corresponding edge g_j may be repeated in the sequence – we then speak of an *edge repetition*. In this vein, for any edge f of the graph there is the trivial even closed walk $\{f, f\}$ – actually, this is the only even closed walk of length 2 in a graph. In particular, by swinging back and forth arbitrarily often one finds even closed walks of arbitrary length! Thus, a procedure is needed that overlooks such useless nuisances that may creep in as an argument gets more intricate. Such a procedure will be given soon below.

Note that the even closed walk $\mathfrak{w}$ may also be given by its *vertex sequence*, namely:

$$g_1 = x_{i_1}x_{i_2},\ g_2 = x_{i_2}x_{i_3},\ \ldots,\ g_{2r} = x_{i_{2r}}x_{i_1},$$

with $i_1, \ldots, i_{2r} \in \{1, \ldots, n\}$. Similarly, one may have a *vertex repetition*. Those even closed walks with no vertex repetition are called *even cycles*. Clearly, an edge repetition implies a vertex repetition, but not vice versa as the simple example in Fig. 1 illustrates.

Remark 3.1. Let $\{g_1, \ldots, g_{2r}\}$ be the structural edge sequence of an even closed walk $\mathfrak{w}$.

1. For all i, $1 \leq i < 2r$, the sequence $\{g_{i+1}, \ldots, g_{2r}, g_1, \ldots, g_i\}$, obtained by cyclically permuting the edges of the original edge sequence, defines the same even closed walk $\mathfrak{w}$ as before. Therefore, by suitably reordering the elements in an edge sequence defining $\mathfrak{w}$, one can arbitrarily choose which of the variables $x_{i_1}, \ldots, x_{i_{2r}}$ comes first in a vertex sequence of $\mathfrak{w}$.
2. There may be more ways of permuting the edges in a given edge sequence of $\mathfrak{w}$ – always preserving the property that the least common multiple of two consecutive elements is not. Thus, in the above example of two triangles $\{f_1, f_2, f_3\}$ and $\{f_4, f_5, f_6\}$ with a common vertex belonging to f_1, f_3, f_4 and f_6, the even closed walk with edge sequence $\{f_1, \ldots, f_6\}$ can also be described by the sequence $\{f_1, f_2, f_3, f_6, f_5, f_4\}$.

Despite this lack of uniqueness of ordering of edge or vertex sequences, we speak of them as if they were uniquely defined by the corresponding even closed walk.

From the first of these observations follows in particular that, given even closed walks $\mathfrak{w}_1 = \{g_1, \ldots, g_{2r}\}$ and $\mathfrak{w}_2 = \{g'_1, \ldots, g'_{2s}\}$ who share at least one vertex, one can always assume that this common vertex is the first element in their vertex sequences as observed above. We then denote by $\mathfrak{w}_1 \sqcup \mathfrak{w}_2$ the even closed walk whose edge sequence is $\{g_1, \ldots, g_{2r}, g'_1, \ldots, g'_{2s}\}$. Conversely, one ought to consider those even closed walks that split this way. We make this into a precise definition.

Definition 3.2. We say that an even closed walk $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ *splits* if it has a vertex repetition, say the first element x_{i_1} in its vertex sequence, and if there exists s, $1 \leq s < r$, such that $\mathfrak{w}_1 = \{g_1, \ldots, g_{2s}\}$ and $\mathfrak{w}_2 = \{g_{2s+1}, \ldots, g_{2r}\}$ are the edge sequences of two smaller even closed walks. When this occurs, we have that $\mathfrak{w} = \mathfrak{w}_1 \sqcup \mathfrak{w}_2$. We say that $\mathfrak{w}$ *splits into* $\mathfrak{w}_1$ *and* $\mathfrak{w}_2$, and also that $\mathfrak{w}$ *splits at the vertex* x_{i_1}. An even closed walk that does not split is said to be *non-split*.

Remark 3.3. Even closed walk splitting has an obvious parallel in other algebraic theories: an even closed walk containing an even closed *subwalk* – in the sense of a proper subset of the given edge sequence being the edge sequence of an even closed walk – may not split into this and another even closed subwalk.

By definition, an even closed walk has a vertex repetition if it splits. The converse fails as the example in Fig. 1 shows. The next lemma gives the behavior of a repeated vertex in the vertex sequence of a non-split even closed walk. It shows in particular that the vertices involved in a non-split even closed walk cannot occur more than twice along its vertex sequence.

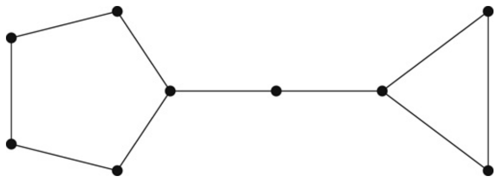


Fig. 2. Typical bow tie.

Lemma 3.4. *Let $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ be a non-split even closed walk in $\mathcal{G}(\mathbf{f})$ with $g_1 = x_{i_1}x_{i_2}, g_2 = x_{i_2}x_{i_3}, \ldots, g_{2r} = x_{i_{2r}}x_{i_1}$ ($r \geq 2$). Let x_{i_j} be a repeated vertex in this sequence, say $x_{i_j} = x_{i_l}$, with $1 \leq j < l \leq 2r$. Then:*

(1) *(Uniqueness of recurrence) $x_{i_k} \neq x_{i_j}$ for all $k \neq j, l$.*
(2) *(Parity condition) $l - j \equiv 1 \pmod 2$.*

Proof. For a suitable edge ordering, one may assume that $j = 1$. Next choose l to be the smallest index such that $x_{i_l} = x_{i_1}$. Since $\mathfrak{w}$ is non-split, l has to be even, otherwise $\mathfrak{w}$ splits into $\{g_1, \ldots, g_{l-1}\}$ and $\{g_l, \ldots, g_{2r}\}$; hence (2) follows. Now, if $x_{i_k} = x_{i_1}$ for some $k, l < k \leq 2r$, then by the same reasoning k has to be even, in which case $\mathfrak{w}$ splits into $\{g_l, \ldots, g_{k-1}\}$ and $\{g_k, \ldots, g_{2r}, g_1, \ldots, g_{l-1}\}$; hence (1) holds as well. □

A similar result holds as regards edge repetitions in a non-split even closed walk. Again it shows that any edge along the edge sequence of a non-split even closed walk occurs at most twice.

Lemma 3.5. *Let $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ be a non-split even closed walk in $\mathcal{G}(\mathbf{f})$ with $g_1 = x_{i_1}x_{i_2}, g_2 = x_{i_2}x_{i_3}, \ldots, g_{2r} = x_{i_{2r}}x_{i_1}$ ($r \geq 2$). If it has an edge repetition, say $g_j = g_l$ for $1 \leq j < l \leq 2r$, then the following three conditions hold:*

(1) *(Sense-reversing recurrence) $x_{i_j} = x_{i_{l+1}}$ and $x_{i_{j+1}} = x_{i_l}$.*
(2) *(Uniqueness of recurrence) $g_k \neq g_j$ for all $k \neq j, l$.*
(3) *(Parity condition) $l - j \equiv 0 \pmod 2$.*

Proof. For a suitable edge ordering, one may assume that $j = 1$.

(1) Since $g_l = g_1$, one has either $x_{i_1} = x_{i_l}$ and $x_{i_2} = x_{i_{l+1}}$, or $x_{i_1} = x_{i_{l+1}}$ and $x_{i_2} = x_{i_l}$. If $x_{i_1} = x_{i_l}$ and $x_{i_2} = x_{i_{l+1}}$, note that the edge sequence $\{g_2, \ldots, g_{l-1}, g_{2r}, g_{2r-1}, \ldots, g_{l+1}, g_l, g_1\}$ also defines the even closed walk $\mathfrak{w}$, hence

$$\mathfrak{w} = \{g_2, \ldots, g_{l-1}, g_{2r}, g_{2r-1}, \ldots, g_{l+1}\} \sqcup \{g_l, g_1\}.$$

(2) It follows from Lemma 3.4, (1).
(3) This is clear since if $l = 2s$ for some s, $1 < s < r$, then $\mathfrak{w}$ splits into $\{g_1, \ldots, g_{2s}\}$ and $\{g_{2s+1}, \ldots, g_{2r}\}$. □

3.2. Supporting configurations

Associated to an even closed walk $\mathfrak{w}$ in $\mathcal{G}(\mathbf{f})$ there is a connected subgraph of $\mathcal{G}(\mathbf{f})$ whose edges are the distinct elements in the edge sequence of $\mathfrak{w}$. This subgraph of $\mathcal{G}(\mathbf{f})$ is called the *support* of $\mathfrak{w}$. Clearly, an even cycle in $\mathcal{G}(\mathbf{f})$ is exactly the configuration that supports a non-split even closed walk with no vertex repetition. As a rule, we make no distinction between an even cycle and its naturally associated non-split even closed walk and, by the same abuse, we will identify an even closed walk with its support. We now proceed to survey a few more configurations that support non-split even closed walks.

3.2.1. Bow ties

The following configuration was introduced in [13].

Definition 3.6. (1) A *bow tie* of $\mathcal{G}(\mathbf{f})$ is the (connected) subgraph $\mathcal{B}$ of $\mathcal{G}(\mathbf{f})$ consisting of two odd cycles whose sets of edges are disjoint, connected by a unique non-empty path. One allows for either cycle to degenerate into a loop — in this case, we speak of a *looped bow tie*.

(2) One allows the connecting path to be formed by one single edge – in which case we call the configuration a *monedge bow tie* – or to degenerate into a single vertex — in which case we refer to the bow tie as being *path-degenerate*. Note that, in particular, a looped bow tie can also be a monedge (looped) bow tie, with either or both cycles being loops; similarly, a looped bow tie can be a path-degenerate (looped) bow tie with one of the cycles (but not both, of course) being a loop.

Such configurations are depicted in Figs. 1–3. Note that a bow tie is the support of a non-split even closed walk with a vertex repetition — indeed, an edge repetition unless it is path-degenerate.

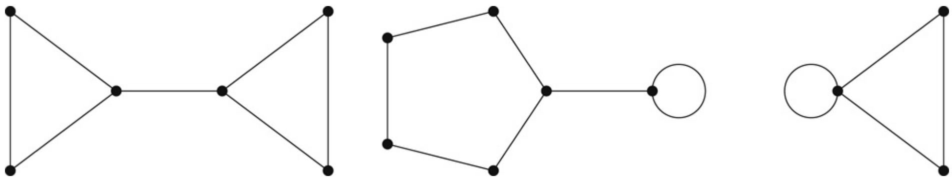


Fig. 3. Monedge, monedge looped and path-degenerate looped bow ties.

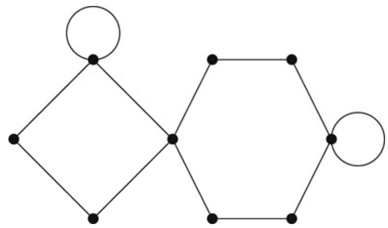


Fig. 4. Cycle arrangement.

Remark 3.7. These configurations were introduced in [13] in order to build the integral closure of the algebra $k[\mathbf{f}]$ in case $\mathcal{G}(\mathbf{f})$ had no loops. The *Hochster monomial* associated to a bow tie is the product of the variables corresponding to the totality of the vertices of the two cycles. We give the notion some flexibility in the sense that we do not a priori require $\mathcal{B}$ to be an *induced* subgraph, i.e., $\mathcal{G}(\mathbf{f})$ may have edges that do not belong to the bow tie configuration and that connect the two structural odd cycles, or one vertex on one odd cycle to one vertex on the path, or one vertex on the path to another vertex on the path. The two approaches differ in that by taking the induced subgraph definition the Hochster monomial is a fresh generator of the integral closure of $k[\mathbf{f}]$, while our present notion allows for the Hochster monomial to belong to $k[\mathbf{f}]$ (i.e., to be a product of edges). Otherwise, the notion is the same as in [13]. We will have more to say on this theme later.

We now introduce two basic configurations in a graph which will play a central role in this part, provided they support even closed walks. The first one includes path-degenerate bow ties, while the second of these configurations will be a generalized version of a bow tie which is not path-degenerate.

3.2.2. Cycle arrangements

The following configuration can be thought of as an extension of the notion of a cycle in a graph.

Definition 3.8. A *cycle arrangement* of a graph $\mathcal{G}(\mathbf{f})$ is a connected subgraph of $\mathcal{G}(\mathbf{f})$ consisting of a set of (even or odd) cycles, here called the *constituent cycles* of the cycle arrangement, satisfying the following properties:

(C_1) Any two constituent cycles have mutually disjoint edges;
(C_2) Any two constituent cycles share at most one vertex;
(C_3) Any vertex of the configuration belongs to at most two constituent cycles.

The following information ought to be kept in mind:

- The vertices of a cycle arrangement belonging to only one of its constituent cycles are called *simple*;
- We will say that a cycle arrangement is *even* or *odd* according to whether the total number of edges in its configuration is even or odd, respectively;
- An even cycle arrangement supports an even closed walk. As often done, by abuse, we will also refer to this even closed walk as an even cycle arrangement. For example, the cycle arrangement in Fig. 4 is a non-split even closed walk;
- An even cycle arrangement has the property that its simple vertices are exactly the non-repeated vertices along its vertex sequence;
- The non-simple vertices of a cycle arrangement belong to exactly two constituent cycles by (C_3), and hence all vertex repetitions in a cycle arrangement satisfy the recurrence condition Lemma 3.4, (1).

