A Necessary and Sufficient Condition for M-Matrices and Its Relation to Block LU Factorization

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ABSTRACT

We present a necessary and sufficient condition for M-matrices in terms of a special diagonal dominance. Then we use the new result to show that if the block comparison matrix of a block matrix $A$ is an M-matrix, there exists a block permutation matrix $P$ such that block LU factorization applied to $A = P^TAP$ is stable—i.e., the norms of the block multipliers $-A_{i,k}^{(k+1)}A_{k,k}^{(k-1)}$ are bounded by 1. We also present a collection of tools in the literature related to the subject matter. We define incomplete M-matrices, prove a necessary and sufficient condition for such matrices, and present their implications for block LU factorization.

1. INTRODUCTION

Let $A = (A_{i,j})$ be a block matrix with an $N \times N$ block structure and with nonsingular diagonal blocks. We define $N \times N$ real matrix $B(A) = (b_{i,j})$ such that $b_{i,j} = -\|A_{i,j}\|$ for $i \neq j$ and $b_{i,i} = (\|A_{i,i}^{-1}\|)^{-1}$. Here $B(A)$ is referred to as the block comparison matrix of the block matrix $A$ [13].

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have studied the block comparison matrix of block matrices in order to understand the stability of block LU factorization and fortuitously found a necessary and sufficient condition for M-matrices in terms of a special diagonal dominance, presented in Section 2. In the same section, we also present a class of \( n \times n \) matrices which we call incomplete M-matrices by \( r \), \( r < n \), and prove a necessary and sufficient condition for such matrices. In Section 3, we present a collection of tools in the literature related to M-matrices, block matrices, and block LU factorization. In Section 4 we prove that there exists a stable block LU factorization for block matrices whose block comparison matrices are M-matrices.

2. M-MATRICES AND INCOMPLETE M-MATRICES

We define M-matrices and strict diagonal dominance in the classical sense [14, 15]:

**Definition 1 (M-matrix [14, 15]).** A real matrix \( A = (a_{i,j}) \) with \( a_{i,j} < 0 \) for \( i \neq j \) is an M-matrix if and only if \( A^{-1} \) exists and has all its entries nonnegative.

**Definition 2 (Strict diagonal dominance).** An \( n \times n \) matrix \( A \) is strictly diagonally dominant if and only if

\[
|a_{i,i}| > \sum_{i=1, i \neq j}^{N} |a_{i,j}|
\]

for \( i = 1, \ldots, n \). If the inequality (2.1) is only true for the index \( i \), we say row \( i \) of the matrix \( A \) is strictly diagonally dominant.

Demmel et al. [4] refer to the inequality (2.1) as strict diagonal dominance by row. If the indices of the off-diagonal terms of this inequality are interchanged, the inequality defines strict diagonal dominance by column.

We state without proof some pertinent classical results:

**Fact 1.**

(1) Given a real matrix \( A \) with \( a_{i,j} \leq 0 \) for \( i \neq j \), the following two statements are equivalent:

- \( A \) is an M-matrix.
- There exists a vector \( g \) with positive entries (we write \( g > 0 \)) such that \( Ag > 0 \) [15].
(2) If $A$ is an $M$-matrix, all its principal submatrices are $M$-matrices and their determinants are greater than zero [15].

(3) If a real matrix $A$ with $a_{i,j} \leq 0$ for $i \neq j$ is also irreducible and strictly diagonally dominant, then $A$ is an $M$-matrix [14].

