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Graph-based multi-agent decision making

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ABSTRACT

In lots of practical multi-criteria decision making (MCDM) problems, there exist various and changeable relations among the criteria which cannot be handled well by means of the existing methods. Considering that graphic or netlike structures can be used to describe the relationships among several individuals, we first introduce the graphic structure into MCDM and formalize the relations among criteria. Then, we develop a new tool, called graph-based multi-agent decision making (GMADM) model, to deal with a kind of MCDM problems with the interrelated criteria. In the model, the graphic structure is paid sufficient attention to in two main aspects: (1) how the graphic structure has influence on the benefits of agents (or the criteria values); and (2) the relation between the graphic structure and the importance weights of agents (criteria). In this case, we can select the best plan(s) (or alternative(s)) according to the overall benefits (the overall criteria values) resulting from the model. Moreover, a fuzzy graph-based multi-agent decision making (FGMADM) method is developed to solve a common kind of situations where the graphic structure of agents is uncertain (confidential or false). Three examples are used to illustrate the feasibility of these two developed methods.

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1. Introduction

When a hacker attacks a network, what he/she wants to do is to damage the network as serious as possible. The hacker may have several attack plans, which are made by experts or formed according to current situations. Suppose that there are n agents $(a_1, a_2, \ldots, a_n)$ and a certain graphic structure in the network (see Fig. 1). Meanwhile, the hacker wants to pick out a plan from the set of plans $\{p_1, p_2, \ldots, p_m\}$ so as to damage the network worst. In this purpose, the hacker needs to assess each of plans and then select the best one. Undoubtedly, if a plan is selected and put into action, some or all agents will be damaged, and the overall damage of the network can be calculated. But for different plans, the damage degrees of the network are different which can be used to evaluate the plans.

We assume that the agents a_j $(j = 1, 2, \ldots, n)$ will suffer some damage d_{ij}, if the hacker puts the plans p_i $(i = 1, 2, \ldots, m)$ into action. Thus, we can construct a damage matrix $D = (d_{ij})_{m \times n}$ of all plans based on which we try to pick out the best plan. Similar to the multi-criteria aggregation methods, we derive the corresponding m overall damage degrees for all plans by utilizing an aggregation method. In this case, we can aggregate each row of the damage matrix into an overall damage degree for the corresponding plan, and the overall damage degrees can be used to assess the plans. But as differing from multi-criteria decision making, there exist some relations among the agents in the network which are formalized as a graphic structure. Therefore, if the hacker wants to select the best plan(s), he/she needs to take the graphic structure into account.

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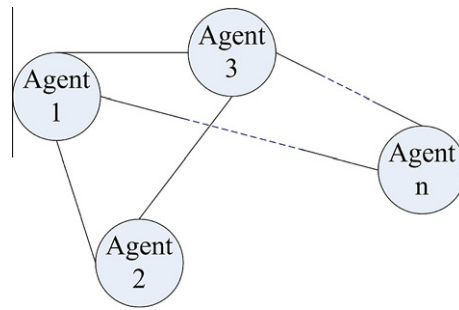 E-mail addresses: yua2006@126.com (X. Yu), xu_zeshui@263.net (Z. Xu).

Fig. 1. A network with n agents.

Like the above plan-selecting problem for attacking a network, how to make decision will be discussed in this paper when the relations of multiple agents are figured by a graph. We call this kind of problems the graph-based multi-agents decision making (GMADM) problems. The GMADM problems occur broadly in almost all fields, such as politics, economy and military, and so on. For example, a situation of battlefield can be figured by a graph, where each combat unit is considered as a vertex (i.e., an agent), and there exits an edge between two agents if the corresponding combat units are hostile or coordinative. Any commander or command department wants to work out an operational plan so as to obtain as much benefit as possible. The GMADM model, developed in this paper, can help the commander or the command department to judge several operational plans and select the best one(s). The GMADM model can also be applied to assist a government in drawing up its policies or an enterprise in making its sales strategies. Therefore, how to solve these kinds of GMADM problems is meaningful and shall be discussed in detail in the following sections.

Generally speaking, the decision factors of a MCDM problem consist of decision makers, alternatives and criteria, etc., and most of the MCDM methods are to serve a kind of problems that there exist no relations among any decision factors (it is independent between any two decision makers, alternatives or criteria). However, the relations among these decision factors are more or less existent in lots of the actual applications. Some authors have already paid attention to this issue. Fan and Feng [1] proposed a MCDM method using the individual and collaborative attribute data so as to solve the actual MCDM problems with both the individual attribute data of a single alternative and the collaborative attribute data of pairwise alternatives. In their contribution, the possible relations between two alternatives have been explored in some actual MCDM problems. As mentioned in [1], when selecting a team leader, the decision maker (DM) usually uses the individual attribute data of candidates such as leadership, management experience and professional expertise. Additionally, the DM should take into consideration of the collaborative attribute data of each pair of candidates such as communication, knowledge sharing, and temperament compatibility and so on. The method for depicting the relations between two alternatives with respect to the collaborative attributes and handling the corresponding MCDM problems has been introduced firstly in [1], but more general frame to establish various relations among alternatives has been not involved in the literature. Antuchevičiene et al. [2] integrated the Mahalanobis distance instead of the Euclidean distance into the usual algorithm of TOPSIS in the process of MCDM, which offers an option to take the correlations among the criteria into consideration. Generally speaking, the taller a person is, the heavier he/she is. Thus if we assess several persons by using height and weight as the main criteria independently, the result will be distorted. Antuchevičiene et al. [2] have added correlation coefficients into the similar dependent criteria so as to eliminate the distortion. Besides, Xu [3] used the Choquet integral to propose some intuitionistic fuzzy aggregation operators, which not only can consider the importance of the elements or their ordered positions, but also can reflect the correlations of the elements or their ordered positions. The method in [3] is also used to deal with the MCDM problems with the dependent criteria like height and weight aforementioned similar to the method in [2]. But differing from the method in [2] that establishes the relations between the criteria by using the correlation coefficients, the method in [3] uses the Choquet integral to calculate the weights of criteria or their ordered positions. The latter is more flexible but subjective than that in [2] because of the more unrestricted fuzzy measure of the Choquet integral in [3]. By considering the multi-criteria aggregation problems where there exists a prioritization relationship over the criteria, Yager [4,5] introduced a number of prioritized aggregation operators which can be conveniently used in practical applications. For example, when we expect to select a basketball player from several persons by considering their heights and weights, obviously the height is a prioritized criterion rather than the weight, i.e., their heights are usually paid more attention than their weights. A stochastic simulation model, which is based on decision variables and stochastic parameters with the given distributions, was constructed to solve the MCDM problems in [6]. The simulation model determines a joint probability distribution for the criteria to quantify the uncertainties and their interrelations. The method in [6] can well handle the MCDM problems with the dependent criteria, but sometimes the relevant joint probability distribution is not given and is hard to obtain similar to the correlation coefficients in [2]. The above existing methods are aiming at the respective specific kinds of MCDM problems with the relative decision factors. Thus, it is necessary to develop a common method in order to describe the relations among the decision factors and solve the relevant MCDM problems. The GMADM model, which needs to be developed, can be a common tool for solving a kind of MCDM problems with the relative criteria. Similar to the example that a person is selected