We refer to [16, Example 8.4.14] for an example of an even cycle arrangement which gives rise to a non-superfluous polynomial relation of the corresponding edge-algebra — we will have more to say later about this sort of matter.

Remark 3.9. An even cycle arrangement may split. It is clear that this happens whenever the cycle arrangement branches out into two even cycle arrangements as shown in the two examples in Fig. 5.

This sort of operation will be made clear later. In Lemma 3.12 a characterization will be given of when an even cycle arrangement is non-split. The first example in Fig. 5 illustrates a trivial obstruction for an even cycle arrangement to be non-split: *a constituent cycle that is connected to exactly one other constituent cycle must be odd*. The second example of Fig. 5

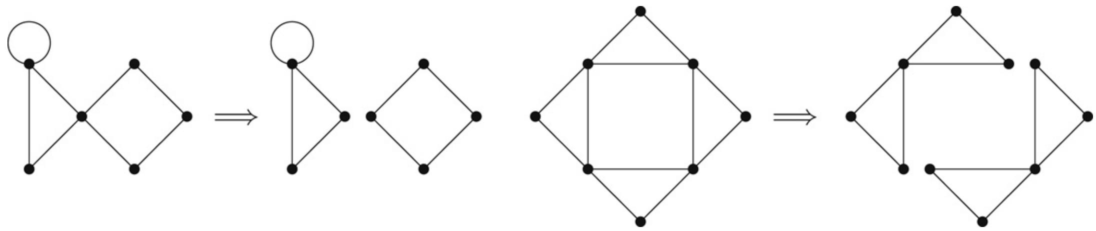


Fig. 5. Even cycle arrangements that split.

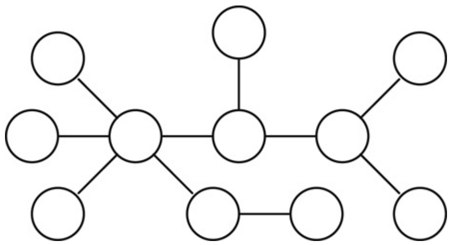


Fig. 6. Shadow of molecule.

puts in evidence yet another obstruction for an even cycle arrangement to be non-split: *the constituent cycles must be the only cycles of the arrangement as a subgraph* — thus, the inner square and the outer octagon are not constituent cycles though they are cycles of the containing graph. This latter necessary condition, which is however non-obvious, will be proved in Corollary 3.13.

3.2.3. Molecules

We now introduce the second configuration. Recall that a *path* of a graph is a non-closed walk without vertex repetition. The first and last vertices of a path are called *extremal*.

Definition 3.10. A *molecule* of a graph $\mathcal{G}(\mathbf{f})$ is a connected subgraph of $\mathcal{G}(\mathbf{f})$ consisting of a set of r cycle arrangements ($r \geq 2$) – its *structural cycle arrangements* – and a set of $r-1$ paths – its *structural paths* – satisfying the following properties:

(M_1) Any two structural cycle arrangements have mutually disjoint edges;
(M_2) Any two structural paths have mutually disjoint vertices (hence mutually disjoint edges as well);
(M_3) A structural cycle arrangement and a structural path have at most one vertex in common (in particular, have no common edges);
(M_4) Every structural cycle arrangement meets at least one structural path and every structural path meets exactly two structural cycle arrangements;
(M_5) Every vertex of the configuration belongs to at most two structural cycle arrangements;
(M_6) A vertex that belongs to two structural cycle arrangements is a simple vertex of both and a vertex that belongs to a structural cycle arrangement and a structural path is a simple vertex of the first and an extremal vertex of the second.

If one draws schematically a circle for each structural cycle arrangement and a line for each structural path, then the *shadow* of a typical molecule is a tree (because the number of structural paths is, by definition, one less than the number of structural cycle arrangements), as depicted in Fig. 6.

As in the case of a cycle arrangement, the following basic information on molecules ought to be kept in mind:

- The constituent cycles of the structural cycle arrangements of a molecule are simply called its *constituent cycles*;
- A molecule is said to be *even* or *odd* according as to whether the number of edges in all its constituent cycles is even or odd, respectively;
- An even molecule is the support of an even closed walk, of which all edge repetitions correspond to the edges in the structural paths. Again, we identify an even molecule and the even closed walk supported on it;
- Each vertex in the vertex sequence of a molecule belongs to either: one single cycle; exactly two cycles; one single cycle and one single path; or one single path;
- The vertex repetitions along the vertex sequence of an even molecule satisfy the recurrence property Lemma 3.4, (1), and all its edge repetitions satisfy the recurrence and sense-reversing properties Lemma 3.5, (1), (2).

The previous bow tie configuration (see, e.g., Fig. 2) is a molecule with two structural cycle arrangements consisting each of one single odd cycle (or loop), and a single path that connects these two cycle arrangements – the only exception is a path-degenerate bow tie, which is a cycle arrangement (see Definition 3.6 and the comments at the end of the paragraph).

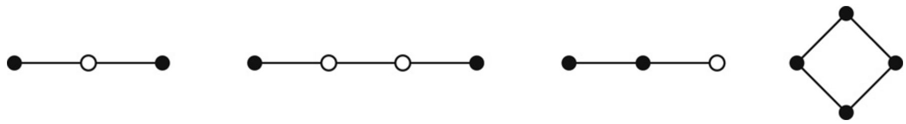


Fig. 7. Skeleton of an even molecule and of even cycle arrangements.

3.2.4. Skeletons of cycle arrangements and molecules

Next one characterizes when even cycle arrangements and even molecules are non-split. For this purpose one introduces the following notion:

Definition 3.11. Let $\mathcal{B}$ be either an even cycle arrangement or an even molecule of a graph $\mathcal{G}(\mathbf{f})$. The *skeleton* $\mathcal{T}(\mathcal{B})$ of $\mathcal{B}$ is a connected graph whose vertices fall under two disjoint sets, the one of the *black* vertices and the one of the *white* vertices (represented respectively by dots and circles), defined as follows:

(S_1) To every constituent cycle of $\mathcal{B}$ there corresponds a vertex of $\mathcal{T}(\mathcal{B})$ and this vertex is black (respectively white) if the cycle is odd (respectively, even);
(S_2) If $\mathcal{B}$ is a molecule then to every edge of a structural path of $\mathcal{B}$ there corresponds a white vertex of $\mathcal{T}(\mathcal{B})$;
(S_3) Two vertices of $\mathcal{T}(\mathcal{B})$ are connected by an edge if and only if the corresponding subconfigurations of $\mathcal{B}$ – whether constituent cycles or edges in a structural path – meet.

Note that the constituent cycles and the structural paths of $\mathcal{B}$ uniquely determine $\mathcal{T}(\mathcal{B})$.

The set of repeated vertices of $\mathcal{B}$ is in bijection with the set of edges of $\mathcal{T}(\mathcal{B})$. Moreover, $\mathcal{T}(\mathcal{B})$ has always an even number of black vertices.

The skeletons of some of the earlier configurations are depicted in Fig. 7. The first one is the skeleton of any monedge bow tie. The second is the skeleton of the non-split cycle arrangement in Fig. 4. The last two are the skeletons of the two split even cycle arrangements in Fig. 5.

Note that if $\mathcal{B}$ is a molecule, its even constituent cycles and the edges in its structural paths are represented in the same way in $\mathcal{T}(\mathcal{B})$, namely, by a white vertex. This is because an edge in a structural path of a molecule can be considered as a degenerate even cycle with two vertices and two edges that coincide.

Our next result characterizes non-split even cycle arrangements and even molecules in terms of its skeleton.

Lemma 3.12. *Let $\mathcal{B}$ be either an even cycle arrangement or an even molecule of a graph $\mathcal{G}(\mathbf{f})$, and let $\mathcal{T}(\mathcal{B})$ be its skeleton. The following are equivalent:*

(1) *$\mathcal{B}$ is non-split ;*
(2) *No edge deletion from $\mathcal{T}(\mathcal{B})$ gives rise to two connected graphs with an even number of black vertices each;*
(3) *$\mathcal{T}(\mathcal{B})$ is a tree and any one edge deletion gives rise to two trees with an odd number of black vertices each.*

Proof. The contrapositive of the implication (1) $\Rightarrow$ (2) is straightforward by recalling that an edge of $\mathcal{T}(\mathcal{B})$ corresponds to a vertex repetition in $\mathcal{B}$, and that an even closed walk that splits will do so at one of its vertex repetitions. Actually, the negation of (2) is a reformulation of the phenomenon described in Remark 3.9.

(3) $\Rightarrow$ (1): Assume that $\mathcal{T}(\mathcal{B})$ is a tree and that $\mathcal{B}$ splits. As already observed, this will happen at one of its vertex repetitions and hence, removing the corresponding edge of $\mathcal{T}(\mathcal{B})$, one obtains two trees with an even number of black vertices each.

(2) $\Rightarrow$ (3): We will be done if we show that (2) implies that $\mathcal{T}(\mathcal{B})$ is a tree because the second part of (3) then trivially holds. Let us assume that $\mathcal{T}(\mathcal{B})$ has at least one cycle and show that (2) fails. This will be proved by induction on the number of cycles of $\mathcal{T}(\mathcal{B})$. If it has one single cycle, say $\mathcal{T}_1$, then removing any two edges of $\mathcal{T}_1$, one gets two trees, and one only has to prove that one can always choose two edges of $\mathcal{T}_1$ such that both trees have an even number of black vertices. If there exists one vertex $\mathbf{v}$ in $\mathcal{T}_1$ such that, removing the two edges of $\mathcal{T}_1$ going through $\mathbf{v}$, one gets two trees with an even number of black vertices, we are done. Otherwise, each vertex in $\mathcal{T}_1$ satisfies that, removing the two edges of $\mathcal{T}_1$ going through it, one gets two trees with an odd number of black vertices. Now consider any two consecutive vertices of $\mathcal{T}_1$ and remove from $\mathcal{T}_1$ the edge that goes through each of them and which is distinct from the edge that connects them. One gets the two expected trees. Finally, if $\mathcal{T}(\mathcal{B})$ has more than one cycle, note that removing one edge in one of its cycles, one gets a connected graph with one cycle less, and by induction we are done. $\square$

Corollary 3.13. *A non-split even cycle arrangement or a non-split even molecule of a graph $\mathcal{G}(\mathbf{f})$ includes no other cycle of $\mathcal{G}(\mathbf{f})$ other than its constituent cycles.*

Proof. By the previous lemma, the skeleton of a non-split even cycle arrangement or a non-split even molecule $\mathcal{B}$ is a tree, hence there cannot be any additional cycles of $\mathcal{G}(\mathbf{f})$ in $\mathcal{B}$ other than its constituent cycles. $\square$

A non-split even cycle arrangement has vertex repetitions (unless it is a single cycle) and no edge repetition. A non-split even molecule has always edge repetitions. The following result states that these are all possible non-split even closed walks in $\mathcal{G}(\mathbf{f})$.

Proposition 3.14. *A non-split even closed walk in a graph $\mathcal{G}(\mathbf{f})$ is either an even cycle arrangement or an even molecule.*

This result is a direct consequence of Lemmas 3.4 and 3.5 and the following two lemmas:

Lemma 3.15. *Let $\mathfrak{w} = \{g_1, \ldots, g_t\}$ be a closed walk (even or odd) in $\mathcal{G}(\mathbf{f})$ with $g_1 = x_{i_1}x_{i_2}, g_2 = x_{i_2}x_{i_3}, \ldots, g_t = x_{i_t}x_{i_1}$. Assume that $\mathfrak{w}$ has no edge repetition, and that any vertex repetition $x_{i_j} = x_{i_l}$ for $1 \le j < l \le t$ satisfies the recurrence condition of Lemma 3.4. Then, $\mathfrak{w}$ is a cycle arrangement.*

Proof. The proof is by induction on the number $s \ge 0$ of vertex repetitions in $\mathfrak{w}$. If $s = 0$, then $\mathfrak{w}$ is a cycle. If $s \ge 1$, one can assume without loss of generality that x_{i_1} is a vertex repetition, i.e., $x_{i_1} = x_{i_l}$ for some l, $1 < l \le t$, and that $x_{i_1}, \ldots, x_{i_{l-1}}$ are all distinct (there is always a vertex repetition with this property). Then, $\mathfrak{w}' := \{g_1, \ldots, g_{l-1}\}$ is a cycle in $\mathcal{G}(\mathbf{f})$ whose vertex sequence contains x_{i_1}, and $\mathfrak{w}'' := \{g_l, \ldots, g_t\}$ is a closed walk in $\mathcal{G}(\mathbf{f})$ whose vertex sequence contains x_{i_1} with $s - 1$ vertex repetitions (x_{i_1} is not a vertex repetition in $\mathfrak{w}''$) that satisfies the recurrence condition of Lemma 3.4. Applying the recursive hypothesis we are done. □

Lemma 3.16. *Let $\mathfrak{w} = \{g_1, \ldots, g_t\}$ be a closed walk (even or odd) in $\mathcal{G}(\mathbf{f})$ with $g_1 = x_{i_1}x_{i_2}, g_2 = x_{i_2}x_{i_3}, \ldots, g_t = x_{i_t}x_{i_1}$ satisfying that $g_j \ne g_{j+1}$ for all $j = 1, \ldots, t$ (with the proviso $g_{t+1} = g_1$). Assume that any vertex repetition $x_{i_j} = x_{i_l}$ for $1 \le j < l \le t$ satisfies the recurrence condition of Lemma 3.4, and that any edge repetition $g_j = g_l$ for $1 \le j < l \le t$ satisfies the sense-reversing property in Lemma 3.5. Then, $\mathfrak{w}$ is a molecule.*