In spite of the long history of $M$-matrices as research material and their popularity as a textbook topic, we have not seen the following result published in textbooks or in well-known journals:

**Theorem 1.** A real, irreducible $n \times n$ matrix $\bar{A} = (\bar{a}_{i,j})$ with $\bar{a}_{i,j} \leq 0$ for $i \neq j$ and $\bar{a}_{i,i} > 0$ is an $M$-matrix if and only if there exists a permutation matrix $P$ such that $P^T \bar{A} P = A = (a_{i,j})$ with

$$|a_{i,i}| > h_i \sum_{j<i} |a_{i,j}| + k_i \sum_{j>i} |a_{i,j}|,$$

with $h_1 = 0$, $h_i \leq 1$ for $i = 2, \ldots, n$ and $k_i \geq 1$ for $i = 1, \ldots, n-1$, $k_n = 0$, and a solution exists for the following $2(n-1) \times (n-1)$ overdetermined system:

$$
\begin{bmatrix}
-h_2 & -h_3 \sum_{i<3} |a_{3,i}| & \cdots & \cdots & -h_n \sum_{i<n} |a_{n,i}|
|a_{3,2}| & a_{n,3} & \cdots & \cdots & |a_{1,n}|
|a_{n,2}| & |a_{n,3}| & \cdots & \cdots & |a_{2,n}|
|a_{1,2}| & |a_{1,3}| & \cdots & \cdots & |a_{2,n}|
-k_2 \sum_{i>2} |a_{2,i}| & |a_{2,3}| & \cdots & \cdots & |a_{2,n}|
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
-1 & -|a_{3,1}| & \cdots & -|a_{n,1}| & k_1 \sum_{i>1} |a_{1,i}|
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4 \\
\vdots \\
x_n
\end{bmatrix}.
$$

(2.3)
If \( h_i = 1 \) for \( i = 2, \ldots, n \) and \( k_i = 1 \) for \( i = 1, \ldots, n - 1 \) in (2.2), the inequality defines strict diagonal dominance of the matrix. As stated in Fact 1, an irreducible strictly diagonal dominant matrix is an M-matrix, but the converse is not true.

**Proof of Theorem 1.** To prove the necessary condition, assume \( \tilde{A} \) is an M-matrix. By Fact 1, there exists a vector \( y > 0 \) such that \( \tilde{A}y > 0 \). Let \( P \) be the permutation matrix whose transpose reorders the vector \( y \) in ascending order: i.e., \( x = P^Ty \) is such that \( x_1 \preceq x_2 \preceq \cdots \preceq x_n \). Permutation matrices are orthonormal; i.e., \( PP^T = I \). Therefore \( y = Px \). The matrix \( P \), applied to the right of \( A \), permutes the columns of \( A \), and \( P^T \), applied to the left of \( AP \), permutes the rows of \( AP \) in the same order. The diagonal elements of \( \tilde{A} \) are still diagonal elements in \( A = P^TAP \), although the order they appear is permuted. \( Ay > 0 \) implies \( P(Ay) = Ax = P^TAPx > 0 \). The last equality implies the following:

\[
|a_{i,i}| > \frac{1}{x_i} \sum_{j<i} |a_{i,j}|x_j + \frac{1}{x_i} \sum_{j>i} |a_{i,j}|x_j
\]  

for \( i = 2, \ldots, n \) and \( i = 1, \ldots, n - 1 \). Let

\[
h_i = \frac{\sum_{j<i} |a_{i,j}|x_j}{x_i \sum_{j<i} |a_{i,j}|},
\]

\[
k_i = \frac{\sum_{j>i} |a_{i,j}|x_j}{x_i \sum_{j>i} |a_{i,j}|}
\]

Then Equation (2.4) implies that

\[
|a_{i,i}| > h_i \sum_{j<i} |a_{i,j}| + k_i \sum_{j>i} |a_{i,j}|
\]  

for \( i = 1, \ldots, n \), with \( h_1 = 0 \) and \( k_n = 0 \). Since \( x_j/x_i \leq 1 \) when \( j < i \), we have \( h_i \leq 1 \) for \( i = 2, \ldots, n \). Similarly, since \( x_j/x_i \geq 1 \) when \( j > i \), we have \( k_i \geq 1 \) for \( i = 1, \ldots, n - 1 \).