to play basketball from several candidates by considering their heights and weights, the height is a prioritized criterion and meanwhile there exists interrelation between the height and the weight. Therefore, we cannot deal with the MCDM problem by using the method in the above literature. In this case, we can design a solution by means of the GMADM model because the GMADM model can establish various relations among the criteria (such as the prioritization relationships, the correlations, etc.) on the basis of changeable graphic structures. In order to do that, we organize the rest of the paper as follows. In Section 2, we introduce some basic concepts and terminologies of multi-criteria decision making and graph theory. Then we analyze how the graphic structure affects the results of decision making, and develop a method to solve the GMADM problems by discussing the relations between the graphic structure and the importance weights or the benefits of agents in Section 3. In addition, fuzzy graph-based multi-agent decision making (FGMADM) is discussed in Section 4, and the practicability of FGMADM is also explained. The final section ends the paper with conclusions.

2. Preliminaries

In this section, we introduce some basic concepts and terminologies which will be used in the following sections.

2.1. Graph theories

According to Diestel [7], a graph is a pair of sets, $G = (V,E)$, satisfying $E \subseteq [V]^2$, i.e., the elements of E are 2-element subsets of V. The elements of V are the vertices of the graph G, and the elements of E are its edges. Especially in this paper, the vertices can be the agents of multi-agent decision making problems. Besides, a useful concept of a graph is introduced as follows:

Definition 2.1 ([7]). Let $G = (V,E)$ be a (non-empty) graph, then the degree $d(v)$ of a vertex $v \in V$ is the number $|E(v)|$ of edges at v; by the definition of a graph, this is equal to the number of neighbors of v, where the set of the neighbors of v is denoted as N_v in this paper.

It is quite well-known that a graph is a convenient way of representing information involving the relationship between the objects which are represented by vertices and relations by edges. When there is fuzziness in the description of the vertices or in their relations or in both, fuzzy graph model is put forward naturally.

Definition 2.2 ([8]). A fuzzy graph $\widetilde{G} = (V, \mu, \rho)$ is a non-empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\rho : V \times V \to [0,1]$, such that $\rho(x,y) \leqslant \mu(x) \wedge \mu(y)$ for all $x, y \in V$, where the symbol $\wedge$ stands for *Min*. We call μ the fuzzy vertex set of $\widetilde{G}$ and ρ the fuzzy edge set of $\widetilde{G}$.

Similarly, there are the following concepts for a fuzzy graph:

Definition 2.3 ([8]). Let $\widetilde{G} = (V, \mu, \rho)$ be a fuzzy graph. Degree of a vertex $v \in V$ is defined as:

$$d(v) = \sum_{u \in V \setminus \{v\}} \rho(u, v) \tag{1}$$

Definition 2.4 ([8]). Let $\widetilde{G} = (V, \mu, \rho)$ be a fuzzy graph. Then a path in the fuzzy graph is a sequence of distinct vertices $u_0, u_1, \ldots, u_n$ (except possibly u_0 and u_n), such that $\rho(u_{i-1}, u_i) > 0$, $1 \leqslant i \leqslant n$. The strength of the path is defined as $\wedge_{i=1}^n \rho(u_{i-1}, u_i)$, i.e., the strength of a path is defined as the degree of membership of a weakest edge of the path.

Note. In this paper, the fuzzy relation between the ith and jth vertices or agents, i.e., $\rho(v_i, v_j)$ or $\rho(a_i, a_j)$ can be denoted as ρ_{ij} for convenience if it is not ambiguous.