Proof. The proof is by induction on the number $s \ge 0$ of edge repetitions in $\mathfrak{w}$. The case $s = 0$ is Lemma 3.15. If $s \ge 1$, one can assume without loss of generality that g_1 is an edge repetition, i.e., $g_\ell = g_1$ for some ℓ, $2 < \ell \le t$, and that $g_1, \ldots, g_{\ell-1}$ are all distinct (there is always at least one edge repetition with this property). By the sense-reversing property (1) in Lemma 3.5, $x_{i_\ell} = x_{i_2}$ and $x_{i_{\ell+1}} = x_{i_1}$, and hence $g_{\ell-1} = x_{i_{\ell-1}}x_{i_2}$ and $g_{\ell+1} = x_{i_1}x_{i_{\ell+2}}$. Thus, $\mathfrak{w}' := \{g_2, \ldots, g_{\ell-1}\}$ is a closed walk in $\mathcal{G}(\mathbf{f})$ and since it has no edge repetition, it is a cycle arrangement by Lemma 3.15. Set $\mathfrak{w}'' := \{g_\ell, \ldots, g_t, g_1\}$. It is a closed walk in $\mathcal{G}(\mathbf{f})$ whose vertex and edge repetitions satisfy the same properties as the ones in $\mathfrak{w}$ because $\mathfrak{w} = \mathfrak{w}' \sqcup \mathfrak{w}''$. Since $g_\ell = g_1$, by the sense-reversing property (1) in Lemma 3.5, one has that $\{g_{\ell+1}, \ldots, g_t\}$ is also a closed walk in $\mathcal{G}(\mathbf{f})$. By the same argument, if $g_{\ell+1} = g_t$, then $\{g_{\ell+2}, \ldots, g_{t-1}\}$ is a closed walk in $\mathcal{G}(\mathbf{f})$ and, iterating, we get that for some $k \ge 0$, $\mathfrak{w}''' := \{g_{\ell+1+k}, \ldots, g_{t-k}\}$ is a closed walk in $\mathcal{G}(\mathbf{f})$ with $g_{\ell+1+k} \ne g_{t-k}$. This closed walk satisfies the same conditions as $\mathfrak{w}$ and it has at most $s - 1$ edge repetitions (more precisely it has $s - 1 - k$ edge repetitions). If $s - 1 - k \ne 0$, applying the recursive hypothesis to $\mathfrak{w}'''$, one gets that it is a molecule. Otherwise, it is a cycle arrangement by Lemma 3.15. We conclude observing that the original configuration supporting the closed walk $\mathfrak{w}$ is exactly the one obtained by connecting the cycle arrangement $\mathfrak{w}'$ to the molecule (or cycle arrangement when $s - 1 - k = 0$) $\mathfrak{w}'''$ by the path supported on $\{g_\ell, \ldots, g_{\ell+k}\}$. By the recurrence property (1) in Lemma 3.4, the vertex $x_{i_\ell} (= x_{i_2})$, respectively $x_{i_{\ell+k+1}}$, is not a repetition in the vertex sequence of $\mathfrak{w}'$, respectively $\mathfrak{w}'''$, and hence $\mathfrak{w}$ is a molecule. □

Thus, the non-split even closed walks in a graph $\mathcal{G}(\mathbf{f})$ are exactly its non-split even cycle arrangements and its non-split even molecules which are characterized in Lemma 3.12.

3.3. Indecomposable even closed walks

We now introduce a subtler class of non-split even closed walks that will tie up polarizability of $\mathbf{f}$ to combinatorial properties of the graph $\mathcal{G}(\mathbf{f})$.

Definition 3.17. A non-split even closed walk $\mathfrak{w}$ in a graph $\mathcal{G}(\mathbf{f})$ is *decomposable* if there exist $h_1, \ldots, h_t \in \mathbf{f}$ satisfying the following conditions:

(D_1) $h_1, \ldots, h_t$ are square free;
(D_2) Any variable involved in the monomials $h_1, \ldots, h_t$ corresponds to a vertex along the vertex sequence of $\mathfrak{w}$;
(D_3) By adding twice every h_j to the edge sequence of $\mathfrak{w}$ then, up to conveniently reordering the resulting sequence, one gets an even closed walk that splits into two smaller even closed walks $\mathfrak{w}_1$ and $\mathfrak{w}_2$ that do not contain $\mathfrak{w}$ and whose edge sequences both contain $h_1, \ldots, h_t$.

Fig. 8 provides a few simple examples of decomposable even closed walks to bear in mind.

We say that the set $h_1, \ldots, h_t$ is a *decomposing set* of $\mathfrak{w}$. An *indecomposable* even closed walk is a non-split even closed walk which is not decomposable. In the examples illustrated in Fig. 8 the dotted edges are the decomposing edges in each case. Note that every h_j may belong to the very edge sequence of $\mathfrak{w}$ as the fourth example in Fig. 8 shows.

The following example illustrates the role of condition (D_1) in the definition of decomposability: in the graph in Fig. 9, if one considers the looped bow tie involving the first and third loops, it is indecomposable since one cannot use the second loop to decompose it because the monomial corresponding to a loop is not square free.

Among the non-split even closed walks in $\mathcal{G}(\mathbf{f})$, many are decomposable as the following result shows:

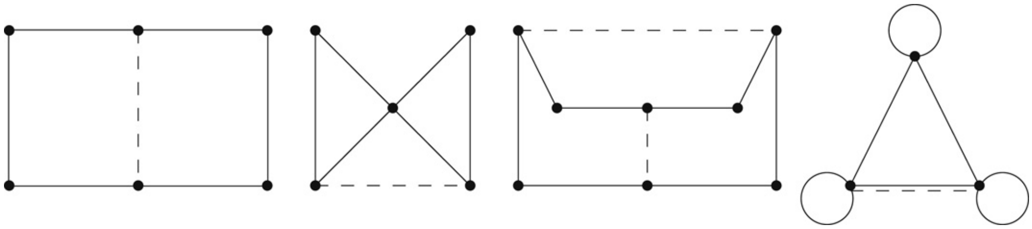


Fig. 8. Decomposable even closed walks.

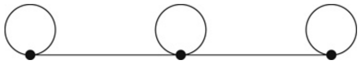


Fig. 9. Indecomposable looped bow tie.

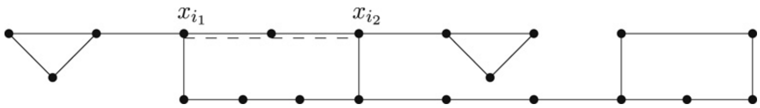


Fig. 10. A non-split even molecule with more than 3 constituent cycles is decomposable.

Lemma 3.18. *Let $\mathfrak{w}$ be a non-split even closed walk in a graph $\mathcal{G}(\mathbf{f})$. Assume that $\mathfrak{w}$ contains a cycle of which at least two vertices are vertex repetitions of $\mathfrak{w}$. Then, $\mathfrak{w}$ is decomposable.*

Proof. In order to prove the result, we will show that if there exists such a cycle $\mathcal{C}$, one can use the elements in $\mathbf{f}$ corresponding to some of its edges as decomposing set. Note that (D_1) and (D_2) will always be satisfied if $h_1, \dots, h_t$ correspond to edges of $\mathcal{C}$, so we have to select them such that (D_3) holds.

By Proposition 3.14, $\mathfrak{w}$ is either an even cycle arrangement or an even molecule, and $\mathcal{C}$ is one of its constituent cycles by Corollary 3.13. Our assumption is that there are two variables, say x_{i_1} and x_{i_2}, corresponding to vertices along the vertex sequence of $\mathcal{C}$, that are vertex repetitions of $\mathfrak{w}$. By Lemma 3.12 (3), removing from the skeleton $\mathcal{T}(\mathfrak{w})$ of $\mathfrak{w}$ the edge corresponding to the vertex repetition x_{i_1}, one gets two trees with an odd number of black vertices. One of them contains the vertex of $\mathcal{T}(\mathfrak{w})$ associated to the constituent cycle $\mathcal{C}$ of $\mathfrak{w}$, and one does not. Denote by $\mathcal{G}_1$ the subgraph of $\mathfrak{w}$ corresponding to the latter. It is the support of an odd closed walk whose vertex sequence contains x_{i_1} as a nonrepeated vertex. We define similarly $\mathcal{G}_2$ by substituting x_{i_2} for x_{i_1}. Choose any of the two paths in $\mathcal{C}$ connecting x_{i_1} and x_{i_2}, and consider the even molecule $\mathfrak{w}_1$ obtained connecting $\mathcal{G}_1$ to $\mathcal{G}_2$ by this path. On the other hand, consider the even closed walk $\mathfrak{w}_2$ supported by the subgraph of $\mathfrak{w}$ obtained by removing $\mathcal{G}_1$ and $\mathcal{G}_2$. One can now easily check that (D_3) holds for the decomposing set $h_1, \dots, h_t$ corresponding to the edges of the cycle $\mathcal{C}$ connecting x_{i_1} and x_{i_2} that we have chosen before. □

Fig. 10 illustrates the idea of the proof with an example. As a consequence of Lemma 3.18, one gets the following result which establishes a characterization of the subclass of indecomposable even closed walks in parallel to Proposition 3.14:

Proposition 3.19. *An indecomposable even closed walk in a graph $\mathcal{G}(\mathbf{f})$ is either an even cycle or a bow tie.*

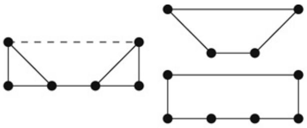
Proof. An indecomposable even walk is non-split and hence, it is either a non-split even cycle arrangement or a non-split even molecule by Proposition 3.14. Moreover, it cannot have more than two constituent cycles, otherwise at least one of them would satisfy the hypothesis in Lemma 3.18. If it has one, it is a cycle. Otherwise, it is a bow tie (that can be path-degenerate or not). □

One can now tell exactly all the indecomposable even closed walks. We will say that a subgraph of $\mathcal{G}(\mathbf{f})$ is *induced* if it is obtained by deleting a set of vertices and all the edges that go through them and/or by deleting a set of loops (leaving of course the base vertex of the loop). One realizes that this is the usual concept for simple graphs, taking care in addition of loops as well.

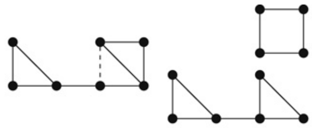
Proposition 3.20. *The indecomposable even closed walks of a graph $\mathcal{G}(\mathbf{f})$ are its indecomposable even cycles and its induced bow ties.*

Proof. Using Proposition 3.19 and observing that induced bow ties are indecomposable, we will be done once is shown that every non-induced bow tie is decomposable. Given a non-induced bow tie, there is at least one edge in $\mathcal{G}(\mathbf{f})$ which is not an edge of the bow tie and that connects two vertices of the bow tie. Depending on the kind of vertices connected by this extra edge, one gets four distinct situations:

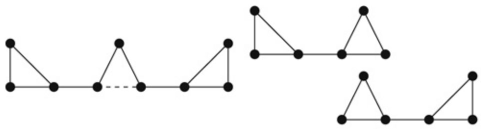
(1) one vertex on one structural odd cycle and the second on the other;
(2) one vertex on one structural odd cycle and the other on the structural path;
(3) both vertices on the structural path;
(4) both vertices on the same structural odd cycle.



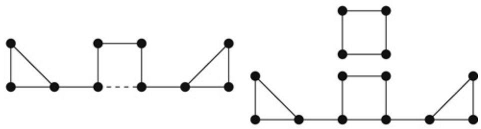
(1) Non induced bow tie disassembled into 2 even cycles.



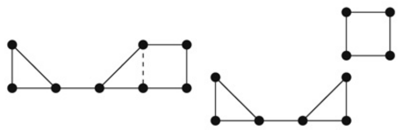
(2) Non induced bow tie disassembled into 1 bow tie and 1 even cycle.



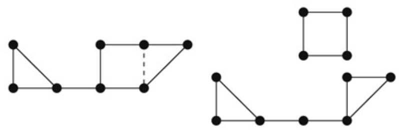
(3.a) Non induced bow tie disassembled into 2 bow ties.



(3.b) Non induced bow tie disassembled into 1 even molecule and 1 even cycle.



(4.a) Non induced bow tie disassembled into 1 bow tie and 1 even cycle.



(4.b) Non induced bow tie disassembled into 1 bow tie and 1 even cycle.

Fig. 11. Non-induced bow ties are decomposable.

Fig. 11 illustrates these four situations. The dot edge is, in each situation, the extra edge that makes the bow tie non-induced. Observe that the decomposition may depend on the parity of the number edges in some specific part of the configuration. For example, in situation (3), depending on the parity of the number of edges on the structural path that connect the two vertices joined by the dot edge, one gets (3a) or (3b). In (4), the dot edge is a chord of one of the structural odd cycles and hence it divides it into an even cycle and an odd cycle. When the odd cycle is connected to the structural path of the non-induced bow tie, one has (4a), otherwise one has (4b).

Note that in each situation, the dot edge is used as decomposing set except in (4.b) where the decomposing set contains the dot edge and some edges of the non-induced bow tie. $\square$

Remark 3.21. Induced even cycles are certainly indecomposable but also cycles having a chord may be indecomposable. Of course, the existence of a chord subdividing the induced subgraph associated to the cycle vertices into smaller even cycles makes it decomposable as the first example in Fig. 8 shows. But a cycle can also be decomposable if this condition is not fulfilled as the third example in Fig. 8 illustrates.