We have shown that if \( \tilde{A} \) is an M-matrix, there exist a permutation matrix \( P \), \( h_i \leq 1 \) for \( i = 2, \ldots, n \), and \( k_i \geq 1 \) for \( i = 1, \ldots, n - 1 \) such that if \( A = (a_{i,j}) = P^T\tilde{A}P \), the inequality (2.2) holds. If we set \( x_1 = 1 \), a solution exists for Equation (2.5) by construction. If we write Equation (2.5) in matrix notation, we obtain Equation (2.3).
To prove the sufficient condition, assume that there exists a permutation matrix $P$, $h_i < 1$ for $i = 2, \ldots, n$, and $k_i \geq 1$ for $i = 1, \ldots, n - 1$ such that if $A = (a_{i,j}) = P^TAP$, the inequality (2.2) holds and a solution exists for Equation (2.3). If a solution exists for (2.3), by elementary algebraic manipulation we can show $x_2 = 1/h_2 \geq 1$ and $x_i \geq 1/h_i \geq 1$ for $i = 3, \ldots, n$. Thus the solution vector is positive. If we set $x_1 = 1$, the equations in (2.5) and the inequality (2.4) are satisfied. The inequality (2.4) implies $Ax \succ 0$. By Fact 1, $A$ is an $M$-matrix.

If the indices relating to the off-diagonal terms in the inequality (2.2) are interchanged, the theorem is still true, because the transpose of an $M$-matrix is also an $M$-matrix.

Statement (1) of Fact 1 implies that $A$ is an $M$-matrix if and only if there exists a diagonal matrix $D$ such that $AD$ is diagonal dominant. [The statement also implies that $D = \text{diag}(g_1, \ldots, g_n)$.] Theorem 1 is interesting in that $M$-matrices are matrices which are partially diagonal dominant in the rows/columns. Note that the inequality (2.2) alone does not guarantee the matrix to be an $M$-matrix. Consider

$$A = \begin{bmatrix} 2 & -1.5 \\ -1.5 & 1 \end{bmatrix}$$

Choosing $k_1 = 1$ and $h_2 = 1/1.51$ satisfies the inequality (2.2), but $A$ is not an $M$-matrix.

The following corollary is immediate from the theorem:

**Corollary 1.** If $\widetilde{A} = (\tilde{a}_{i,j})$ is a nonsingular $M$-matrix,

1. at least one row of $\widetilde{A}$ is strictly diagonal dominant;
2. at least one row of every principal submatrix of $\widetilde{A}$ is strictly diagonally dominant.

**Proof of Corollary 1.** By Theorem 1, there exists a permutation matrix $P$ such that if $A = P^T\widetilde{A}P$, the inequality (2.2) is true. Since diagonal dominance is invariant under row and column interchanges, we can assume without loss of generality that $P = I$. For $i = 1$, the first term of the inequality vanishes:

$$|a_{1,1}| > k_1 \sum_{j>1} |a_{1,j}| \geq \sum_{j \neq 1} |a_{1,j}|. \quad (2.7)$$

(The last inequality holds because $k_1 \geq 1$.) Statement (1) of the corollary is proved.
By Fact 1, every principal submatrix of $A$ is an $M$-matrix. Then from statement (1) of the corollary, we can conclude that statement (2) is true. ■

The corollary is still true if the word "row" is replaced by the word "column," because if $A$ is an $M$-matrix, its transpose is also an $M$-matrix.

We say a $n \times n$ real matrix whose off-diagonal elements are nonpositive is an incomplete $M$-matrix by $r$ if one of its $(n - r) \times (n - r)$ principal submatrices is an $M$-matrix. Theorem 2 presents a necessary and sufficient condition for such matrices.

**Theorem 2.** An $n \times n$ real matrix whose off-diagonal elements are nonpositive is an incomplete $M$-matrix by $r$ if and only if there exists a vector $x > 0$ such that $Ax$ has at least $n - r$ positive entries.

