Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 218 (2008) 576–599
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Symplectic resolutions, Lefschetz property and formality

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Received 5 October 2007; accepted 20 January 2008

Available online 14 March 2008

Communicated by Daniel S. Freed

Abstract

We introduce a method to resolve a *symplectic orbifold* (M,ω) into a smooth symplectic manifold $(\tilde{M},\tilde{\omega})$. Then we study how the formality and the Lefschetz property of $(\tilde{M},\tilde{\omega})$ are compared with that of (M,ω). We also study the formality of the symplectic blow-up of (M,ω) along symplectic submanifolds disjoint from the orbifold singularities. This allows us to construct the first example of a simply connected compact symplectic manifold of dimension 8 which satisfies the Lefschetz property but is not formal, therefore giving a counter-example to a conjecture of Babenko and Taimanov.

Keywords: Symplectic resolutions; Symplectic blow-ups; Lefschetz property; Formality

1. Introduction

In [11], Merkulov proved that for a compact symplectic manifold the Lefschetz property is equivalent to the $d\delta$-lemma, a property similar to the dd^c-lemma for Kähler manifolds. Later

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.01.012

Babenko and Taimanov studied formality of symplectic manifolds in [1]. There, they produced families of non-formal symplectic manifolds in dimensions strictly greater 8, all of which failed to satisfy the Lefschetz property. Due to the fact that the ordinary dd^c-lemma implies formality [3], they were led to conjecture that the $d\delta$-lemma (or equivalently the Lefschetz property) implied formality of symplectic manifolds.

Using symplectic blow-up, the first author proved this conjecture false [2] in all dimensions strictly greater than 2 and, for simply connected spaces, in all dimensions strictly greater than 8. Further, due to a well-known result of Miller and Neisendorfer [12], any simply connected manifold of dimension 6 or less is formal. Hence the only case where the conjecture still stood was for simply connected symplectic 8-manifolds. As Miller's result suggests, the requirements that the manifold is simply connected and 8-dimensional are strong constraints. Indeed, only recently, in [5], were the first examples of non-formal simply connected symplectic 8-manifolds produced. Now we prove that the conjecture does not hold in 8 dimensions either, therefore completing the study of the relationship between the Lefschetz property and formality.

To show that there is no relation between those properties, we construct an example by merging and improving on techniques from [2] and [5]. The tool we use to detect non-formality is not Massey products, but a new product which depends on an even cohomology class a and which we call a-Massey product. The method for construction of new symplectic manifolds is the *symplectic resolution of singularities*, in the spirit of [5] as well as symplectic blow-up. Putting these together, we study in detail how the a-Massey products and the Lefschetz property behave under symplectic blow-up and under symplectic resolution of singularities.

The way we construct our example consists in taking a quotient of a non-formal symplectic manifold by a (non-free) action of a finite group so that the resulting manifold is a symplectic orbifold with nontrivial a-Massey product. Then we blow-up this orbifold along suitable submanifolds to produce a non-formal orbifold which satisfies the Lefschetz property and finally we resolve the isolated symplectic orbifold singularities. The resulting smooth manifold is a counter-example to the Babenko–Taimanov conjecture.

This paper is organized as follows. In Section 2 we introduce new obstructions to formality called a-Massey products and study their properties. There we also study formality of orbifolds and show that the minimal model for the topological space underlying an orbifold is given by the minimal model for the algebra of orbifold differential forms. Therefore, similarly to the case of manifolds, in order to check formality of an orbifold one can simply work with differential forms, instead of piecewise linear forms on some triangulation.

In Section 3, we introduce the concept of a symplectic resolution and show that any symplectic orbifold with isolated singularities can be resolved into a smooth symplectic manifold. Our method of resolution of singularities of symplectic orbifolds works in more cases than that of [13]. Then, we study the behaviour of a-Massey products and the Lefschetz property under resolutions. We show that both are preserved by resolution of orbifold singularities.

In Section 4, we recall results about the behaviour of the Lefschetz property under symplectic blow-up and give conditions for a-Massey products to be preserved under blow-up. Finally, in the last section we put these ingredients together to produce the counter-example to the Babenko–Taimanov conjecture.