4. Combinatorics and polar syzygies

In this section we establish the nature of generators of both the differential syzygy module $\mathcal{Z}$ and its counterpart, the polar syzygy module $\mathcal{P}$ – see Section 2 for the needed terminology.

4.1. Even closed walks induce syzygies

Recall that, as in Section 2, the elements of $\mathcal{Z}$ (respectively, of $\mathcal{P}$) are named differential (respectively, polar) syzygies of $\mathbf{f}$. We have the following basic result.

Lemma 4.1. *Let $\mathbf{f} \subset R$ be a set of monomials of degree 2 and let $\mathfrak{w} = \{g_1, \dots, g_{2r}\}$ be an even closed walk of $\mathcal{G}(\mathbf{f})(r \geq 2)$. Then the transpose of the vector*

$$\widetilde{z_{\mathfrak{w}}} := \left(\frac{g}{g_1}, -\frac{g}{g_2}, \frac{g}{g_3}, \dots, -\frac{g}{g_{2r}}\right)$$

is a differential syzygy of the edge sequence $\{g_1, \dots, g_{2r}\}$, where g stands for the least common multiple of the distinct monomials in the sequence $g_1, \dots, g_{2r}$.

Proof. Assume that $g_1 = x_{i_1}x_{i_2}, g_2 = x_{i_2}x_{i_3}, \ldots, g_{2r} = x_{i_{2r}}x_{i_1}$. One has

$$\begin{aligned}\frac{g}{g_1}\,\mathrm{d}g_1 - \frac{g}{g_2}\,\mathrm{d}g_2 &= \frac{g}{x_{i_1}}\mathrm{d}x_{i_1} + \frac{g}{x_{i_2}}\mathrm{d}x_{i_2} - \left(\frac{g}{x_{i_2}}\mathrm{d}x_{i_2} + \frac{g}{x_{i_3}}\mathrm{d}x_{i_3}\right)\\ &= \frac{g}{x_{i_1}}\mathrm{d}x_{i_1} - \frac{g}{x_{i_3}}\mathrm{d}x_{i_3}\end{aligned}$$

as elements of $\sum_{i=1}^{n} R\mathrm{d}x_i$. Inducting, one gets at the $(2r-2)$nd step

$$\frac{g}{g_1}\mathrm{d}g_1 - \frac{g}{g_2}\mathrm{d}g_2 + \cdots - \frac{g}{g_{2r-2}}\mathrm{d}g_{2r-2} = \frac{g}{x_{i_1}}\mathrm{d}x_{i_1} - \frac{g}{x_{i_{2r-1}}}\mathrm{d}x_{i_{2r-1}}.$$

Applying two more steps and recalling that $g_{2r} = x_{i_{2r}}x_{i_1}$, it is clear that

$$\frac{g}{g_1}\mathrm{d}g_1 - \frac{g}{g_2}\mathrm{d}g_2 + \cdots - \frac{g}{g_{2r}}\mathrm{d}g_{2r} = 0. \quad \square$$

We associate to an even closed walk $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ of $\mathcal{G}(\mathbf{f})$, a vector $\mathrm{z}_{\mathfrak{w}}$ in R^n as follows: denoting by $(\widetilde{z_{\mathfrak{w}}})_j$ the jth entry of the vector $\widetilde{z_{\mathfrak{w}}}$ defined in Lemma 4.1 ($1 \le j \le 2r$), the ith entry of $\mathrm{z}_{\mathfrak{w}}$ ($1 \le i \le n$) is $\sum_{j\,/\,g_j=f_i}(\widetilde{z_{\mathfrak{w}}})_j$ (understanding that this is 0 if f_i does not belong to the edge sequence of $\mathfrak{w}$). Note that if the even closed walk $\mathfrak{w}$ is non-split, the ith entry of $\mathrm{z}_{\mathfrak{w}}$ is 0 if and only if f_i does not belong to the edge sequence of $\mathfrak{w}$ by Lemma 3.5 (3). Moreover, by Lemma 3.5 (2), the nonzero entries of $\mathrm{z}_{\mathfrak{w}}$ are pure monomials in R with a factor ± 1 or ± 2.

Example 4.2. Consider $\mathbf{f} = \{f_1, \ldots, f_5\} \subset R = K[x_1, x_2, x_3]$ with $f_1 = x_1^2, f_2 = x_1x_2, f_3 = x_2^2, f_4 = x_2x_3, f_5 = x_3^2$ whose associated graph $\mathcal{G}(\mathbf{f})$ is shown in Fig. 9. If $\mathfrak{w}$ is the induced looped bow tie in $\mathcal{G}(\mathbf{f})$ involving the first and the third loops, then $\mathrm{z}_{\mathfrak{w}} = (x_2x_3^2, -2x_1x_3^2, 0, 2x_1^2x_3, -x_1^2x_2)^t \in R^5$.

The following result is one of the basic bridging devices between combinatorics and polarizability. Keeping the just introduced notation, one has:

Theorem 4.3. *Let $\mathbf{f} \subset R$ be a set of monomials of degree 2. Then the differential syzygy module $\mathcal{Z}$ of $\mathbf{f}$ is generated by the vectors $\mathrm{z}_{\mathfrak{w}}$, for all non-split even closed walks $\mathfrak{w}$ of length ≥ 4 of the graph $\mathcal{G}(\mathbf{f})$.*

Proof. By Lemma 4.1, for every even closed walk $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ the transpose of $\widetilde{z_{\mathfrak{w}}}$ is a syzygy of the differentials of the edge sequence $\{g_1, \ldots, g_{2r}\}$. Suppose that $\mathfrak{w}$ is non-split. For any edge repetition $g_j = g_l$ in the edge sequence, identify the corresponding differentials $\mathrm{d}g_j, \mathrm{d}g_l$ and, accordingly, introduce a factor of ± 2 as coefficient of the corresponding coordinate of $\widetilde{z_{\mathfrak{w}}}$ because $j \equiv l \pmod 2$. Next, complete the transpose of $\widetilde{z_{\mathfrak{w}}}$ to a full vector of R^m by placing 0 at every coordinate corresponding to an $f_j \notin \{g_1, \ldots, g_{2r}\}$. In this way, the resulting vector of R^m clearly belongs to $\mathcal{Z}$.

Conversely, let $\mathrm{z} \in \mathcal{Z}$ be a differential syzygy of $\mathbf{f}$. Since $\mathbf{f}$ is a set of monomials of the same degree, the transposed Jacobian module $\mathcal{D}(\mathbf{f})$ in its natural embedding in $\sum_{i=1}^{n} R\,\mathrm{d}x_i$ is graded with respect to the fine grading. Therefore, it has a minimal $\mathbb{Z}^n$-graded free resolution and, in particular, z is an R-linear combination of vectors $\mathrm{z}_1, \ldots, \mathrm{z}_t$ in $\mathcal{Z} \subset R^m$ whose coordinates are terms $\alpha\mathbf{x}^{\mathbf{a}} \in R$ with $\alpha \in \mathbb{Q}$. Multiplying each z_i by an integer, one can assume without loss of generality that any differential syzygy is an R-linear combination of vectors in $\mathcal{Z} \subset R^m$ whose coordinates are terms $\alpha\mathbf{x}^{\mathbf{a}} \in R$ with $\alpha \in \mathbb{Z}$. Thus, assume that the given differential syzygy z is already of the latter form, so that one has a relation of the form $\alpha_1\mathbf{x}^{\mathbf{a}_1}\mathrm{d}f_1 + \alpha_2\mathbf{x}^{\mathbf{a}_2}\mathrm{d}f_2 + \cdots + \alpha_m\mathbf{x}^{\mathbf{a}_m}\mathrm{d}f_m = 0$ with $\alpha_i \in \mathbb{Z}$. In other words, one can assume that the given differential syzygy z gives a relation of the form

$$\epsilon_1 M_1\mathrm{d}g_1 + \epsilon_2 M_2\mathrm{d}g_2 + \cdots + \epsilon_s M_s\mathrm{d}g_s = 0 \tag{4}$$

where $g_1, \ldots, g_s \in \mathbf{f}$, $\epsilon_1, \ldots, \epsilon_s \in \{-1, +1\}$, and $M_1, \ldots, M_s$ are monomials in R such that $\gcd(M_1, \ldots, M_s) = 1$ and $M_i = M_j$ (and $\epsilon_i = \epsilon_j$) whenever $g_i = g_j$ for some $1 \le i < j \le s$. Moreover, one can also assume that this relation is shortest for $\mathrm{d}g_1, \ldots, \mathrm{d}g_s$. In this situation we claim that $\mathrm{z} = \mathrm{z}_{\mathfrak{w}}$ for some non-split even closed walk $\mathfrak{w}$.

Indeed, write $g_1 = x_{i_1}x_{i_2}$. Then, by the same token as in the proof of Lemma 4.1, one has $M_1\mathrm{d}g_1 = M_1x_{i_2}\,\mathrm{d}x_{i_1} + M_1x_{i_1}\,\mathrm{d}x_{i_2}$ (including the collapsing case $i_1 = i_2$, whereby $M_1\mathrm{d}g_1$ has one single non-zero coordinate, namely, $2M_1x_{i_1}$ as coefficient of $\mathrm{d}x_{i_1}$). Now (4), forces the existence of an index ℓ, $2 \le \ell \le s$, such that $\epsilon_\ell = -1$ and that one of the two non-zero coordinates of the vector $M_\ell\mathrm{d}g_\ell$ is $M_1x_{i_1}$ as coefficient of $\mathrm{d}x_{i_2}$. Moreover, the other non-zero coordinate cannot be a coefficient of $\mathrm{d}x_{i_1}$. In other words, upon reordering the g_j' s if necessary, one can assume that $\epsilon_2 = -1$, that $g_2 = x_{i_2}x_{i_3}$ for some $i_3 \ne i_1$, and $M_1x_{i_1} = M_2x_{i_3}$. Then $M_1\mathrm{d}g_1 - M_2\mathrm{d}g_2 = M_1x_{i_2}\,\mathrm{d}x_{i_1} - M_2x_{i_2}\,\mathrm{d}x_{i_3}$, with $i_3 \ne i_1$. By the same argument, there exists ℓ, $3 \le \ell \le s$, such that $\epsilon_\ell = +1$ and with the property that one of the non-zero coordinates of the vector $M_\ell\mathrm{d}g_\ell$ is $M_2x_{i_2}$ as coefficient of $\mathrm{d}x_{i_3}$. Again, upon reordering the g_j's if necessary, one can assume that $\ell = 3$, i.e., $g_3 = x_{i_3}x_{i_4}$ for some $i_4 \ne i_2$, and $M_2x_{i_2} = M_3x_{i_4}$. Then $M_1\mathrm{d}g_1 - M_2\mathrm{d}g_2 + M_3\mathrm{d}g_3 = M_1x_{i_2}\,\mathrm{d}x_{i_1} + M_3x_{i_3}\,\mathrm{d}x_{i_4}$. Iterating this process and reordering the g_i's at each step if necessary, one gets that $g_j = x_{i_j}x_{i_{j+1}}$ for all $j = 1, \ldots, s$. Note that in order to get the zero vector, s has to be even, and $i_{j+1} = i_1$. In other words, $\{g_1, \ldots, g_s\}$ is an even closed walk. Moreover, the condition that has to be satisfied by the monomials M_j at each step is

$$M_j \frac{g_j}{\gcd(g_j, g_{j+1})} = M_{j+1} \frac{g_{j+1}}{\gcd(g_j, g_{j+1})}, \quad \forall j = 1, \ldots, s. \tag{5}$$

Setting $M := M_1 g_1$, one has that $M = M_j g_j$ for all $j = 1, \ldots, s$. Moreover, $\frac{g_{j+1}}{\gcd(g_j, g_{j+1})}$ divides M_j, and hence lcm (g_j, g_{j+1}) divides M. Letting g stand for the least common multiple of the distinct monomials in the sequence $g_1, \ldots, g_s$, this implies the existence of a monomial $N \in R$ such that $M = gN$. Then, for all $j = 1, \ldots, s$, $M_j = \frac{g}{g_j} N$. Since we have assumed that the monomials M_j have no non-trivial common factor, one has that $N = 1$, and hence $\mathrm{z} = \mathrm{z}_{\mathfrak{w}}$ for the even closed walk $\mathfrak{w} := \{g_1, \ldots, g_s\}$.

Finally, observe that if an even closed walk $\mathfrak{w}$ splits into two smaller even closed walks $\mathfrak{w}_1$ and $\mathfrak{w}_2$, then $\mathrm{z}_{\mathfrak{w}} = \frac{g}{\ell_1} \mathrm{z}_{\mathfrak{w}_1} + \frac{g}{\ell_2} \mathrm{z}_{\mathfrak{w}_2}$ where g, ℓ_1 and ℓ_2 are the least common multiples of the monomials in the edge sequences associated to $\mathfrak{w}$, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively. $\square$

Let $P \subset k[\mathbf{T}]$ be the presentation ideal of $k[\mathbf{f}]$ relative to the given generators $\mathbf{f}$. We formally introduce a construct that is a special polar syzygy to play a central role in the discussion.

Definition 4.4. Let $\mathfrak{w}$ denote an even closed walk of the graph $\mathcal{G}(\mathbf{f})$. To it one associates the binomial relation $p_{\mathfrak{w}} = \mathbf{T}_{\mathfrak{w}^+} - \mathbf{T}_{\mathfrak{w}^-} \in P$ in a notation mimicking that of [16, 7.1.4]. Define the *associated polar syzygy* $\tau_{\mathfrak{w}}$ to be the differential of $p_{\mathfrak{w}}$ further evaluated at the edges of $\mathfrak{w}$. In further detail, regarding $\tau_{\mathfrak{w}}$ as a column vector, its jth coordinate is the T_jth derivative of $p_{\mathfrak{w}}$ (hence, a monomial) further evaluated at the corresponding edge g_j in the edge sequence of the walk $\mathfrak{w}$.