**Proof of Theorem 2.** To prove the necessary condition, assume that $A$ is an incomplete $M$-matrix by $r$. The proof is by induction on $r$. First, assume $r = 1$. By definition, there exists an $(n - 1) \times (n - 1)$ principal submatrix $A_1$ and a vector $x$ of length $(n - 1)$, $x > 0$, such that $A_1x > 0$. Without loss of generality, assume $A_1$ to be the principal submatrix consisting of the first $n - 1$ rows and first $n - 1$ columns. $A_1x > 0$ for some $x > 0$ implies that the diagonal entries of $A_1$ must be positive, because the off-diagonal entries of $A_1$ are nonpositive. Thus

$$0 < |a_{i,i}|x_i - \sum_{j \neq i}^{n-1} |a_{i,j}|x_j \equiv b_i$$

for $i = 1, \ldots, n - 1$. Let $x_n$ be a positive number such that

$$x_n < \frac{\min_i b_i}{\max_i |a_{i,n}|}$$

for $1 \leq i \leq n - 1$. This expression and the definition of $b_i$ in Equation (2.8) imply

$$b_i - |a_{i,n}|x_n > 0$$

for $1 \leq i \leq n - 1$. Substituting the definition of $b_i$ in Equation (2.8) into this expression, we obtain

$$|a_{i,i}|x_i - \sum_{j \neq i}^n |a_{i,j}|x_j > 0$$
for \(1 \leq i \leq n - 1\), which implies that the first \(n - 1\) entries of \(Ax\) are positive. Suppose this necessary condition is true for \(r = r_0\). We want to show that it is also true for \(r = r_0 + 1\). If a matrix \(A\) is an incomplete \(M\)-matrix by \(r_0 + 1\), one of its \((n - r_0 - 1) \times (n - r_0 - 1)\) principal submatrices, say \(A_1\), is an \(M\)-matrix. Without loss of generality assume that the first \((n - r_0 - 1) \times (n - r_0 - 1)\) principal submatrix of \(A\) is an \(M\)-matrix. There exist a vector \(x\) of length \(n - r_0 - 1\) and an \(x > 0\) such that \(A_1x > 0\). That is,

\[
0 < |a_{i,i}|x_i - \sum_{j \neq i}^{n-r_0-1} |a_{i,j}|x_j = c_i. \tag{2.9}
\]

Let \(x_{r_o}\) be a positive number such that

\[
x_{r_o} < \frac{\min_i c_i}{\max_i |a_{i,r_o}|}
\]

for \(1 \leq i \leq n - r_0 - 1\). This expression and the definition of \(c_i\) in Equation (2.9) imply

\[
c_i - |a_{i,r_o}|x_{r_o} > 0
\]

for \(1 \leq i \leq n - r_0 - 1\). Substituting the definition of \(c_i\) in Equation (2.9) into this expression, we obtain

\[
|a_{i,i}|x_i - \sum_{j \neq i}^{n-r_0-1} |a_{i,j}|x_j > 0
\]

for \(1 \leq i \leq n - r_0 - 1\). Let \(d\) be a positive number such that

\[
(d + a_{r_0,r_0})x_{r_o} - \sum_{j \neq r_0} a_{r_0,j}|x_j| > 0.
\]

Let \(\hat{A}\) be a matrix equal to \(A\) except for the \((r_0, r_0)\)th entry. Let \(\hat{a}_{r_0,r_0} = d + a_{r_0,r_0}\). Then \(\hat{A}\) is an incomplete \(M\)-matrix by \(r_0\), because its first \((n - r_0) \times (n - r_0)\) principal submatrix is an \(M\)-matrix. Then by the induction hypothesis, there exist a vector \(x\) of length \(n - r_0\) and an \(x > 0\) such that \(A^{}x\) has at least \(n - r_0\) positive entries. But \(\hat{A}^{}x\) only differs from \(A^{}x\) by the \(r_0\)th row. Therefore \(Ax\) has at least \(n - r_0 - 1\) positive entries, and we complete the proof of the necessary condition for incomplete \(M\)-matrices.
The proof of the sufficient condition is a much simpler. Assume that there is a positive vector \( x > 0 \) such that \( Ax \) has at least \( n - r \) positive entries. Without loss of generality assume the first \( n - r \) entries of \( Ax \) are positive. Since the off-diagonal terms of \( A \) are nonpositive, the condition that the first \( n - r \) entries of \( Ax \) are positive for some \( x > 0 \) implies that \( a_{i,i} > 0 \) for \( i = 1, \ldots, n - r \). Then

\[
|a_{i,i}|x_i - \sum_{j \neq i}^n |a_{i,j}|x_j > 0.
\]