Definition 2.5 ([9]). In a connected fuzzy graph $\widetilde{G} = (V, \mu, \rho)$, the ρ-distance $\delta(u, v)$ is the smallest ρ-length of a path $\widehat{P} = (u = u_0, u_1, \ldots, u_{n-1}, u_n = v)$:

$$l\left(\widehat{P}\right) = \sum_{i=1}^n \frac{1}{\rho(u_{i-1}, u_i)} \tag{2}$$

If $n = 0$, then we stipulate $l\left(\widehat{P}\right) = 0$. Thus, the eccentricity $e(u)$ of a vertex u is:

$$e(u) = \max_{v \in V} \{\delta(u, v)\} \tag{3}$$

2.2. Multi-criteria decision making

Suppose that there is a set of alternatives, X, and a set of criteria, C. How to choose an alternative in X, which satisfies these criteria most, is called multi-criteria decision making (MCDM) [10]. Generally, a MCDM problem can be solved on the basis of aggregation techniques, and the processes are modeled as follows (see Fig. 2):
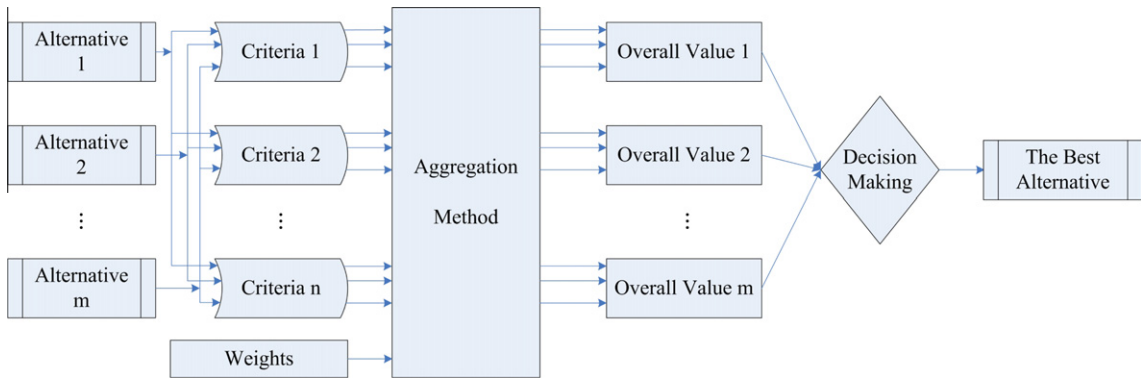
Fig. 2. The MCDM model.

In a MCDM problem, suppose that there are m alternatives x_i $(i = 1, 2, \ldots, m)$ in X and n criteria c_j $(j = 1, 2, \ldots, n)$ in C, and all assessment information constitutes a matrix $M = (\alpha_{ij})_{m \times n}$, where α_{ij} denotes the criterion value of the alternative x_i under the criterion c_j. Each alternative is depicted by n criteria values (assessment information), so it is hard to rank all alternatives directly and a decision making method is needed. According to the main idea of MCDM, we can choose a proper aggregation technique to aggregate the criteria values into an overall one for an alternative, and then rank the alternatives by means of all overall criteria values. For example, for the alternatives x_i $(i = 1, 2, \ldots, m)$, there are n criteria values $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$. By using an aggregation method [11–16] (such as the weighted averaging operator and the ordered weighted averaging operator, and so on), we integrate the criteria values $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$ into an overall one α_i. We finally rank all alternatives and select the most desirable one(s) by comparing the overall criteria values α_i $(i = 1, 2, \ldots, m)$.

In what follows, we introduce a common aggregation operator which will be used in the next sections:

Definition 2.6 ([17]). Let $WA : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. If

$$WA_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} w_i \alpha_i \qquad (4)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is weight vector of $(\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $w_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$, then the function WA is called a weighted averaging (WA) operator.

3. Graph-based multi-agent decision making

In some cases, there exist some relations (like cooperation and competition) among the criteria in a MCDM problem, which cannot be solved well by means of the existing methods. When the graph theory is used to formalize the relations among the criteria, this kind of MCDM problems are then called graph-based multi-agent decision making (GMADM) problems in this paper, and the criteria are just like vertices in a graph, nodes in a network or agents in a mission. Hence, suppose that there are a set of plans $P = \{p_1, p_2, \ldots, p_m\}$ and a set of agents $A = \{a_1, a_2, \ldots, a_n\}$ which is the set of vertices in a graph $G = (A, E)$, where $E \subseteq [A]^2$ is the set of relations among agents in A. Then to choose the best plan(s) so as to produce the maximal benefit is what GMADM wants. Similar to MCDM in Fig. 2, we develop a GMADM model as follows (see Fig. 3):
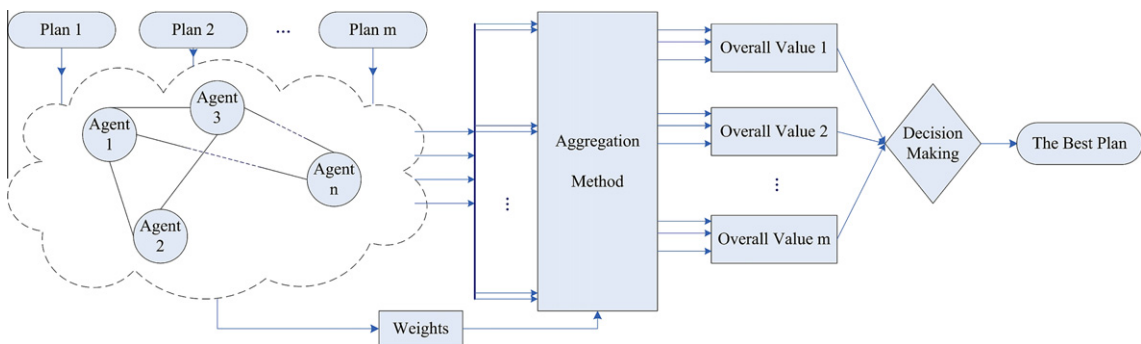

Fig. 3. The GMADM model.

According to the GMADM model, we shall choose a proper aggregation method to obtain the best plan(s). In GMADM, we first must be clear that how these plans affect the benefits of agents. For example, in the case of network attack, when the hacker attacks one of the agents, the neighbors of the agent will also be damaged. In this case, we shall calculate the benefit of each agent under every plan in accordance with the graphic structure (i.e., the relations among the agents). But in some problems, we need not to take the graph structure into account, when calculating the benefits. Thus, we shall take different measures for different situations.