2. Formality and *a*-Massey products

2.1. Formality of differential graded algebras

In this section we review the notion of formality [3,14] and Massey products, which are well-known obstructions to formality. Then we introduce a new product which depends on an even cohomology class and is similar to Massey products. This new product also provides obstructions to formality, much in the spirit of Massey products, but in some situations they are simpler to compute than higher order Massey products. We finish with some comments about the formality of manifolds and orbifolds.

We work with differential graded commutative algebras, or DGAs, over the field of real numbers, $\mathbb{R}$. We denote the degree of an element a of a DGA by $|a|$. A DGA $(\mathcal{A}, d)$ is *minimal* if:

1. $\mathcal{A}$ is free as an algebra, that is, $\mathcal{A}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus V^i$, and
2. there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well-ordered index set I, such that $|a_\mu| \leqslant |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ $(\mu < \tau)$. This implies that da_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

Given a differential algebra $(\mathcal{A}, d)$, we denote its cohomology by $H(\mathcal{A})$. The cohomology of a differential graded algebra $H(\mathcal{A})$ is naturally a DGA with the product induced by that on $\mathcal{A}$ and with differential identically zero. The DGA $\mathcal{A}$ is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$.

A differential algebra $(\mathcal{M}, d)$ is a *minimal model* of $(\mathcal{A}, d)$ if $(\mathcal{M}, d)$ is minimal and there exists a morphism of differential graded algebras $\rho : (\mathcal{M}, d) \to (\mathcal{A}, d)$ inducing an isomorphism $\rho^* : H(\mathcal{M}) \to H(\mathcal{A})$ in cohomology. In [7] Halperin proved that any connected differential algebra $(\mathcal{A}, d)$ has a minimal model unique up to isomorphism.

A DGA $\mathcal{A}$ with minimal model $\mathcal{M}$ is *formal* if there is a morphism of differential algebras $\psi : \mathcal{M} \to H(\mathcal{A})$ which induces an isomorphism in cohomology. In this case $\mathcal{M}$ is simultaneously the minimal model for $\mathcal{A}$ and $H(\mathcal{A})$.

In order to detect non-formality, instead of computing the minimal model, which usually is a lengthy process, we can use Massey products, which are obstructions to formality. The simplest type of Massey product is the triple (also known as ordinary) Massey product, which we define next.

Let $\mathcal{A}$ be a DGA and $a_i \in \mathcal{A}$, $1 \leqslant i \leqslant 3$, be three closed elements such that $a_1 \wedge a_2$ and $a_2 \wedge a_3$ are exact. The (*triple*) *Massey product* of the a_i is the set

$$\langle a_1, a_2, a_3 \rangle = \big\{ \big[a_1 \wedge a_{2,3} + (-1)^{|a_1|+1} a_{1,2} \wedge a_3 \big] \;\big|\; da_{1,2} = a_1 \wedge a_2, da_{2,3} = a_2 \wedge a_3 \big\} \subset H^{|a_1|+|a_2|+|a_3|-1}(\mathcal{A}),$$

where $|a_i|$ is the degree of a_i. This set depends only on the cohomology classes of the a_i and not on the a_i themselves, hence this expression also defines a product for cohomology classes.[1]

[1] In the literature it is also usual to call the induced product in cohomology Massey product. This difference is purely semantic and does not change any of the arguments used in the paper.

Given $a_{1,2}$ and $a_{2,3}$ as above, we can add any closed elements $\alpha_{1,2}$ and $\alpha_{2,3}$ to them and we still have the equalities

$$d(a_{1,2}+\alpha_{1,2}) = a_1 \wedge a_2 \quad \text{and} \quad d(a_{2,3}+\alpha_{2,3}) = a_2 \wedge a_3,$$

hence we see that $\langle a_1, a_2, a_3\rangle$ is a set of the form $c + ([a_1] \wedge H^{|a_2|+|a_3|-1}(\mathcal{A}) + H^{|a_1|+|a_2|-1}(\mathcal{A}) \wedge [a_3])$. So the Massey product gives a well-defined element in

$$\frac{H^{|a_1|+|a_2|+|a_3|-1}(\mathcal{A})}{[a_1] \wedge H^{|a_2|+|a_3|-1}(\mathcal{A}) + H^{|a_1|+|a_2|-1}(\mathcal{A}) \wedge [a_3]}.$$