This implies

\[
|a_{i,i}|x_i - \sum_{j \neq i}^{n-r} |a_{i,j}|x_j > 0.
\]

By Fact 1, the first \((n - r) \times (n - r)\) principal submatrix of \( A \) is an \( M \)-matrix. •

3. BLOCK COMPARISON MATRICES, MINORANTS

Before we discuss how the results in Section 1 can be applied to block \( LU \) factorization, we summarize the wealth of material relating an \( N \times N \) block matrix \( A = (A_{i,j}) \) to certain \( N \times N \) real matrices of the form \( B = (b_{i,j}) \).

In 1973, Carlson and Varga [2] defined \( \mathcal{F}_\pi(A) = (b_{i,j}) \), where \( b_{i,j} = -m(A_{i,j})M(A_{i,j}^{-1}A_{i,j}) \) for all \( i \neq j \), \( b_{i,i} = F_i(A) \), with the operators \( M(\cdot) \) and \( m(\cdot) \) as matrix norms and "reciprocal norms," and \( F_i \) the \( i \)-th component of the \( G \)-function \( F \). [A function \( F = (F_1, F_2, \ldots, F_n) \) is a \( G \)-function if and only if for each \( i = 1, 2, \ldots, n \), \( F_i \) maps the set of \( n \times n \) complex matrices to the real line and \( F_i \) has the property that \( m(A_{i,i}) > F_i(A) \).] Note that if \( A_{i,i} \) is nonsingular, \( m(A_{i,i}) = M(A_{i,i}^{-1})^{-1} \).

Carlson and Varga proved that a function \( F = (F_1, F_2, \ldots, F_n) \) is a \( G \)-function if and only if \( D + \mathcal{F}(A) \), where \( D \) is any diagonal matrix with positive entries, is an \( M \)-matrix for every \( n \times n \) complex matrix \( A \).

Replacing the diagonal entries of \( \mathcal{F}(A) \) with \( m(A_{i,i}) \) yields a variant of the block comparison matrix. The fact that the nonsingularity of this variant guarantees the nonsingularity of the corresponding block matrix was proved in 1962 by Fiedler and Pták [7] and in 1971 by Johnston [10].

In 1983 Dahlquist [3] defined an \( N \times N \) real matrix \( \hat{A} = (\hat{a}_{i,j}) \) to be a majorant of \( A \) if and only if \( \hat{a}_{i,j} \geq \|A_{i,j}\| \). He defined an \( N \times N \) real matrix
\( \tilde{A} = (\tilde{a}_{i,j}) \) to be a minorant of \( A \) if and only if \( \tilde{a}_{i,i} \leq \| A_{i,j}^{-1} \|^{-1} \) and \( \tilde{a}_{i,j} \leq -\| A_{i,j} \| \) when \( i \neq j \). He proved the following elegant result, which we will use in the next section:

**Theorem 3.** Let \( \tilde{A} \) be an \( N \times N \) minorant of the \( N \times N \) block matrix \( A = (A_{i,j}) \). Assume \( \tilde{A} \) is an M-matrix. Then \( \tilde{A} \) has a triangular factorization, \( \tilde{A} = \tilde{L} \tilde{U} \), such that \( \tilde{L} \) has unit diagonal elements and \( \tilde{U} \) has positive diagonal elements. \( A \) has a unique block triangular factorization, \( A = LU \), with unit matrices in the block diagonal of \( L \). The matrices \( \tilde{L}, \tilde{U} \) are minorants of \( L, U \). The inverses of \( \tilde{L}, \tilde{U} \), and \( \tilde{A} \) are majorants of, respectively, the inverses of \( L, U, A \).