In this purpose, we can calculate the benefit of each agent for a plan by improving Galeotti et al.'s method [18]. Consider n agents, each of them is identified with a vertex in a graph. For a plan, we are clear whether every agent takes an action or not (just as being attacked by a hacker). When an agent a_i take an action, we denote by $t_i = 1$; otherwise, $t_i = 0$. Then the benefit of a_i can be calculated by

$$b_i = t_i + \bar{t}_{N_i}, \quad \text{for } i = 1, 2, \ldots, n \tag{5}$$

where N_i is the set of the agent a_i's neighbors, and $\bar{t}_{N_i} = \sum_{j \in N_i} \xi_{ij} t_j (\xi_{ij} \in [0, 1]$ can be interpreted as the influence coefficient between the relevant agents). If $\xi_{ij} = 0$, for $j \in N_i$ $(i = 1, 2, \ldots, n)$, then $b_i = t_i$, that is to say, the benefits of the agents have no reference to their neighbors and it is not necessary to take the graphic structure into account.

Besides, the importance of agents may be different. If we can find the difference exactly, the results of the GMADM model will be more worth being trusted. In the actual GMADM problems, how to determine the importance weights of agents is usually relevant to the graphic structure, so we shall discuss how to calculate the weights of agents according to some basic concepts of the graph theory.

In some cases, we take it for granted that the more neighbors an agent has, the more important it is. In other cases, if the connectivity of the graph is reduced because of removal of an agent, we regard the agent as an important one. Thus, we can analyze the degree or the connectivity of each vertex in order to obtain the importance weight of each agent. Taking the network attacking as an example, the hacker prefers to attack a node with more neighbors so as to damage the network more seriously. We assume that there are n vertices $A = (a_1, a_2, \ldots, a_n)$ in a graph $G = (A, E)$, and we can obtain the degree of each vertex $d(a_i)$ according to Definition 2.1, then the weights of vertices can be calculated as:

$$w_i = \frac{d(a_i)}{\sum_{j=1}^{n} d(a_j)}, \quad \text{for } i = 1, 2, \ldots, n \tag{6}$$

For example, in a graph $G = (A, E)$, where $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $E = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_1, a_5), (a_2, a_3), (a_2, a_4)\}$ (see Fig. 4).
According to Definition 2.1, we have

$$d(a_1) = 4, \quad d(a_2) = 3, \quad d(a_3) = 2, \quad d(a_4) = 2, \quad d(a_5) = 1$$

Then, we can calculate the weights of vertices by (6):

$$w_1 = \frac{1}{3}, \quad w_2 = \frac{1}{4}, \quad w_3 = \frac{1}{6}, \quad w_4 = \frac{1}{6}, \quad w_5 = \frac{1}{12}$$

Furthermore, during the solutions of some GMADM problems, we must take some subjective factors into account. For example, the decision maker has already ranked the importance degrees of all agents. He/she considers that more benefit will be produced if the agent (considered to be important) takes an action. Generally speaking, there exist the subjective differences among the agents in a graph, and some agents usually are regarded as prior to others, such as the server in a network, the command department in a battlefield, etc. In this case, some agents shall be considered as a matter of priority and the loss of their benefit cannot be compensated by other agents'. In order to make a proper decision in this kind of problems, we can first construct the prioritization relations among the agents, and then calculate the benefit of each plan by using the prioritized aggregation operators [4,5]. In what follows, we develop an operator by considering both the degrees of agents and their prioritization relations in GMADM.

We assume that all agents $A = \{a_1, a_2, \ldots, a_n\}$ can be partitioned into q distinct categories $H_1, H_2, \ldots, H_q$ such that $H_i = \{a_{i1}, a_{i2}, \ldots, a_{in_i}\} (\sum_{i=1}^{q} n_i = n)$ in a graph $G = (A, E)$, and there exists a prioritization among the categories $H_1 \succ H_2 \succ \cdots \succ H_q$ where the symbol "$\succ$" denotes "prior to". According to the graphic structure, we can derive the degree of each agent $d(a_i)$, which can be regularized by
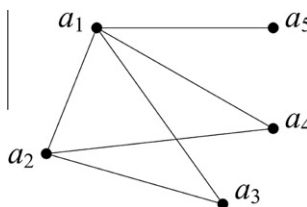

Fig. 4. A graph with five vertices.

$$\bar{d}(a_i) = \frac{d(a_i)}{\max_j(d(a_j))} \tag{7}$$

and then for each category H_i we calculate

$$S_i = \begin{cases} 1, & i = 0 \\ \min_j(\bar{d}(a_{ij})), & otherwise \end{cases} \tag{8}$$

based on which we calculate the importance weight of each category:

$$\omega_i = \prod_{k=1}^{i} S_{k-1}, \quad \text{for } i = 1, 2, \ldots, q \tag{9}$$

By using the above weights, we can obtain the overall benefit of a plan, if we assume that the benefits of agents are b_{ij} ($i = 1, 2, \ldots, q$; $j = 1, 2, \ldots, n_i$):

$$b = \sum_{i,j} \omega_i b_{ij} = \sum_{i=1}^{q} \omega_i \sum_{j=1}^{n_i} b_{ij} \tag{10}$$

We must note that $\sum_i \omega_i \neq 1$ and it is unnecessary to normalize them. That is because that, according to [4], the results of (10) will not satisfy monotonicity any longer if we normalize the weights. Therefore, for each plan, we can calculate its overall benefit by using the above operator, and then rank these plans according to the overall benefits. In what follows, we take a succinct example to illustrate the practicability of the above method:

Example 1. Suppose that there is a graph $G = (A, E)$ with five agents a_i ($i = 1, 2, 3, 4, 5$) and $E = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_1, a_5), (a_2, a_3), (a_2, a_4)\}$ (see Fig. 4). Then we can derive the degree of each agent:

$$d(a_1) = 4, \quad d(a_2) = 3, \quad d(a_3) = 2, \quad d(a_4) = 2, \quad d(a_5) = 1$$

By (7), we have

$$\bar{d}(a_1) = 1, \quad \bar{d}(a_2) = 0.75, \quad \bar{d}(a_3) = 0.5, \quad \bar{d}(a_4) = 0.5, \quad \bar{d}(a_5) = 0.25$$

Suppose that there exist the prioritization relations $\{a_2, a_3\} \succ \{a_1\} \succ \{a_4, a_5\}$, then by (8) we have

$$S_0 = 1, \quad S_1 = 0.5, \quad S_2 = 1, \quad S_3 = 0.25$$

and then by (9) we can calculate the weight of each category:

$$\omega_1 = 1, \quad \omega_2 = 0.5, \quad \omega_3 = 0.5$$

We also assume that there is a plan p, in which just a_1 takes an action, then $t_1 = 1$ and $t_i = 0$ ($i = 2, 3, 4, 5$). If all influence coefficients are 0.5 (i.e., $\xi_{ij} = 0.5$, $i \neq j$ and $i, j = 1, 2, 3, 4, 5$), then we can get the benefits of all agents according to (5):

$$b_1^{(p)} = t_1 + \bar{t}_{N_1} = 1, \quad b_2^{(p)} = b_3^{(p)} = b_4^{(p)} = b_5^{(p)} = 0 + 0.5 \times 1 = 0.5$$

By (10), we can calculate the overall benefit of p:

$$b^{(p)} = \omega_1 \cdot (b_2^{(p)} + b_3^{(p)}) + \omega_2 \cdot b_1^{(p)} + \omega_3 \cdot (b_4^{(p)} + b_5^{(p)}) = 2$$

4. Fuzzy graph-based multi-agent decision making

The first definition of fuzzy graph was proposed by Rosenfeld [19], from the fuzzy relations introduced by Zadeh [20]. Since then, it has been growing fast, and has numerous applications in various fields. Because the fuzzy graph can well describe the uncertainty of all kinds of networks, we analyze the decision making problems on the basis of fuzzy graphic structures in this section, and then deal with this new kind of decision making problems by a method called fuzzy graph-based multi-agent decision making (FGMADM) method in this paper.

First of all, we define this kind of problems as follows:

Suppose that there is a set of n uncertain agents $\widetilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$, which can be described by a fuzzy set $\{(\tilde{a}_i, \mu_i) | i = 1, 2, \ldots, n\}$. μ_i, which can be regarded as the possibility of existence and $0 \leqslant \mu_i \leqslant 1$, is the membership degree of $\tilde{a}_i$. If there exists a fuzzy relation between two agents $\tilde{a}_i$ and $\tilde{a}_j$, we denote the fuzzy relation as $\rho_{ij}(0 < \rho_{ij} \leqslant \mu_i \wedge \mu_j$; $i, j = 1, 2, \ldots, n$); otherwise, $\rho_{ij} = 0$. By means of the definition of fuzzy graphs in Definition 2.2, we formalize the agents and their fuzzy relations as a fuzzy graph $\widetilde{G} = (\widetilde{A}, \mu, \rho)$. Let a set of plans, $P = \{p_1, p_2, \ldots, p_m\}$, and the implementation of any plan will force some or all agents to take actions, during which benefits will be produced. How to choose a best plan so as to derive the maximal benefit is just a FGMADM problem.

We also can take the network attacking as an example. Sometimes, actual networks are confidential, such as internal networks, so a hacker has no idea to make this kind of netlike structures clear. In this case, the hacker can only describe the

uncertain network structures by fuzzy graphs in accordance with partial reliable information. Thus, the hacker shall find a method to solve the FGMADM problem so as to select a plan to damage the network as serious as possible.

In a fuzzy graph $\widetilde{G} = \left(\widetilde{A}, \mu, \rho\right)$, for a plan, suppose that if an agent $\tilde{a}_i \in \widetilde{A}$ takes an action, we let $\tilde{t}_i = 1$; otherwise, $\tilde{t}_i = 0$. Then we can calculate the benefit of each agent $\tilde{a}_i$ by using

$$\tilde{b}_i = \mu_i \tilde{t}_i + \bar{\bar{t}}_{N_i}, \quad \text{for } i = 1, 2, \ldots, n \tag{11}$$

where N_i is the set of the agent a_i's neighbors, and $\bar{\bar{t}}_{N_i} = \sum_{j \in N_i} \rho_{ij} \xi_{ij} \tilde{t}_j$ ($\xi_{ij} \in [0, 1]$ can be interpreted as influence coefficient between relevant agents).

If the weights of all agents are given, we can obtain the overall benefit of the plan by using an aggregation operator. We assume that the weighted averaging operator (see Definition 2.6) is chosen, and then the overall benefit of the plan can be calculated by

$$\tilde{b} = \sum_{i=1}^{n} w_i \tilde{b}_i, \quad \text{for } i = 1, 2, \ldots, n \tag{12}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector.

However, if the weights of agents are not given, we shall calculate them according to some known information, such as fuzzy graphic structure, the benefits of agents, etc. In what follows, we develop a method to derive the weights by means of the fuzzy graphic structure (the degrees of vertices in a fuzzy graph):

$$w_i = \frac{d(\tilde{a}_i)}{\sum_{j=1}^{n} d(\tilde{a}_j)}, \quad \text{for } i = 1, 2, \ldots, n \tag{13}$$

where $d(\tilde{a}_i) = \sum_{\tilde{a}_j \in N_i} \rho_{ij}$ indicates the degree of $\tilde{a}_i$ (see Definition 2.3). As putting the results of (13) into (12), we can obtain the overall benefit of a plan which will help to select the best plan. In addition, we can use the other concepts of fuzzy graphs to derive the weights. For example, if it is paid special attention to whether removing a vertex will reduce the connectivity of a fuzzy graph, we can derive the weights by the concepts of cut-vertex.