Example 4.5. If $\mathfrak{w}$ is the induced bow tie considered in Example 4.2, then $p_{\mathfrak{w}} = T_1 T_4^2 - T_2^2 T_5$. The associated polar syzygy is $\tau_{\mathfrak{w}} = (x_2^2 x_3^2, -2x_1 x_2 x_3^2, 0, 2x_1^2 x_2 x_3, -x_1^2 x_2^2)^t$. Note that this polar syzygy is related to the differential syzygy $\mathrm{z}_{\mathfrak{w}}$ determined in Example 4.2 by $\tau_{\mathfrak{w}} = x_2 \mathrm{z}_{\mathfrak{w}}$. As we shall argue in Lemma 4.7, this relation is not accidental.

Of a similar nature is the following counterpart to Theorem 4.3.

Theorem 4.6. *Let $\mathbf{f} \subset R$ be a set of monomials of degree 2. Then the polar syzygy module $\mathcal{P}$ of $\mathbf{f}$ is generated by the vectors $\tau_{\mathfrak{w}}$, for all non-split even closed walks $\mathfrak{w}$ of length ≥ 4 of the graph $\mathcal{G}(\mathbf{f})$.*

Proof. Since there is no particular claim about a minimal set of generators, it will suffice to argue that: (1) the presentation ideal P as above is generated by the polynomials $p_{\mathfrak{w}}$, for all even closed walks $\mathfrak{w}$ of the graph $\mathcal{G}(\mathbf{f})$; (2) if an even closed walk $\mathfrak{w}$ splits into smaller cycles $\mathfrak{w}_1$ and $\mathfrak{w}_2$, then the corresponding polynomial $p_{\mathfrak{w}}$ is superfluous in the sense that it belongs to the subideal generated by the polynomials $p_{\mathfrak{w}_1}$ and $p_{\mathfrak{w}_2}$.

We deal with the second claim first as it is visible offhand. Namely, one has in the previous notation $p_{\mathfrak{w}} = (\mathbf{T}_{\mathfrak{w}_2^+})(\mathbf{T}_{\mathfrak{w}_1^+} - \mathbf{T}_{\mathfrak{w}_1^-}) + (\mathbf{T}_{\mathfrak{w}_1^-})(\mathbf{T}_{\mathfrak{w}_2^+} - \mathbf{T}_{\mathfrak{w}_2^-}) = (\mathbf{T}_{\mathfrak{w}_2^+}) p_{\mathfrak{w}_1} + (\mathbf{T}_{\mathfrak{w}_1^-}) p_{\mathfrak{w}_2}$.

As for the first claim, we note that it is [16, Proposition 8.1.2(a)] when the graph $\mathcal{G}(\mathbf{f})$ is simple. In general, if loops are taken into consideration, the same proof works with minor adaptation. Indeed, setting $\mathcal{B} := \{p_{\mathfrak{w}} \mid \mathfrak{w} \text{ is an even closed walk}\}$, one has $(\mathcal{B}) \subset P$ as already pointed out before. Denoting by P_s the part of the toric ideal P of degree s, we show by induction on $s \geq 2$ that $P_s \subset (\mathcal{B})$. Thus, let $p \in P_2$ be any binomial, say $p = T_{i_1} T_{i_2} - T_{i_3} T_{i_4}$ for $1 \leq i_1, i_2, i_3, i_4 \leq m$ with $i_1 \neq i_3, i_4$ and $i_2 \neq i_3, i_4$. At least one of the monomials $f_{i_1}, f_{i_2}, f_{i_3}, f_{i_4}$ is square free (otherwise $f_{i_1} = f_{i_3}$ or $f_{i_1} = f_{i_4}$). One can assume without loss of generality that $f_{i_1} = x_1 x_2$, that x_1 divides f_{i_3} and that x_2 divides f_{i_4}, i.e., $f_{i_3} = x_1 x_j$ and $f_{i_4} = x_2 x_k$ for $1 \leq j, k \leq n$ such that $j \neq 2$ and $k \neq 1$. If $j = 1$ and $k = 2$, then $p = p_{\mathfrak{w}}$ where $\mathfrak{w}$ is a monedge bow tie whose structural cycles are loops. If $j = 1$ (and $k \neq 2$), or $k = 2$ (and $j \neq 1$), or $j = k$ (and $j \neq 1$, $k \neq 2$), then $p = p_{\mathfrak{w}}$ where $\mathfrak{w}$ is a path-degenerate looped bow tie whose structural cycle is a 3-cycle. Finally, if $j \neq 1$, $k \neq 2$ and $j \neq k$, $p = p_{\mathfrak{w}}$ where $\mathfrak{w}$ is a 4-cycle. Thus, $P_2 \subset (\mathcal{B})$. In order to show that $P_s \subset (\mathcal{B})$ once we assume that $P_t \subset (\mathcal{B})$ for all $t < s$, we use an argument similar to the one used in loc. cit. when the graph $\mathcal{G}(\mathbf{f})$ is simple. Let $p = T_{i_1} \cdots T_{i_s} - T_{j_1} \cdots T_{j_s}$ be a binomial in P_s with $1 \leq i_1, \ldots, i_s, j_1, \ldots, j_s \leq m$. If, relabeling the generators, one has that $f_{i_1} \cdots f_{i_r} = f_{j_1} \cdots f_{j_r}$ for some $r < s$, then the relation $p = T_{i_{r+1}} \cdots T_{i_s} (T_{i_1} \cdots T_{i_r} - T_{j_1} \cdots T_{j_r}) + T_{j_1} \cdots T_{j_r} (T_{i_{r+1}} \cdots T_{i_s} - T_{j_{r+1}} \cdots T_{j_s})$ and the induction hypothesis imply that $p \in (\mathcal{B})$. Assume now that $f_{i_1} \cdots f_{i_r} \neq f_{j_1} \cdots f_{j_r}$ for all $r < s$ and any relabeling of the elements $f_{i_1}, \ldots, f_{i_s}, f_{j_1}, \ldots, f_{j_s}$. Since $f_{i_1} \cdots f_{i_s} = f_{j_1} \cdots f_{j_s}$, relabeling $f_{j_1}, \ldots f_{j_s}$ is necessary, one can assume without loss of generality that $f_{i_1} = x_{k_1} x_{l_1}$ and $f_{j_1} = x_{l_1} x_{k_2}$ for $1 \leq k_1, l_1, k_2 \leq n$ such that $k_1 \neq k_2$ (note that if f_{i_1}, respectively f_{j_1}, corresponds to a loop in $\mathcal{G}(\mathbf{f})$, then $k_1 = l_1$, respectively $l_1 = k_2$). Thus, x_{k_2} divides $f_{i_2} \cdots f_{i_s}$ and one can assume that $f_{i_2} = x_{k_2} x_{l_2}$ for some $1 \leq l_2 \leq n$. One has that $f_{i_1} f_{i_2} = x_{k_1} x_{l_2} f_{j_1}$, and hence x_{l_2} divides $f_{j_2} \cdots f_{j_s}$. At this step, one has that $f_{i_1} = x_{k_1} x_{l_1}$, $f_{j_1} = x_{l_1} x_{k_2}$, $f_{i_2} = x_{k_2} x_{l_2}$, and one can assume that $f_{j_2} = x_{l_2} x_{k_3}$ for $k_3 \neq k_1$ unless $s = 2$. Iterating the argument, one gets an even closed walk $\mathfrak{w}$ of the graph $\mathcal{G}(\mathbf{f})$ such that $p = p_{\mathfrak{w}}$. $\square$

Next we clarify the precise relation between the polar syzygy $\tau_{\mathfrak{w}}$ and its differential counterpart $\mathrm{z}_{\mathfrak{w}}$, for a given non-split even closed walk $\mathfrak{w}$.

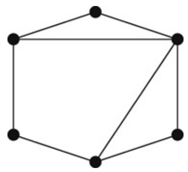


Fig. 12. Non-split even closed walk providing a superfluous differential syzygy.

Lemma 4.7. *Let $\mathfrak{w}$ denote a non-split even closed walk on a graph $\mathcal{G}(\mathbf{f})$. Then,*

$$\mathrm{T}_{\mathfrak{w}} = M\mathrm{z}_{\mathfrak{w}} ,$$

where M is the product of the repeated vertices in the closed walk obtained by removing from $\mathfrak{w}$ the loops ($M = 1$ if it has no vertex repetition). In particular, this applies to the following particular configurations:

(1) *If $\mathfrak{w}$ is either an even cycle, a path-degenerate looped bow tie, or a monedge bow tie whose structural cycles are loops, then $\mathrm{T}_{\mathfrak{w}} = \mathrm{z}_{\mathfrak{w}}$.*
(2) *If $\mathfrak{w}$ is a path-degenerate bow tie which is not looped, and if x_i is the common vertex of its two structural cycles, then $\mathrm{T}_{\mathfrak{w}} = x_i\mathrm{z}_{\mathfrak{w}}$.*
(3) *If $\mathfrak{w}$ is a bow tie which is neither path-degenerate nor a monedge bow tie whose structural cycles are loops, and if N is the product of the vertices of the structural connecting path excluding the base vertex of the structural odd cycle when the latter is a loop, then $\mathrm{T}_{\mathfrak{w}} = N\mathrm{z}_{\mathfrak{w}}$.*

Proof. Consider a non-split even closed walk $\mathfrak{w} = \{g_1, \ldots, g_{2r}\}$ on $\mathcal{G}(\mathbf{f})$. On the one hand, recall that in order to get $\mathrm{T}_{\mathfrak{w}}$ one takes $\mathbf{T}$-derivatives of the binomial $p_{\mathfrak{w}} = T_1T_3 \cdots T_{2r-1} - T_2T_4 \cdots T_{2r}$ and evaluate every T_j on the corresponding edge g_j in the edge sequence of the walk (see Definition 4.4) – as a slight check, note that the $\mathbf{T}$-degree of $p_{\mathfrak{w}}$ is r, hence the $\mathbf{x}$-degree of $\mathrm{T}_{\mathfrak{w}}$ is the even integer $2(r-1) = 2r-2$. Thus, typically, the first coordinate reads $g_3g_5 \cdots g_{2r-1}$ (respectively, $2g_1g_3g_5 \cdots \widehat{g_{2j+1}} \cdots g_{2r-1} = 2g_3g_5 \cdots g_{2r-1}$) if g_1 is not repeated (respectively, if $g_1 = g_{2j+1}$ for some $j \geq 1$). On the other hand, the least common multiple of $\{g_1, \ldots, g_{2r}\}$ is the monomial

$$g = \frac{x_{i_1} \cdots x_{i_{2r}}}{M} = \frac{g_1g_3 \cdots g_{2r-1}}{M} = \frac{g_2g_4 \cdots g_{2r}}{M}$$

and one readily obtains the required relation.

Of course, (1), (2) and (3) follow readily from the general statement. □

4.2. Minimal sets of generators

In this part we seek to squeeze down the previous slightly loose sets of generators to minimal sets of generators of the modules $\mathcal{Z}$ and $\mathcal{P}$.

First a word about sets of minimal generators of these modules. Since $\mathcal{Z}$ is the module of syzygies of the transposed Jacobian module $\mathcal{D}(\mathbf{f}) \subset \sum_{i=1}^{n} R\,\mathrm{d}x_i$ and the latter is a graded $k[\mathbf{x}]$-module with respect to the standard graded polynomial ring $k[\mathbf{x}]$, then $\mathcal{Z}$ is a graded submodule, say, $\mathcal{Z} = \oplus_{s\geq 0} \mathcal{Z}_s$. Theorem 4.3 tells us a set of generators of $\mathcal{Z}$. This set can in theory be squeezed to a minimal set of generators; for any such set of generators, the number of elements in each graded piece $\mathcal{Z}_s$ is invariant as is the highest possible degree s of an element in it. We will agree to say that an element of $\mathcal{Z}$ is *superfluous* in the sense that it does not belong to any set of minimal generators of $\mathcal{Z}$ (not just lying outside a specific such set). This is of course the counterpart to the usual notion of an absolute minimal generator z which, in our setting, reads as $z \in \mathcal{Z} \setminus (\mathbf{x})\mathcal{Z}$. Of course, a test for knowing that z is superfluous is that its degree be larger than the uniquely defined highest generation degree of the module $\mathcal{Z}$; however, we in general have no theoretic hold of this degree.

A similar phenomenon happens in $\mathcal{P}$ as the latter is in its turn a graded submodule of $\mathcal{Z}$. Here one can pretest superfluity of an element $\mathrm{T}_{\mathfrak{w}}$ in $\mathcal{P}$ by testing whether the associated binomial $p_{\mathfrak{w}}$ is a minimal generator of the defining ideal P. Unfortunately this works only in one direction in general (see Remark 4.12).

We give some examples to illustrate this order of ideas in our present setting, stressing additionally that even non-split or indecomposable walks may be (absolute) superfluous.