We say that $\langle a_1, a_2, a_3\rangle$ is *trivial* if $0 \in \langle a_1, a_2, a_3\rangle$. The *indeterminacy* of the Massey product is the set

$$\{c - c' \mid c, c' \in \langle a_1, a_2, a_3\rangle\} = [a_1] \wedge H^{|a_2|+|a_3|-1}(\mathcal{A}) + H^{|a_1|+|a_2|-1}(\mathcal{A}) \wedge [a_3].$$

Now we move on to the definition of higher Massey products (see [16]). Given $a_i \in \mathcal{A}$, $1 \leqslant i \leqslant n$, $n \geqslant 3$, the Massey product $\langle a_1, a_2, \ldots, a_n\rangle$, is defined if there are elements $a_{i,j}$ on $\mathcal{A}$, with $1 \leqslant i \leqslant j \leqslant n$, except for the case $(i, j) = (1, n)$, such that

$$\begin{aligned} a_{i,i} &= a_i, \\ da_{i,j} &= \sum_{k=i}^{j-1} \overline{a_{i,k}} \wedge a_{k+1,j}, \end{aligned} \tag{1}$$

where $\bar{a} = (-1)^{|a|}a$. Then the *Massey product* is

$$\langle a_1, a_2, \ldots, a_n\rangle = \left\{ \left[\sum_{k=1}^{n-1} \overline{a_{1,k}} \wedge a_{k+1,n} \right] \,\middle|\, a_{i,j} \text{ as in (1)} \right\} \subset H^{|a_1|+\cdots+|a_n|-(n-2)}(\mathcal{A}).$$

We say that the Massey product is *trivial* if $0 \in \langle a_1, a_2, \ldots, a_n\rangle$. Note that for $\langle a_1, a_2, \ldots, a_n\rangle$ to be defined it is necessary that the lower order Massey products $\langle a_1, \ldots, a_i\rangle$ and $\langle a_{i+1}, \ldots, a_n\rangle$ with $2 < i < n-2$ are defined and trivial. As before, the indeterminacy of the Massey product is

$$\{c - c' \mid c, c' \in \langle a_1, a_2, \ldots, a_n\rangle\}.$$

However, in contrast with the triple products, in general there is no simple description of this set.

The relevance of Massey products to formality comes from the following well-known result.

Theorem 2.1. *(See [3,16].) A DGA which has a non-trivial Massey product is not formal.*

2.2. *a-Massey products*

Next, we introduce another obstruction to the formality, which generalizes the triple Massey products and has the advantage of being simpler for computations than the higher order Massey products.

Proposition 2.2. *Let $\mathcal{A}$ be a DGA and let $a, b_1, \dots, b_n \in \mathcal{A}$ be closed elements such that the degree $|a|$ of a is even and $a \wedge b_i$ is exact, for all i. Let ξ_i be any form such that $d\xi_i = a \wedge b_i$. Then the form*

$$c = \sum_i \overline{\xi_1} \wedge \dots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \dots \wedge \xi_n \tag{2}$$

is closed, where $\overline{\xi} = (-1)^{|\xi|}\xi$.

Definition 2.3. In the situation above, the *nth order a-Massey product* of the b_i (or just *a-product*) is the subset

$$\langle a; b_1, \dots, b_n \rangle := \left\{ \left[\sum_i \overline{\xi_1} \wedge \dots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \dots \wedge \xi_n \right] \,\middle|\, d\xi_i = a \wedge b_i \right\} \subset H(\mathcal{A}).$$

We say that the a-Massey product is *trivial* if $0 \in \langle a; b_1, \dots, b_n \rangle$.

If $n = 2$, the product introduced above is just the triple Massey product $\langle b_1, a, b_2 \rangle$, but for higher values of n these products are different to the higher order Massey products. In the applications we will use the 3rd order a-product with b_i even degree elements, so that the product can be written as

$$\langle a; b_1, b_2, b_3 \rangle = \big\{ [b_1 \wedge \xi_2 \wedge \xi_3 + \text{c.p.}] \mid d\xi_i = a \wedge b_i \big\}, \tag{3}$$

where c.p. stands for cyclic permutations. The product (3) appeared before in [5] in the same context we will use it later.

Now we study the indeterminacy of this product and show that the a-product is an obstruction to formality.