In 1987, Polman [13] defined the block comparison matrix of \( A \) as the one with elements \( b_{i,j} = -\| A_{i,j} \| \) when \( i \neq j \), and \( b_{i,i} = (\| A_{i,j} \|)^{-1} \). If the blocks \( A_{i,j} \) are \( 1 \times 1 \) blocks, \( B(A) \) is the usual comparison matrix of \( A \). A matrix is called an H-matrix if and only if its comparison matrix is an M-matrix. Polman defined a block H-matrix as a block matrix \( A \) such that the \( B(DAE) \) is an M-matrix for some nonsingular block diagonal matrices \( D \) and \( E \).

In 1992, Dieci and Lorenz [5] defined a block M-matrix: a real \( N \times N \) block matrix \( A = (A_{i,j}) \) of the form \( A_{i,j} = \sigma_{i,j} I \) with \( \sigma_{i,j} \leq 0 \) for \( i \neq j \) is a block M-matrix if and only if there exist a real vector \( e > 0 \) and a real scalar \( \sigma \) such that the quadratic form defined by \( \sum_{j=1}^{N} e_{j} A_{i,j} - \sigma I \) is positive definite.

In 1992, Nabben [12] defined generalized M-matrices: a complex \( N \times N \) block matrix \( A = (A_{i,j}) \), where the diagonal blocks are Hermitian and the off-diagonal blocks are negative semidefinite, is a generalized M-matrix if there exists a positive vector \( u \) such that \( \sum_{j=1}^{N} u_{j} A_{i,j} \) is positive definite.

4. BLOCK TRIANGULAR FACTORIZATION (BLU)

Block triangular factorization (BLU) can be described recursively:

\[
A^{(k+1)} = L_{A}^{(k+1)} A^{(k)},
\]

where \( A^{(0)} = A \), where \( L_{A}^{(k+1)} \) is a matrix which is the sum of the identity matrix and a matrix with one nonzero block column, namely the \( (k + 1) \)th block column, and that block column only has nonzero blocks below the
diagonal block:

\[
L_A^{(k+1)} = \begin{bmatrix}
I & 0 \\
\vdots & \vdots \\
- A_{k+2,k+1}^{(k)} & \cdots & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
- A_{N,k+1}^{(k)} & \cdots & 0 & \cdots & I
\end{bmatrix} (4.11)
\]

We refer to Equations (4.10) and (4.11) as the kth block pivotal step of the BLU, the matrices \(A^{(k)}\) as the intermediate matrices of the BLU, and the subdiagonal blocks of \(L_A^{(k)}\) as the block multipliers. If all the block multipliers have norms less than or equal to 1, we say BLU applied to \(A\) is stable.

Note that this scheme fails if there exists \(k\) such that \(A_{k+1,k+1}^{(k)}\) is singular.

If the subblocks are \(1 \times 1\), the BLU becomes the classical triangular factorization without pivoting. Pivoting ensures numerical stability but can become computationally expensive if the matrix is too large to be stored in memory and disk storage has to be used. We note an ongoing effort to expand the set of matrices to which LU without pivoting can be safely applied.

It is well known that\(LU\) without pivoting will be stable when \(A\) is positive definite. Golub et al. and Mathias showed that \(LU\) without pivoting is stable when the Hermitian part of \(A\) is positive definite (see [9] and [11]).

If the block comparison matrix \(B(A)\) of the block matrix \(A\) is an \(M\)-matrix, Theorem 3 in Section 3 assures that BLU will not fail. Demmel et al. [4] proved that if \(A\) is diagonally dominant by column, \(BLU\) is stable. In this section, we extend this result to the case when the block comparison matrix of \(A\) is an \(M\)-matrix.

**Theorem 4.** Let \(\bar{A} = (\bar{A}_{i,j})\) be an \(N \times N\) block matrix with nonsingular diagonal blocks. Suppose the block comparison matrix \(B(A)\) of \(A\) is an \(M\)-matrix. Then there exists a block permutation matrix \(P\) such that BLU applied to \(A = P^T\bar{A}P\) is stable.