Example 4.8. Consider the simple graph in Fig. 12. Here the hexagon $\mathfrak{w}$ is non-split, but the corresponding $\mathrm{z}_{\mathfrak{w}}$ is deep inside the submodule generated by the vectors corresponding to the square and the path-degenerate bow tie and these two form a set of minimal generators of $\mathcal{Z}$. How do we know that $\mathrm{z}_{\mathfrak{w}}$ is (absolute) superfluous? Simply because, by definition, its degree is 4 which is larger than the generation degree 3 of $\mathcal{Z}$. As an additional remark, since $\mathrm{z}_{\mathfrak{w}} = \mathrm{T}_{\mathfrak{w}}$ for a hexagon and $\mathrm{T}_{\mathfrak{w}}$ is part of a minimal set of generators of $\mathcal{P}$, then the edges of the graph do not form a polarizable set.

Example 4.9. For an example in $\mathcal{P}$, consider the graph in Fig. 9. The even closed walk $\mathfrak{w}$ of length 6 supported by the bow tie involving the first and third loops is such that $p_{\mathfrak{w}}$ belongs to the ideal $(p_{\mathfrak{w}_1}, p_{\mathfrak{w}_2}) \subset k[\mathbf{T}]$, where $\mathfrak{w}_1, \mathfrak{w}_2$ are the two even closed walks of length 4 supported by the other two bow ties. By a previous observation above $\mathrm{T}_{\mathfrak{w}}$ is not an absolute minimal generator of $\mathcal{P}$. Note that $\mathfrak{w}$ is non-split (and even indecomposable as observed right before Lemma 3.18), and that $\mathrm{z}_{\mathfrak{w}}$ is actually a minimal generator of $\mathcal{Z}$; of course, necessarily, $\tau_{\mathfrak{w}} \neq \mathrm{z}_{\mathfrak{w}}$.

One can now improve on the result of Theorem 4.3 as a first approximation to describing a minimal generating set of the differential syzygy module $\mathcal{Z}$.

Theorem 4.10. *Keeping the previous notation, the syzygy module $\mathcal{Z}$ of $\mathcal{D}(\mathbf{f})$ is generated by the vectors $\mathrm{z}_{\mathfrak{w}}$, for all even cycles and induced bow ties $\mathfrak{w}$ of the graph $\mathcal{G}(\mathbf{f})$.*

Proof. By Theorem 4.3, $\mathcal{Z}$ is generated by the vectors $\mathrm{z}_{\mathfrak{w}}$, for all non-split even closed walks $\mathfrak{w}$ of the graph $\mathcal{G}(\mathbf{f})$ of length ≥ 4.

We first show that if $\mathfrak{w}$ is a non-split even closed walk that contains a cycle of which at least two vertices are vertex repetitions of $\mathfrak{w}$, then $\mathrm{z}_{\mathfrak{w}} \in (\mathbf{x})\mathcal{Z}$. This statement is proved using the decomposition used in the proof Lemma 3.18: in this situation one can readily check that if $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are the even closed walks introduced there, then $\mathrm{z}_{\mathfrak{w}} = \frac{g}{\ell_1}\mathrm{z}_{\mathfrak{w}_1} + \frac{g}{\ell_2}\mathrm{z}_{\mathfrak{w}_2}$ where g, ℓ_1 and ℓ_2 are the least common multiples of the monomials in the edge sequences associated to $\mathfrak{w}$, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively. Since the sets of variables involved in the vertex sequences of $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are both strictly contained in the set of variables involved in the vertex sequence of $\mathfrak{w}$, one has that $\frac{g}{\ell_1}, \frac{g}{\ell_2} \neq 1$, and hence $\mathrm{z}_{\mathfrak{w}} \in (\mathbf{x})\mathcal{Z}$.

As a consequence, following the argument in the proof of Proposition 3.19 we deduce that $\mathcal{Z}$ is at least generated by the vectors $\mathrm{z}_{\mathfrak{w}}$, for all even cycles and bow ties $\mathfrak{w}$ of the graph $\mathcal{G}(\mathbf{f})$.

To complete the proof we show that this set of generators can be further shrunk. Namely, we now show that if $\mathfrak{w}'$ is a non-induced bow tie, then the differential syzygy $\mathrm{z}_{\mathfrak{w}'}$ belongs to the submodule generated by the vectors $\mathrm{z}_{\mathfrak{w}}$ for all cycles and induced bow ties $\mathfrak{w}$ of $\mathcal{G}(\mathbf{f})$. We induct on the number of the induced edges of the graph adjacent to vertices of $\mathfrak{w}'$, off the structural edges of $\mathfrak{w}'$. If this number is zero - i.e., no additional such edges, then the bow tie is non-induced, hence the result is vacuously satisfied.

In order to apply the inductive hypothesis, refer back to the decomposition of $\mathfrak{w}'$ into two even closed walks $\mathfrak{w}_1$ and $\mathfrak{w}_2$ as in the proof of Proposition 3.20. Note that this provides a relation $\mathrm{z}_{\mathfrak{w}'} = \frac{g}{\ell_1}\mathrm{z}_{\mathfrak{w}_1} + \frac{g}{\ell_2}\mathrm{z}_{\mathfrak{w}_2}$ where g, ℓ_1 and ℓ_2 are the least common multiples of the monomials along the structural edge sequences of $\mathfrak{w}'$, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively. This holds for any of the basic ways described in Fig. 11 in which a non-induced bow tie can decompose. Now, with one single exception, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are even cycles or bow ties. The exception is when, say, $\mathfrak{w}_1$ is an even molecule (see Fig. 11, (3.b)). But then $\mathfrak{w}_2$ is an even cycle, and the molecule $\mathfrak{w}_1$ again decomposes further into a bow tie and an even cycle which is $\mathfrak{w}_2$.

Thus, in all situations, one has $\mathrm{z}_{\mathfrak{w}'} = \lambda\frac{g}{\ell_1}\mathrm{z}_{\mathfrak{w}_1} + \frac{g}{\ell_2}\mathrm{z}_{\mathfrak{w}_2}$ where $\mathfrak{w}_1, \mathfrak{w}_2$ are even cycles or bow ties, g, ℓ_1 and ℓ_2 are the least common multiples of the monomials along the structural edge sequences of $\mathfrak{w}'$, $\mathfrak{w}_1$ and $\mathfrak{w}_2$ respectively, and $\lambda = 1$ except in the basic situation (3.b) where $\lambda = 2$, $\mathfrak{w}_1$ is an even cycle and $\mathfrak{w}_2$ a bow tie.

If, say, $\mathfrak{w}_1$ is a non-induced bow tie, the number of the induced edges of the graph adjacent to vertices of $\mathfrak{w}_1$, off the structural edges of $\mathfrak{w}_1$, is strictly smaller than the analogous number corresponding to $\mathfrak{w}'$. Therefore, we can apply the inductive hypothesis and the result follows suit. $\square$

Remark 4.11. In general, one cannot replace even cycles by indecomposable even cycles in Theorem 4.10. Consider the graph whose edges $\mathbf{f}$ are those of a decagon, i.e., a 10-cycle with vertices labeled $x_1, x_2, \ldots, x_{10}$, and in addition the chords x_2x_8 and x_3x_7. A straightforward calculation shows that $\mathbf{f}$ is polarizable – see also Theorem 4.14. Moreover, the differential syzygy module is minimally generated by the 4-cycle $\{x_2, x_8, x_7, x_3, x_2\}$ and the entire 10-cycle. To be in conformity with the result of Theorem 4.10, note that the monedge bow tie whose structural odd cycles are both of length 5 and whose structural path is the edge x_2x_3 is decomposable – using as decomposing set the edge x_7x_8 as in Fig. 11, (1). On the other hand, the 10-cycle is decomposable with decomposing set the edges x_2x_8, x_2x_3 and x_3x_7, by which it disassembles into the 4-cycle and the monedge bow tie.

Remark 4.12. A point that would require further clarification is a criterion for the inequality $\mu(\mathcal{P}) \leq \mu(P)$ to be an equality. An example where a decomposable even closed walk provides a superfluous generator of $\mathcal{P}$ while the binomial $p_{\mathfrak{w}}$ is a non-superfluous generator of the presentation ideal P of $k[\mathbf{f}]$ is illustrated by the graph of Example 5.21. In this example $\mathbf{f}$ is polarizable. The cycle arrangement in [16, Ex. 8.4.14] provides us with the same phenomenon and is moreover non-polarizable.

The following result soups-up the previous result by capturing a class of even closed walks $\mathfrak{w}$ whose associated syzygies $\tau_{\mathfrak{w}}$ (respectively, $\mathrm{z}_{\mathfrak{w}}$) are part of a minimal set of generators of $\mathcal{P}$ (respectively, $\mathcal{Z}$).

Lemma 4.13. *If $\mathfrak{w}$ is an induced bow tie on a graph $\mathcal{G}(\mathbf{f})$, then the associated syzygy $\tau_{\mathfrak{w}}$ (respectively, $\mathrm{z}_{\mathfrak{w}}$) is part of a minimal set of generators of $\mathcal{P}$ (respectively, $\mathcal{Z}$).*

Proof. By Theorem 4.6, $\mathcal{P}$ is generated by the set of syzygies $\tau_{\mathfrak{w}}$ where $\mathfrak{w}$ runs through the set of non-split even closed walks. Since $\mathcal{P}$ is a graded $k[\mathbf{x}]$-module, this set contains a subset $\mathcal{M}$ forming a minimal set of generators of $\mathcal{P}$. We claim that if $\mathfrak{w}$ is a non-induced bow tie in $\mathcal{G}(\mathbf{f})$ then $\mathfrak{w} \in \mathcal{M}$.

One can assume without loss of generality that the first monomial f_1 in $\mathbf{f}$ corresponds to the first edge in the edge sequence of $\mathfrak{w}$; in particular, the first coordinate of the vector $\tau_{\mathfrak{w}}$ is nonzero.

Suppose then that $\mathfrak{w} \notin \mathcal{M}$. Write $\mathfrak{w}$ as a $k[\mathbf{x}]$-linear combination of $\tau_{\mathfrak{w}_1}, \dots, \tau_{\mathfrak{w}_\ell}$, where $\mathfrak{w}_1, \dots, \mathfrak{w}_\ell \in \mathcal{M}$. Then, f_1 belongs to the edge sequence of at least one of those even closed walks, say $\mathfrak{w}_1 = \{f_1, g_2, \dots, g_{2r}\}$ with $g_2, \dots, g_{2r} \in \mathbf{f}$, and the first coordinate of $\tau_{\mathfrak{w}_1}$ divides the first coordinate of $\tau_{\mathfrak{w}}$.

Now, since $\mathfrak{w}_1$ does not coincide with $\mathfrak{w}$ because we are assuming that $\mathfrak{w} \notin \mathcal{M}$ and since $\mathfrak{w}$ does not contain any proper even closed subwalk because it is a bow tie, it follows that at least one of the monomials in the edge sequence of $\mathfrak{w}_1$, say g_i for some $i \in \{2, \dots, 2r\}$, does not belong to the edge sequence of $\mathfrak{w}$. If $g_i = x_jx_k \in \mathbf{f}$ for $1 \leq j \leq k \leq n$, we claim that both x_j and x_k belong to the vertex sequence of $\mathfrak{w}$. If x_j does not belong to the vertex sequence of $\mathfrak{w}$, x_j does not divide any of the nonzero coordinates of $\tau_{\mathfrak{w}}$, in particular it does not divide its first coordinate. On the other hand, recall that the first coordinate of $z_{\mathfrak{w}_1}$ is $\frac{g}{f_1}$ where g stands for the least common multiple of $f_1, g_2, \dots, g_{2r}$, and hence x_j divides the first coordinate of the vector $z_{\mathfrak{w}_1}$ (it divides g_i and does not divide f_1 because it does not belong to the vertex sequence of $\mathfrak{w}$). By Lemma 4.7, $\tau_{\mathfrak{w}_1} = M z_{\mathfrak{w}_1}$ for some monomial $M \in R$, and hence x_j divides the first coordinate of $\tau_{\mathfrak{w}_1}$ which in turn divides the first coordinate of the vector $\tau_{\mathfrak{w}}$, a contradiction.

We have thus shown that there exists an edge $x_jx_k \in \mathbf{f}$, not belonging to the edge sequence of $\mathfrak{w}$, and such that both x_j and x_k belong to the vertex sequence of $\mathfrak{w}$. Therefore the bow tie $\mathfrak{w}$ is non-induced. This wraps up the proof for a polar syzygy $\tau_{\mathfrak{w}}$.

The proof for $z_{\mathfrak{w}}$ is similar by drawing upon a set of generators of $\mathcal{Z}$ such as given in Theorem 4.3. □

We can now give a complete combinatorial characterization of polarizability.

Theorem 4.14. *A set $\mathbf{f} \subset R$ of monomials of degree 2 is polarizable if and only if every induced bow tie of the associated graph $\mathcal{G}(\mathbf{f})$ is one of the following:*

(1) *A monedge bow tie whose structural cycles are loops;*
(2) *A path-degenerate looped bow tie.*

In particular, if $\mathbf{f}$ consists only of squarefree monomials – i.e., if the graph $\mathcal{G}(\mathbf{f})$ is simple – then $\mathbf{f}$ is polarizable if and only if $\mathcal{G}(\mathbf{f})$ does not have any induced bow tie.

Proof. By Lemma 4.7, a bow tie supporting an even closed walk $\mathfrak{w}$ satisfies $\tau_{\mathfrak{w}} = z_{\mathfrak{w}}$ if and only if it is one of the types (1) or (2) in the present statement.

Now assume that the only induced bow ties in $\mathcal{G}(\mathbf{f})$ are of these types. Then any generator $z_{\mathfrak{w}}$ of $\mathcal{Z}$ as in Theorem 4.10 belongs to the polar syzygy module $\mathcal{P}$, hence $\mathbf{f}$ is polarizable.