Analogically, if there exist the prioritization relations among the agents in the FGMADM problems, we shall solve this kind of problems by using the prioritized aggregation operators [4,5] together with the necessary fuzzy graph structure.

Suppose that, in a fuzzy graph $\widetilde{G} = \left(\widetilde{A}, \mu, \rho\right)$, there are q sets with a prioritization $\widetilde{H}_1 \succ \widetilde{H}_2 \succ \ldots \succ \widetilde{H}_q$ which partitions the set of agents, $\widetilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\}$, into q distinct categories, and each of the categorys $\widetilde{H}_i = \{\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in_i}\}$ ($i = 1, 2, \ldots, q$). If the degrees of agents are used to deal with the FGMADM problem, the solution is the same as the method for solving the GMADM problems with the prioritized agents in the last section. Here, we develop a method to handle the FGMADM problems by means of the prioritized aggregation operators together with the eccentricities of agents if the corresponding fuzzy graph is connected.

According to Definition 2.5, we first calculate the eccentricities of all agents $e(\tilde{a}_i)$ ($i = 1, 2, \ldots, n$), which can be regularized by

$$\bar{e}(\tilde{a}_i) = \frac{\min_j(e(\tilde{a}_j))}{e(\tilde{a}_i)} \tag{14}$$

and then for each category $\widetilde{H}_i$, we calculate

$$\widetilde{S}_i = \begin{cases} 1, & i = 0 \\ \min_j(\bar{e}(\tilde{a}_{ij})), & \text{otherwise} \end{cases} \tag{15}$$

After that, we can calculate the importance weight of each category:

$$\omega_i = \prod_{k=1}^{i} \widetilde{S}_{k-1}, \quad \text{for } i = 1, 2, \ldots, q \tag{16}$$

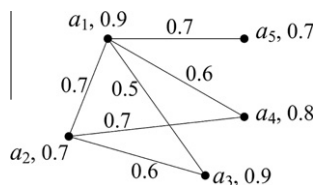By using the above weights, we then can obtain the overall benefit of a plan:

Fig. 5. A fuzzy graph with five vertices.

$$\tilde{b} = \sum_{i,j} \omega_i \tilde{b}_{ij} = \sum_{i=1}^{q} \omega_i \sum_{j=1}^{n_i} \tilde{b}_{ij} \qquad (17)$$

if the benefits of agents are $\tilde{b}_{ij}$ $(i = 1, 2, \ldots, q; \ j = 1, 2, \ldots, n_i)$.

We can also illustrate the practicability of the above method by a brief example similar to Example 1.

Example 2. Suppose that there is a fuzzy graph $\widetilde{G} = \left(\widetilde{A}, \mu, \rho \right)$ with five agents $\tilde{a}_i$ $(i = 1, 2, \ldots, 5)$ (see Fig. 5), where

$$\mu_1 = 0.9, \quad \mu_2 = 0.7, \quad \mu_3 = 0.9, \quad \mu_4 = 0.8, \quad \mu_5 = 0.7$$

and

$$\rho_{12} = 0.7, \quad \rho_{13} = 0.5, \quad \rho_{14} = 0.6, \quad \rho_{15} = 0.7, \quad \rho_{23} = 0.6, \quad \rho_{24} = 0.7$$

Then according to Definition 2.5, we can derive the eccentricity of each agent:

$$e(\tilde{a}_1) = 2, \quad e(\tilde{a}_2) = 2.86, \quad e(\tilde{a}_3) = 3.43, \quad e(\tilde{a}_4) = 3.10, \quad e(\tilde{a}_5) = 3.43$$

By (14), we have

$$\bar{e}(\tilde{a}_1) = 1, \quad \bar{e}(\tilde{a}_2) = 0.699, \quad \bar{e}(\tilde{a}_3) = 0.583, \quad \bar{e}(\tilde{a}_4) = 0.645, \quad \bar{e}(\tilde{a}_5) = 0.583$$

Suppose that there exist the prioritization relations $\{\tilde{a}_2, \tilde{a}_3\} \succ \{\tilde{a}_1, \tilde{a}_4\} \succ \{\tilde{a}_5\}$, then by (15) we have

$$S_0 = 1, \quad S_1 = 0.583, \quad S_2 = 0.645, \quad S_3 = 0.583$$

and then by (16), we can calculate the weights of each category:

$$\omega_1 = 1, \quad \omega_2 = 0.583, \quad \omega_3 = 0.376$$

We assume that there is a plan p, in which only $\tilde{a}_1$ takes an action, and then $\tilde{t}_1 = 1$ and $\tilde{t}_i = 0$ $(i = 2, 3, 4, 5)$. If all influence coefficients are 0.5, i.e., $\xi_{ij} = 0.5$ for $i, j = 1, 2, \ldots, 5$ and $i \neq j$, then we can get the benefits of all agents according to (11):

$$\tilde{b}_1^{(p)} = \mu_1 \tilde{t}_1 + \bar{\bar{t}}_{N_1} = 0.9$$
$$\tilde{b}_2^{(p)} = \mu_2 \tilde{t}_2 + \bar{\bar{t}}_{N_2} = \rho_{12} \xi_{21} \tilde{t}_1 = 0.35$$
$$\tilde{b}_3^{(p)} = \mu_3 \tilde{t}_3 + \bar{\bar{t}}_{N_3} = \rho_{13} \xi_{31} \tilde{t}_1 = 0.25$$
$$\tilde{b}_4^{(p)} = \mu_4 \tilde{t}_4 + \bar{\bar{t}}_{N_4} = \rho_{14} \xi_{41} \tilde{t}_1 = 0.3$$
$$\tilde{b}_5^{(p)} = \mu_5 \tilde{t}_5 + \bar{\bar{t}}_{N_5} = \rho_{15} \xi_{51} \tilde{t}_1 = 0.35$$