We need the following lemma to prove Theorem 4:

**Lemma 1.** If \(\bar{B} = (\bar{B}_{i,j})\) is an \(M\)-matrix, there exists a permutation matrix \(P\) such that the \(LU\) factorization (without pivoting) applied to \(B = P^T\bar{B}P\) has multipliers with magnitudes less than or equal to 1.
Proof of Lemma 1. Lemma 1 can be considered as a corollary to the proof of Theorem 1. Since $\bar{B}$ is an $M$-matrix, its transpose is also an $M$-matrix, and there exists a positive vector $y$ such that $y^T \bar{B} > 0$. Let $P$ be the permutation matrix such that the entries of $x^T = y^T P$ are in ascending order: $x_1 \leq x_2 \leq \ldots \leq x_N$. As in the proof of Theorem 1 we see that

$$x^T B > 0,$$

$$b_{1,1} > \sum_{j=2}^{N} |b_{j,1}| \frac{x_j}{x_1} \geq \sum_{j=2}^{N} |b_{j,1}|.$$  

(4.13)

The multipliers of the first pivotal steps are of the form $-b_{j,1}/b_{1,1}$ for $j > 1$. By the inequality (4.13), these quantities have magnitude less than 1. Consider the second pivotal step. The inequality (4.12) can be written as

$$x^T (L^{(1)})^{-1} L^{(1)} B > 0.$$

where $L^{(1)}$ eliminates the subdiagonal elements in the first column. Write $x^{(1)} = x^T (L^{(1)})^{-1}$ and $B^{(1)} = L^{(1)} B$, and we can write

$$(x^{(1)})^T B^{(1)} > 0.$$  

(4.14)

Note that $x_2^{(1)} = x_2, \ldots, x^{(1)} = x_N$. Then $x_2^{(1)} \leq x_3^{(1)} \leq \ldots \leq x_N^{(1)}$. Also note that $b_{i,1}^{(1)} = 0$ for $i > 1$. The inequality (4.14) implies

$$b_{2,2}^{(1)} > \sum_{j=3}^{N} |b_{j,2}^{(1)}| \frac{x_j^{(1)}}{x_2^{(1)}} \geq \sum_{j=3}^{N} |b_{j,2}^{(1)}|.$$  

(4.15)

The multipliers of the second pivotal steps are of the form $-b_{j,2}/b_{2,2}$ for $j > 2$. As with the multipliers of the first pivotal step, from the inequality (4.15) we can conclude that these quantities have magnitude less than 1. The proof regarding the magnitudes of the multipliers of the other pivotal steps is analogous.

Proof of Theorem 4. Since $B(\bar{A})$ is an $M$-matrix, according to Lemma 1 there exists a $N \times N$ permutation matrix, say $P = (p_{i,j})$, such that the $LU$ factorization (without pivoting) applied to $P^T B(\bar{A}) P$ is stable. Let $Q = (Q_{i,j})$ be the block permutation matrix conformable to $P$, i.e., $Q_{i,j} = p_{i,j} I$, where $I$
is the identity matrix. If we write \( A = Q^T \hat{A} Q = (A_{i,j}) \), then \( B(A) = P^T B(\hat{A}) P \). By the definition of \( P \) and Lemma 1, \( B(A) \) has a triangular factorization, \( B(A) = \hat{L} \hat{U} \), such that \( \hat{L} \) has unit diagonal elements and \( \hat{U} \) has positive diagonal elements. Lemma 1 also assures that the subdiagonal entries of \( L \) have magnitudes less than 1. From the proof of Theorem 3 (see [3]), the subdiagonal entries of \( \hat{L} = \hat{Y}_{i,j} \) are nonpositive. Theorem 3 also assures that \( A \) has a unique block triangular factorization, \( A = L U \), with unit matrices in the block diagonal of \( L \), and with \( \hat{L} \) as a minorant of \( L \). That is,

\[-\| \hat{Y}_{i,j} \| < -\| L_{i,j} \|\]

for \( i \neq j \). This implies

\[ \| L_{i,j} \| \leq \| \hat{Y}_{i,j} \| \leq 1 \]

for \( i \neq j \). Thus \( BLU \) applied to \( A \) is stable.