Conversely, let $\mathfrak{w}$ be an even closed walk in $\mathcal{G}(\mathbf{f})$ supported by an induced bow tie. Again if $\mathfrak{w}$ is neither of the two types in the statement then $\tau_{\mathfrak{w}} = M z_{\mathfrak{w}}$ for some monomial $M \neq 1$. By Lemma 4.13, $\tau_{\mathfrak{w}}$ is part of a minimal set of generators of $\mathcal{P}$, hence $\tau_{\mathfrak{w}} \notin (\mathbf{x})\mathcal{P}$. Therefore we must conclude that $z_{\mathfrak{w}} \notin \mathcal{P}$, hence $\mathbf{f}$ is not polarizable. □

5. Applications

5.1. Veronese, squarefree Veronese, bipartite

Corollary 5.1. *Let $\mathbf{f} \subset R$ be either the set of all monomials of degree* 2*, or the set of all squarefree monomials of degree* 2*. Then $\mathbf{f}$ is polarizable.*

Proof. In both cases, the result is a direct consequence of Theorem 4.14. We first treat the squarefree case. The corresponding graph is a complete simple graph (no loops). In particular it has no induced bow ties as any induced subgraph of a complete graph is itself complete. As for the 2-Veronese embedding, the corresponding graph is a complete graph with a loop based at every vertex. This clearly forces any induced bow tie to be either a triangle with a loop based at one of its vertices or two loops connected by an edge. □

Another consequence is a more conceptual proof of one of the main results of [10].

Corollary 5.2. *Let $\mathcal{G}(\mathbf{f})$ denote a connected bipartite graph on edges $\mathbf{f}$. Then $\mathbf{f}$ is polarizable.*

Proof. It follows from Theorem 4.14 because $\mathcal{G}(\mathbf{f})$ has no odd cycle by [16, Prop. 6.1.1] and hence has no bow ties. □

Connected bipartite graphs admit various characterizations in the graph literature and also in algebraic combinatorics (see, e.g., [12]). We will next give yet another characterization based solely on the underlying edge-algebra. We say that a graph with loops is connected if the underlying graph removing all loops is a connected simple graph.

Let $A = k[\mathbf{f}] \subset R$ with $\mathbf{f}$ distinct monomials of degree 2 in $n \geq 2$ variables. To any two variables x_i, x_j with $i \neq j$ we associate the k-algebra surjection

$$\pi_{i,j} : R \twoheadrightarrow S = k[x_1, \dots, \widehat{x_j}, \dots, x_n]$$
$$\pi_{i,j}(x_k) = x_k \quad (k \neq j)$$
$$\pi_{i,j}(x_j) = x_i.$$

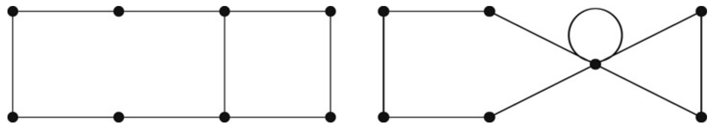


Fig. 13. Edge-pinching a chordal even cycle.

Clearly, $\ker(\pi_{i,j}) = (x_j - x_i)$. Set $B := \pi_{i,j}(A) \subset S$ for the image of the restriction of this map to the k-subalgebra A. Then B is generated by the images of $\mathbf{f}$, hence is still generated by monomials $\mathbf{f}'$ of degree 2.

If $\mathcal{G} = \mathcal{G}(\mathbf{f})$ and $\mathcal{G}' = \mathcal{G}(\mathbf{f}')$ denote the respective associated graphs (with loops) then we say that the corresponding graph-theoretic process is an *edge-pinching operation* (see [15, Corollary 4.9] where this notion has been considered in a special case).

Proposition 5.3. *Let $\mathcal{G}$ be a connected graph, possibly with loops, and let $A \subset k[x_1, \ldots, x_n]$ ($n \geq 2$) denote its associated edge-algebra. Let $\mathcal{G}'$ denote the graph obtained by an edge-pinching operation on any proper edge (i.e., not a loop). Then $\mathcal{G}$ is bipartite (in particular, has no loops) if and only if the corresponding restriction map $A \longrightarrow S$ is injective.*

We need the following technical result.

Lemma 5.4. *Let $\mathcal{G}$ be a connected graph with n vertices having at least one loop, and let $A \subset R = k[x_1, \ldots, x_n]$ denote its associated edge-algebra. Then $\dim A = n$.*

Proof. Fix a loop, say, $x_1^2 \in A$. Let $\widetilde{\mathcal{G}}$ denote the graph obtained from $\mathcal{G}$ by keeping all vertices and removing the loop x_1^2. Clearly, $\widetilde{\mathcal{G}}$ is still connected. If it still contains a loop we are done by induction on the total number of edges and loops. Thus, we may assume that $\widetilde{\mathcal{G}}$ has no loops. If $\widetilde{\mathcal{G}}$ is not bipartite then its associated edge-ideal has dimension n, hence so does A. If $\widetilde{\mathcal{G}}$ is bipartite, its log-matrix M has rank $n-1$. Therefore, by adding further a column $(0, 1, \ldots, 0)^t$ increases the rank of M by one, hence the log-matrix of $\mathcal{G}$ is at least n, as required. □

Proof of the proposition. Suppose first that $\pi_{|A} : A \longrightarrow S$ is injective. In particular π does not collapse two distinct generators (edges) of A, hence the images of the generators are all distinct and correspond to a graph with at least one loop (e.g., x_{n-1}^2) whose associated edge-algebra is $\pi(A)$. Clearly, this graph is still connected. By Lemma 5.4, $\dim \pi(A) = n-1$. But then $\dim A = n-1$ as well. In particular, again by Lemma 5.4, $\widetilde{\mathcal{G}}$ has no loops, hence must be a bipartite graph.

Conversely, if $\mathcal{G}$ is bipartite then $\dim A = n-1$. Once more by Lemma 5.4, $\dim \pi(A) = n-1$. But then the restriction $\pi_{|A} : A \twoheadrightarrow \pi(A)$ must have kernel zero since A is a domain. □

The following result shows that polarizability depends on the chosen embedding $A \subset R$, hence is not an invariant property of the algebra A.

Proposition 5.5. *Let $\mathcal{G}$ be a graph (or an induced subgraph) consisting of an even cycle with one single chord inducing a decomposition in two smaller even cycles. Then the graph $\mathcal{G}'$ obtained by pinching the chord (see Fig. 13) is not polarizable.*

Proof. By edge-pinching we have created an induced path-degenerate bow tie whose structural cycles are not loops and the result follows from Theorem 4.14. □

Corollary 5.6. *Polarizability is not an invariant property of the algebra A.*

Proof. Consider $\mathbf{f}$ such that the graph $\mathcal{G}(\mathbf{f})$ is an even cycle with one single chord inducing a decomposition in two smaller even cycles and $\mathbf{f}'$ whose associated graph $\mathcal{G}(\mathbf{f}')$ is obtained by edge-pinching the chord of $\mathcal{G}(\mathbf{f})$. Then $\mathbf{f}$ is polarizable by Theorem 4.14 while $\mathbf{f}'$ is not polarizable by Proposition 5.5. Nevertheless, $k[\mathbf{f}] \simeq k[\mathbf{f}']$ (the defining ideals of both k-subalgebras coincide). □

Remark 5.7. The actual reason why polarizability is not an invariant property of the algebra A is that a k-algebra isomorphism may not preserve certain crucial configurations of the corresponding graph. Thus, e.g., in Fig. 13 the path-degenerate bow tie in the right most graph, whose odd cycles are a pentagon and a triangle, is not preserved under the above isomorphism of algebras.

5.2. Polarizability versus normality

Recall the notion of a cohesive set of monomials.

Definition 5.8 (*[15, Definition 4.2]*). The set $\mathbf{f}$ is said to be *cohesive* if there is no partition $\mathbf{x} = \mathbf{y} \cup \mathbf{z}$ of the variables such that $\mathbf{f} = \mathbf{g} \cup \mathbf{h}$, where the monomials in the set $\mathbf{g}$, resp. $\mathbf{h}$, involve only the $\mathbf{y}$-variables, resp. $\mathbf{z}$-variables.

One clearly has that $\mathbf{f}$ is cohesive if and only if $\mathcal{G}(\mathbf{f})$ is connected. The following characterization for the normality of $k[\mathbf{f}]$ has essentially been obtained (independently) in [13] and [9].

Proposition 5.9. *Let $A = k[\mathbf{f}] \subset R$ be generated by a cohesive set $\mathbf{f}$ of monomials of degree 2 and let $\mathcal{G}(\mathbf{f})$ denote the corresponding graph. The following conditions are equivalent:*

(1) *A is integrally closed;*
(2) *$\mathcal{G}(\mathbf{f})$ satisfies the so-called odd cycle condition, i.e., for any two odd cycles which are induced (i.e., no chords) in $\mathcal{G}(\mathbf{f})$ and have mutually disjoint vertex sets, there exists an edge of $\mathcal{G}(\mathbf{f})$ joining a vertex of one cycle to a vertex of the other.*
(3) *Any induced bow tie of $\mathcal{G}(\mathbf{f})$ is either a path-degenerate bow tie or a monedge bow tie (possibly including the respective looped versions).*

Proof. (1) $\Leftrightarrow$(2) is (i)$\Leftrightarrow$ (iii) in [9, Corollary 2.3].

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) Given two odd cycles as stated – called for convenience *non-chordal* – there must be a path connecting the two since we are assuming that $\mathcal{G}(\mathbf{f})$ is connected. This yields a bow tie $\mathcal{B}$ in the graph, and we may assume that $\mathcal{B}$ has a connecting path of smallest length ℓ among all bow ties in the graph whose structural odd cycles are non-chordal. Assume, as if it were, that $\ell \geq 2$. If $\mathcal{B}$ is induced, it would be a contradiction to (3). If it is not induced, let e be an edge between two vertices of $\mathcal{B}$. Since the two odd cycles are non-chordal, e must connect vertices across the two cycles or across a cycle and the path. In the first case, we are done, while the second case is ruled out as it implies a new bow tie with non-chordal cycles such that e is an edge of one of the cycles and admitting a connecting path of length $\leq \ell - 1$. $\square$

The next result explains the precise relationship between the notions of polarizability and normality.

Theorem 5.10. *Let $A = k[\mathbf{f}] \subset R$ be generated by a cohesive set $\mathbf{f}$ of monomials of degree 2 and let $\mathcal{G}(\mathbf{f})$ denote the corresponding graph.*

(i) *If $\mathbf{f}$ is polarizable then A is integrally closed (hence, a Cohen–Macaulay ring).*
(ii) *Conversely, suppose that $\mathcal{G}(\mathbf{f})$ has no configuration of the following kinds:*
  (a) *Induced monedge bow ties (with neither odd cycle degenerating into a loop) ;*
  (b) *Induced monedge looped bow ties (with only one odd cycle degenerating into a loop) ;*
  (c) *Induced path-degenerate bow ties (with neither odd cycle degenerating into a loop).*

If A is integrally closed then $\mathbf{f}$ is polarizable.

Proof. It follows from Proposition 5.9 and Theorem 4.14. $\square$

Remark 5.11. Note that the above result does not conflict with the result of Corollary 5.6 (see also Remark 5.7).

The following consequence for algebras of Veronese type of degree 2 could have been given before with slightly more effort, but having it here stresses the normality of these algebras. Recall that, given an integer $d \geq 1$ and a sequence of integers $1 \leq s_1 \leq \cdots \leq s_n \leq d$, the k-subalgebra $A \subset R$ generated by the set of monomials

$$F = \{x^{a_1} \cdots x^{a_n} \mid a_1 + \cdots + a_n = d;\ 0 \leq a_i \leq s_i \quad \forall\, i\}$$

is called the *algebra of Veronese type* of degree d subordinate to the vector $(s_1, \ldots, s_n)$. These algebras form a subclass of the class of the polymatroidal algebras of maximal rank (see [8], [7]). In the next subsection we will actually show that *all* polymatroidal algebras of degree 2 are polarizable.

Corollary 5.12. *If $\mathbf{f}$ are the defining generators of an algebra $A \subset R$ of Veronese type of degree 2 then $\mathbf{f}$ is polarizable.*

Proof. It is known that A is normal (cf., e.g., [8]; see also [13]). On the other hand, since $d = 2$ the relevant subordinating vectors have $s_i \leq 2$ for all i. It follows that $\mathbf{f}$ consists of all squarefree monomials of degree 2 and possibly some pure powers. It is then self-evident that the associated graph does not admit any induced path-degenerate or monedge bow ties except eventually looped-triangles or two loops joined with an edge. By Theorem 5.10, (ii), $\mathbf{f}$ is polarizable. $\square$

Corollary 5.13. *Let $F : \mathbb{P}^{n-1} - - \to \mathbb{P}^{m-1}$ be a rational map defined by a cohesive set $\mathbf{f}$ of distinct monomials of degree 2. If $\dim k[\mathbf{f}] = n$ and $\mathbf{f}$ is polarizable then F maps $\mathbb{P}^{n-1}$ birationally onto its image. In particular, $k[\mathbf{f}]$ is a rational singularity.*

Proof. First observe that the claim on birationality is equivalent to saying that the ring extension $k[\mathbf{f}] \subset k[(\mathbf{x})_2]$ (2-Veronese) is birational (see, e.g., [14, Proof of Proposition 2.1]). Thus, if for some subset $\mathbf{f}' \subset \mathbf{f}$ the corresponding rational map is birational onto its image then so will be the one defined by $\mathbf{f}$. Let us choose $\mathbf{f}'$ to be the subset of the squarefree monomials in $\mathbf{f}$.