By (17), we can calculate the overall benefit of p:

$$\tilde{b}^{(p)} = \omega_1 \cdot (\tilde{b}_2^{(p)} + \tilde{b}_3^{(p)}) + \omega_2 \cdot (\tilde{b}_1^{(p)} + b_4^{(p)}) + \omega_3 \cdot b_5^{(p)} = 1.43$$

5. Illustrative example

A practical decision making problem involving the prioritization of 10 information technology improvement projects was ever analyzed in [3,21]. In this section, we will take the problem as an example so as to illustrate the practicability of the GMADM model.

The information management steering committee of Midwest American Manufacturing Corp. (MAMC) wants to prioritize for development and implementation a set of ten information technology improvement projects $S = \{s_i | i = 1, 2, \ldots, 10\}$: (1) Quality Management Information (s_1), (2) Inventory Control (s_2), (3) Customer Order Tracking (s_3), (4) Materials Purchasing Management (s_4), (5) Fleet Management (s_5), (6) Design Change Management (s_6), (7) Electronic Mail (s_7), (8) Customer Returns and Complaints (s_8), (9) Employee Skills Tracking (s_9), and (10) Budget Analysis (s_{10}), which have been proposed by area managers. The committee is concerned that the projects are prioritized from highest to lowest potential contribution to the firm's strategic goal of gaining competitive advantage in the industry. In assessing the potential contribution of each project, a set of three factors are considered:

$$X = \{x_1 : productivity, x_2 : differentiation, x_3 : management\}$$

where the productivity factor assesses the potential of a proposed project to increase the effectiveness and efficiency of the firm's manufacturing and service operations, the differentiation factor assesses the potential of a proposed project to fundamentally differentiate the firm's products and services from its competitors' and to make them more desirable to its customers, and the management factor assesses the potential of a proposed project to assist management in improving their planning, controlling and decision-making activities.

The committee evaluates the projects s_i $(i = 1, 2, \ldots, 10)$ in relation to the factors x_j $(j = 1, 2, 3)$, and gives more importance to x_1 and x_2 than to x_3, but, on the other hand, the committee gives some advantage to the projects that are good both in x_3
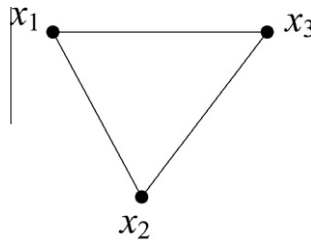
Fig. 6. The relationships among the factors x_j ($j = 1, 2, 3$).

and in either of x_1 and x_2. In this case, we assume that the relationships among the factors x_j ($j = 1, 2, 3$) can be described by a complete graph $G = (A, E)$, where $A = X = \{x_1, x_2, x_3\}$ (see Fig. 6). Similar to (5), we can design some influence coefficients to quantify the relationships among the factors. For example, ξ_{12}, ξ_{13} and ξ_{23} are used to denote the influence coefficients between corresponding two factors respectively, then the overall benefit of a certain project s_i can be calculated by

$$b_{s_i} = w_1 \cdot (b_{s_i}(x_1) + \xi_{12} b_{s_i}(x_2) + \xi_{13} b_{s_i}(x_3)) + w_2 \cdot (b_{s_i}(x_2) + \xi_{12} b_{s_i}(x_1) + \xi_{23} b_{s_i}(x_3)) + w_3 \cdot (b_{s_i}(x_3) + \xi_{13} b_{s_i}(x_1)$$
$$+ \xi_{23} b_{s_i}(x_2)) \tag{18}$$

where $b_{s_i}(x_j)$ denotes the benefit of the project s_i with respect to the factor x_j if the weights of these factors are w_1, w_2 and w_3 respectively. In (18), we call $\bar{b}_{s_i}(x_1) = b_{s_i}(x_1) + \xi_{12} b_{s_i}(x_2) + \xi_{13} b_{s_i}(x_3)$, $\bar{b}_{s_i}(x_2) = b_{s_i}(x_2) + \xi_{12} b_{s_i}(x_1) + \xi_{23} b_{s_i}(x_3)$ and $\bar{b}_{s_i}(x_3) = b_{s_i}(x_3) + \xi_{13} b_{s_i}(x_1) + \xi_{23} b_{s_i}(x_2)$ the benefits of the project s_i with respect to the factors x_j ($j = 1, 2, 3$) respectively after considering the influence from other factors.