The following theorem can be considered as a corollary to Demmel et al. [4]:

**Theorem 5.** If \( B(A) \) is an M-matrix, there exists a block diagonal matrix \( D \) such that \( BLU \) applied to \( DA \) is stable.

*Proof of Theorem 5.* If \( B(A) \) is an M-matrix, its transpose is an M-matrix. Then by Fact 1, there exists a positive vector \( x > 0 \) such that \( x^T B(A) > 0 \). Let \( D \) be a block diagonal matrix such that \( D_{i,i} = x_i I \). By the choice of \( D \) and the definition of \( B(A) \), \( DA \) is strictly block diagonal dominant in the rows, and by Demmel et al. [4], \( BLU \) applied to \( DA \) is stable.

Theorem 2 in Section 2 implies that if \( B(A) \) is an incomplete M-matrix by \( r \) and if \( r \) is small compared to the dimension of \( B(\hat{A}) \), say \( N \), we can permute the block rows and block columns of \( A \) to form \( \hat{A} \) such that the first \((N - r) \times (N - r)\) principal submatrix of \( B(\hat{A}) \) is an M-matrix. Say \( \hat{A}_1 \) is the principal submatrix of \( \hat{A} \) corresponding to the principal submatrix of \( B(\hat{A}) \), which is an M-matrix. \( BLU \) applied to \( \hat{A}_1 \) is stable. We can first apply \( BLU \) to \( \hat{A}_1 \) and then complete the factorization by applying a more rigorous algorithm to the Schur complement of \( \hat{A}_1 \).

Even without the recommended permutations, or block row scaling, the intermediate matrices \( A^{(k)} \) of \( LU \) on M-matrices, or \( BLU \) on block matrices
whose block comparison matrices are $M$-matrices, still have what Golub and Van Loan [8] called "weighty" diagonals or block diagonals. The following corollary can be considered as a corollary to Theorem 2 of the previous section:

**Corollary 2.** If $A^{(k)} = (A_{i,j}^{(k)})$ is the intermediate matrix in BLU, let $B(A) = (B_{i,j})$ be its block comparison matrix, and $B^{(k)}(A) = (B_{i,j}^{(k)})$ be the intermediate matrix of the LU if $B(A)$. Then for $i \neq j$

$$b_{i,i}^{(k)}b_{j,j}^{(k)} > b_{i,j}^{(k)}b_{j,i}^{(k)},$$

$$||A_{i,j}^{(k)}|| ||A_{j,i}^{(k)}|| > ||A_{i,j}^{(k)}|| ||A_{j,i}^{(k)}||.$$

**Proof of Corollary 2.** Since $B^{(k)}$ is an $M$-matrix, by (2) of Fact 1 all principal submatrices have positive determinant. Consider the $2 \times 2$ principal submatrix

$$T = \begin{bmatrix} b_{i,i}^{(k)} & b_{i,j}^{(k)} \\ b_{j,i}^{(k)} & b_{j,j}^{(k)} \end{bmatrix}.$$ 

The determinant of the matrix $T$ is $b_{i,i}^{(k)}b_{j,j}^{(k)} - b_{i,j}^{(k)}b_{j,i}^{(k)}$. Since this expression is positive, the first inequality stated in the corollary follows immediately. The second inequality follows from Theorem 3 that the triangular factors of $B(A)$ are minorants to the block triangular factors of $A$. ■

5. **CONCLUSION**

We have studied the block comparison matrices of block matrices in order to understand the stability of block triangular factorization of block matrices and found a necessary and sufficient condition for $M$-matrices, which we believe to be new. We also expanded the set of block matrices in which stable BLU is possible.

**REFERENCES**


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