Now, on the one hand Theorem 5.10, (i), implies that $k[\mathbf{f}']$ is normal, while on the other hand, the normality of the squarefree $k[\mathbf{f}']$ is equivalent to the normality of the ideal $(\mathbf{f}')$ in this case (see [16, Corollary 8.7.13]). Therefore, by [14, Proposition 3.1] the extension $k[\mathbf{f}'] \subset k[(\mathbf{x})_2]$ is birational, as required. $\square$

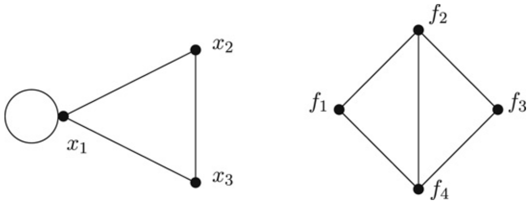


Fig. 14. A graph and its edge graph.

5.3. Polarizability versus linear presentation

We deal here with the case in which $\mathbf{f}$ is linearly presented, i.e. when its module of first syzygies is generated by linear ones. We characterize this property in terms of the diameter of a graph (Lemma 5.16) and show that if $\mathbf{f}$ is linearly presented then it is polarizable (Proposition 5.18).

In order to characterize when $\mathbf{f}$ is linearly presented, we introduce the *edge graph* of $\mathcal{G}(\mathbf{f})$, denoted $\mathcal{L}(\mathbf{f})$ (see [16, Definition 6.6.1]): its vertex set is the set of edges of $\mathcal{G}(\mathbf{f})$, hence can be viewed as the elements of $\mathbf{f}$; two vertices f_i and f_j of $\mathcal{L}(\mathbf{f})$ are adjacent (i.e., form an edge) if and only if f_i and f_j have a common variable (i.e., $\gcd(f_i, f_j) \neq 1$). Observe that the graph $\mathcal{L}(\mathbf{f})$ is always a simple graph (no loops) and that $\mathbf{f}$ is cohesive (see Definition 5.8 previously recalled) if and only if $\mathcal{G}(\mathbf{f})$ is connected, if and only if $\mathcal{L}(\mathbf{f})$ is connected.

Example 5.14. For $\mathbf{f} = \{f_1, \ldots, f_4\}$ with $f_1 = x_1^2, f_2 = x_1x_2, f_3 = x_2x_3$ and $f_4 = x_1x_3$, the graphs $\mathcal{G}(\mathbf{f})$ and $\mathcal{L}(\mathbf{f})$ are given in Fig. 14.

As observed in [15, Lemma 4.1], the lack of cohesiveness is an obstruction for the existence of enough linear syzygies. In the situation we focus on in this section, it is thus natural to assume that $\mathbf{f}$ is cohesive, i.e., that $\mathcal{G}(\mathbf{f})$ and $\mathcal{L}(\mathbf{f})$ are both connected graphs.

Definition 5.15. Given a simple connected graph $\mathcal{G}$, the *distance* between two vertices of $\mathcal{G}$ is the minimum length of a path connecting them, and the *diameter* of $\mathcal{G}$ is the longest distance (i.e., the longest shortest path) between any two of its vertices.

Lemma 5.16. *Assume that $\mathbf{f}$ is cohesive. Then, the ideal $I = (\mathbf{f}) \subset R$ is linearly presented if and only if the graph $\mathcal{L}(\mathbf{f})$ is of diameter ≤ 2.*

Proof. Recall that $\mathbf{f} = \{f_1, \ldots, f_m\}$, denote by $\{e_1, \ldots, e_m\}$ the canonical basis of the free module R^m, and set

$$s_{ij} := \frac{f_j}{\gcd(f_i, f_j)}\, e_i - \frac{f_i}{\gcd(f_i, f_j)}\, e_j \in R^m,$$

for $i, j \in \{1, \ldots, m\}$, $i \neq j$. It is well known (see, e.g., [2, Chapter 5, Thm. 3.2]) that the first syzygy module of I is generated by the set $\mathcal{S}(\mathbf{f}) := \{s_{ij} \mid 1 \leq i < j \leq m\}$. Consider the partition $\mathcal{S}(\mathbf{f}) = \mathcal{LS}(\mathbf{f}) \cup \mathcal{KS}(\mathbf{f})$ where

$$\mathcal{LS}(\mathbf{f}) := \{s_{ij} \mid 1 \leq i < j \leq m, \gcd(f_i, f_j) \neq 1\}$$

and

$$\mathcal{KS}(\mathbf{f}) := \{s_{ij} \mid 1 \leq i < j \leq m, \gcd(f_i, f_j) = 1\}.$$

The syzygies s_{ij} in $\mathcal{LS}(\mathbf{f})$ are linear, and the ones in $\mathcal{KS}(\mathbf{f})$ are Koszul syzygies since $s_{ij} = f_je_i - f_ie_j$ if $\gcd(f_i, f_j) = 1$. The ideal $I = (\mathbf{f})$ has linear syzygies if and only if $\mathcal{KS}(\mathbf{f})$ is contained in the submodule of R^m generated by $\mathcal{LS}(\mathbf{f})$.

First observe that the diameter of the graph $\mathcal{L}(\mathbf{f})$ is 1 (i.e., the graph $\mathcal{L}(\mathbf{f})$ is complete) if and only if $\mathcal{KS}(\mathbf{f}) = \emptyset$. More precisely, for all i, j, $1 \leq i < j \leq m$, one has that the distance between the vertices f_i and f_j of $\mathcal{L}(\mathbf{f})$ is 1 if and only if $s_{ij} \in \mathcal{LS}(\mathbf{f})$.

The result will follow if one shows that if $\mathcal{KS}(\mathbf{f}) \neq \emptyset$ then, for any i, j such that $s_{ij} \in \mathcal{KS}(\mathbf{f})$, the syzygy s_{ij} belongs to the submodule generated by $\mathcal{LS}(\mathbf{f})$ if and only if the distance between the vertices f_i and f_j of $\mathcal{L}(\mathbf{f})$ is 2.

Thus, suppose $\mathcal{KS}(\mathbf{f}) \neq \emptyset$ and let $g \in \mathcal{KS}(\mathbf{f})$. One can assume, without loss of generality, that $g = s_{12}$ and, relabeling the variables if necessary, that $f_1 = x_1x_i$ and $f_2 = x_jx_n$ for some $i \in \{1, \ldots, n-1\}$ and $j \in \{2, \ldots, n\}$ such that $i \neq j$. Then, $g = x_jx_ne_1 - x_1x_ie_2$. If g belongs to the submodule generated by $\mathcal{LS}(\mathbf{f})$, then there exists at least one element in $\mathcal{LS}(\mathbf{f})$ such that one of its two nonzero entries is either x_je_1 or x_ne_1. This implies that either $x_jx_1 \in \mathbf{f}$, or $x_jx_i \in \mathbf{f}$, or $x_nx_1 \in \mathbf{f}$, or $x_nx_i \in \mathbf{f}$, and hence, the distance between the vertices f_1 and f_2 of $\mathcal{L}(\mathbf{f})$ is 2. Conversely, if the distance between the vertices f_1 and f_2 of $\mathcal{L}(\mathbf{f})$ is 2, one has that either $x_jx_1 \in \mathbf{f}$, or $x_jx_i \in \mathbf{f}$, or $x_nx_1 \in \mathbf{f}$, or $x_nx_i \in \mathbf{f}$. Assume for example that $f_3 = x_jx_1$. Then, $s_{13} = x_je_1 - x_ie_3$ and $s_{23} = x_1e_2 - x_ne_3$ are elements in $\mathcal{LS}(\mathbf{f})$, and since $g = x_ns_{13} - x_is_{23}$, we are through. $\square$

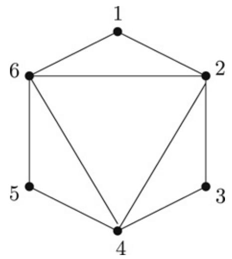


Fig. 15. Linearly presented with cubic relations

Remark 5.17. There is another kind of complementary configuration to a given simple graph $\mathcal{G}(\mathbf{f})$ called the *complement* of $\mathcal{G}(\mathbf{f})$, denoted $\overline{\mathcal{G}(\mathbf{f})}$: it has the same vertex set as $\mathcal{G}(\mathbf{f})$, and the edges are those edges of the complete simple graph on the same vertex set which are not edges of $\mathcal{G}(\mathbf{f})$ (see [16, p. 175]).

Fröberg [5] proved that the ideal $I = (\mathbf{f}) \subset R$ generated by a set $\mathbf{f}$ of squarefree monomials of degree 2 has a linear resolution if and only if the graph $\overline{\mathcal{G}(\mathbf{f})}$ is chordal, i.e., has no induced cycles of length ≥ 4. This result is related to Lemma 5.16 in the following way: if $\mathbf{f}$ is a set of squarefree monomials of degree 2, the graph $\mathcal{L}(\mathbf{f})$ has diameter ≤ 2 if and only if the graph $\overline{\mathcal{G}(\mathbf{f})}$ has no induced 4-cycles. Thus, for simple graphs Lemma 5.16 reproves a piece of Fröberg's result. Actually, there is a refinement of Fröberg's result in [3, Theorem 2.1] which we regrettably have been unaware of. Using it together with [3, Proposition 2.3], one can recover Lemma 5.16. Since the above proof is straightforward and elementary, we decided to keep it (see also [4] for yet another approach).

We can now prove the following fundamental connection between linear presentation and polarizability.

Proposition 5.18. *If the ideal $I = (\mathbf{f}) \subset R$ generated by a set $\mathbf{f}$ of monomials of degree 2 is linearly presented then $\mathbf{f}$ is polarizable.*

Proof. By the characterization in Lemma 5.16, if $I = (\mathbf{f}) \subset R$ is linearly presented, the induced odd cycles (with no chord) in $\mathcal{G}(\mathbf{f})$ (if any) are loops and triangles. Moreover, the induced bow ties in $\mathcal{G}(\mathbf{f})$ (if any) are two loops joined with an edge or a triangle with a loop centered in one of its vertices. By Theorem 4.14, $\mathbf{f}$ is polarizable. $\square$

Corollary 5.19. *If $\mathbf{f}$ is a polymatroidal set of monomials of degree 2 then $\mathbf{f}$ is polarizable.*

Proof. By [1], if $\mathbf{f}$ is ordered in the reverse lexicographic order, then it has linear quotients, i.e., the ideals $(f_1, \ldots, f_{i-1}) : f_i$ are generated by a set of variables, for every $1 \leq i \leq m$. It is self-evident that having linear quotients entice linear presentation, hence the result follows from Proposition 5.18. $\square$

In a curious roundabout fashion we recover [15, Corollary 3.8]:

Corollary 5.20. *Let $F : \mathbb{P}^{n-1} - - \to \mathbb{P}^{m-1}$ be a rational map defined by a cohesive set $\mathbf{f}$ of distinct monomials of degree 2. If $\dim k[\mathbf{f}] = n$ and $(\mathbf{f}) \subset R$ is linearly presented then F maps $\mathbb{P}^{n-1}$ birationally onto its image. In particular, $k[\mathbf{f}]$ is a rational singularity.*

Proof. It follows immediately from Proposition 5.18 and Corollary 5.13. $\square$

We end with a couple of remarks and an example.

Namely, let again $k[\mathbf{T}]/P \simeq k[\mathbf{f}]$. If P happens to be generated by sole quadrics then a minimal set of generators of the polar syzygy module $\mathcal{P}$ is automatically a minimal subset of generators of the differential syzygy module $\mathcal{Z}$. This is of course a favorable situation which one would like to understand better. If the ideal $(\mathbf{f}) \subset R$ is linearly presented then P has "many" quadrics, but still may require generators of higher degrees. In fact, these degrees may be arbitrarily high as the following example shows.

Example 5.21. Consider a complete graph (no loops) with $t \geq 3$ vertices. Mark one of the t-cycles of the graph as the "bounding cycle". For each pair of consecutive vertices v_1, v_2 of the bounding cycle introduce a new vertex v and new edges vv_1 and vv_2. In this way we have constructed a graph on $n = 2t$ vertices equipped with a new bounding n-cycle. It is easy to see that the diameter of the new graph is still ≤ 2, hence its edges correspond to a set $\mathbf{f}$ that is linearly presented, hence polarizable by Proposition 5.18. However the new bounding cycle induces an element of P of degree t that is not contained in the ideal generated by the quadrics in P. The reason it does not induce an extra minimal generator at the level of $\mathcal{Z}$ (or, which is the same, of $\mathcal{P}$) is that it is decomposed by the internal chords of the new bounding cycle. The simplest case ($t = 3$) is depicted in Fig. 15.

A question also naturally arises as to what is the impact on polarizability of $\mathbf{f}$ if the presentation ideal P is actually fully generated in degree 2. Easy examples show that, in general, $\mathbf{f}$ may not be polarizable. However, these examples are such that the ideal $P \subset k[\mathbf{T}]$ is not itself linearly presented. Thus it seems reasonable to pose:

Question 5.22. Suppose that P is generated by quadrics and is linearly presented. Is $\mathbf{f}$ polarizable? More strongly, is $\mathbf{f}$ linearly presented as well?

A special important class of algebras satisfying these hypotheses are the Koszul algebras $A = k[\mathbf{T}]/P$, which are generated by quadrics and have linear resolution.

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