In accordance with the committee's opinion, x_1 and x_2 are more important than x_3, and any project with the better benefits concerning both x_3 and either x_1 or x_2 has a higher priority, thus, we assume that the weights of these factors x_j ($j = 1, 2, 3$) are 0.4, 0.4 and 0.3 respectively, i.e.,

$$w = (w_1, w_2, w_3)^T = (0.4, 0.4, 0.3)^T \tag{19}$$

and the influence coefficients are

$$\xi_{12} = -0.1 \cdot (b_{s_i}(x_1) + b_{s_i}(x_2)), \quad \xi_{13} = 0.05 \cdot (b_{s_i}(x_1) + b_{s_i}(x_3)), \quad \xi_{23} = 0.05 \cdot (b_{s_i}(x_2) + b_{s_i}(x_3)) \tag{20}$$

respectively, for the projects s_i ($i = 1, 2, \ldots, 10$). In this case, the larger both $b_{s_i}(x_1)$ (or $b_{s_i}(x_2)$) and $b_{s_i}(x_3)$, the larger ξ_{13} (or ξ_{23}), and thus the larger both $\bar{b}_{s_i}(x_1)$ and $\bar{b}_{s_i}(x_3)$ (or $\bar{b}_{s_i}(x_2)$). The projects s_i ($i = 1, 2, \ldots, 10$) will have a good overall benefit. Contrarily, if we have the larger $b_{s_i}(x_1)$ and $b_{s_i}(x_2)$, we will have the smaller $\bar{b}_{s_i}(x_1)$ and $\bar{b}_{s_i}(x_2)$ because of the influence of the negative coefficient ξ_{12}. The corresponding projects s_i ($i = 1, 2, \ldots, 10$) will not be considered as a good one according to its overall benefit.

The evaluation information on the projects s_i ($i = 1, 2, \ldots, 10$) under the factors x_j ($j = 1, 2, 3$) can be shown in Table 1.

Because of the limit of the contents, we will calculate the overall benefit of s_1 in detail in the following, but leave out the calculation processes for the other projects and only give final results.

We first calculate three influence coefficients by (20):

$$\xi_{12} = -0.1 \cdot (b_{s_1}(x_1) + b_{s_1}(x_2)) = -0.1 \cdot (0.7 + 0.8) = -0.15$$
$$\xi_{13} = 0.05 \cdot (b_{s_1}(x_1) + b_{s_1}(x_3)) = 0.05 \cdot (0.7 + 0.9) = 0.08$$
$$\xi_{23} = 0.05 \cdot (b_{s_1}(x_2) + b_{s_1}(x_3)) = 0.05 \cdot (0.8 + 0.9) = 0.085$$

and the benefits of the project s_1 with respect to the factors x_j ($j = 1, 2, 3$) respectively after considering the influences from the other factors can be

Table 1
The evaluation information on the projects.

	x_1	x_2	x_3
s_1	0.7	0.8	0.9
s_2	0.6	0.8	0.8
s_3	0.4	0.6	0.5
s_4	0.7	0.8	0.6
s_5	0.5	0.7	0.4
s_6	0.4	0.6	0.8
s_7	0.3	0.4	0.2
s_8	0.6	0.5	0.8
s_9	0.4	0.9	0.3
s_{10}	0.3	0.6	0.4

$$\bar{b}_{s_1}(x_1) = b_{s_1}(x_1) + \xi_{12}b_{s_1}(x_2) + \xi_{13}b_{s_1}(x_3) = 0.652$$
$$\bar{b}_{s_1}(x_2) = b_{s_1}(x_2) + \xi_{12}b_{s_1}(x_1) + \xi_{23}b_{s_1}(x_3) = 0.771$$
$$\bar{b}_{s_1}(x_3) = b_{s_1}(x_3) + \xi_{13}b_{s_1}(x_1) + \xi_{23}b_{s_1}(x_2) = 1.024$$

then by (18), we obtain the overall benefit of s_1: $b_{s_1} = 0.877$.

Similarly, we can calculate the overall benefits of the other projects s_i ($i = 2, \ldots, 10$):

$$b_{s_2} = 0.801, \quad b_{s_3} = 0.545, \quad b_{s_4} = 0.753, \quad b_{s_5} = 0.577, \quad b_{s_6} = 0.661$$

$$b_{s_7} = 0.331, \quad b_{s_8} = 0.697, \quad b_{s_9} = 0.574, \quad b_{s_{10}} = 0.473$$

Therefore, according to the overall benefits of the projects s_i ($i = 1, 2, \ldots, 10$), we rank these projects as:

$$s_1 \succ s_2 \succ s_4 \succ s_8 \succ s_6 \succ s_5 \succ s_9 \succ s_3 \succ s_{10} \succ s_7$$

where "$\succ$" denotes "prior to".

The result is similar to that in [3], and the best projects derived by the methods of both the papers are s_1. Thus, the above simple example illustrates the feasibility and practicability of the GMADM model. In addition, the GMADM model is more practical to solve most of the decision making problems with the interrelated decision factors than the existing decision making methods including the method in [3].

6. Concluding remarks

In this paper, we have used the graphic structure to describe the interrelated criteria in multi- criteria decision making, and thus a GMADM model has been developed. We have properly solved the GMADM problems by analyzing how the graphic structure of agents affects the benefits and the importance weights of agents. Furthermore, considering some situations that the graphic structure is uncertain, we have developed another method called the fuzzy graph-based multi-agent decision making method. This kind of decision making methods are of worth being developed in further research, because they can be well applied into the actual decision making problems.

The novelty of the paper is that a common model of multi-criteria decision making (MCDM) with the interrelated criteria has been developed and various relationships among the criteria have been described by using the corresponding graphical structures. Undoubtedly, some MCDM problems with the interrelated criteria have been solved by the existing methods (like the methods in [2–6]), but these methods can be regarded as the special cases of the GMADM model. Meanwhile, aiming at lots of the unsettled complex MCDM problems with the interrelated criteria, we can clearly depict the relationships among the criteria and then derive a solution by means of the GMADM model.

However, we have just discussed a common process of graph-based multi-agent decision making. As we know, for any practical decision making problems, there exist some larruping points that we need to attach importance to. Thus, there exists no approach that can be used to all decision making problems, and we need to design a concrete solution process for a practical problem, involving the steps of the calculation of weights, the quantificational description of various relations among the decision factors, and the selection of aggregation methods, etc. Therefore, in the next work, we shall try to utilize our method to deal with some complex decision making problems, which cannot be well solved by the existing ones.

Acknowledgements

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