Lemma 2.4. *Let σ be the permutation of $\{1, \dots, n\}$ which is just the transposition of j and $j+1$, for some j. Then, given a, b_i and ξ_i as above, we have $c = (-1)^{(|b_j|+1)(|b_{j+1}|+1)} c_\sigma$, where c is given by Eq.* (2) *and*

$$c_\sigma = \sum_i \overline{\xi_{\sigma(1)}} \wedge \dots \wedge \overline{\xi_{\sigma(i-1)}} \wedge b_{\sigma(i)} \wedge \xi_{\sigma(i+1)} \wedge \dots \wedge \xi_{\sigma(n)}.$$

The proof is a straightforward computation.

Lemma 2.5. *The a-Massey product $\langle a; b_1, \dots, b_n \rangle$ only depends on the cohomology classes $[a]$, $[b_i]$ and not on the particular elements representing these classes.*

Proof. Let $a + d\alpha$ be another representative for the class $[a]$. Then, the generic element in $\langle a + d\alpha; b_1, \dots, b_n \rangle$ is given by the cohomology class of

$$c' = \sum_i \overline{\xi'_1} \wedge \dots \wedge \overline{\xi'_{i-1}} \wedge b_i \wedge \xi'_{i+1} \wedge \dots \wedge \xi'_n \tag{4}$$

with ξ'_i satisfying $d\xi'_i = (a + d\alpha) \wedge b_i$. Thus, $\xi_i = \xi'_i - \alpha \wedge b_i$ satisfies $d\xi_i = a \wedge b_i$. Since a is of even degree, α is of odd degree and hence $\alpha^2 = 0$. Therefore, letting c be given by Eq. (2) we have

$$
\begin{aligned}
c' &= \sum_i (\overline{\xi_1 + \alpha \wedge b_1}) \wedge \ldots \wedge b_i \wedge \ldots \wedge (\xi_n + \alpha \wedge b_n) \\
&= c + \sum_{j<i} \overline{\xi_1} \wedge \ldots \wedge \overline{\xi_{j-1}} \wedge \overline{\alpha \wedge b_j} \wedge \overline{\xi_{j+1}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \ldots \wedge \xi_n \\
&\quad + \sum_{i<j} \overline{\xi_1} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \alpha \wedge b_j \wedge \ldots \wedge \xi_n \\
&= c + \sum_{j<i} \overline{\xi_1} \wedge \ldots \wedge \overline{\xi_{j-1}} \wedge \overline{\alpha \wedge b_j} \wedge \overline{\xi_{j+1}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \ldots \wedge \xi_n \\
&\quad - \sum_{i<j} \overline{\xi_1} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge \overline{\alpha \wedge b_i} \wedge \overline{\xi_{i+1}} \wedge \ldots \wedge \overline{\xi_{j-1}} \wedge b_j \wedge \ldots \wedge \xi_n \\
&= c,
\end{aligned}
$$

where in the second equality we have expanded the expression for c' and used $\alpha^2 = 0$, in the third equality we used that $\eta \wedge \alpha = -\overline{\alpha} \wedge \overline{\eta}$ for any form η, as α is odd, and in the last we used that the two sums are the same, with the roles of i and j reversed. This shows that the a-Massey product only depends on $[a]$.

A similar computation shows that the same is true for the b_i. We do it for $i = 1$. If $b_1 + d\alpha$ is another representative for the class $[b_1]$, then the generic element in $\langle a; b_1 + d\alpha, b_2, \ldots, b_n \rangle$ is given by the cohomology class (4), where ξ_i' satisfies $d\xi_1' = a \wedge (b_1 + d\alpha)$, and $d\xi_i' = a \wedge b_i$ for $i > 1$. Take $\xi_1 = \xi_1' - a \wedge \alpha$ and $\xi_i' = \xi_i$ for $i > 1$, so that $d\xi_i = a \wedge b_i$. Letting c be given by Eq. (2) we have

$$
\begin{aligned}
c' &= (b_1 + d\alpha) \wedge \xi_2 \wedge \ldots \wedge \xi_n + \sum_{i>1} (\overline{\xi_1 + a \wedge \alpha}) \wedge \overline{\xi_2} \wedge \ldots \wedge b_i \wedge \ldots \wedge \xi_n \\
&= c + \left(d\alpha \wedge \xi_2 \wedge \ldots \wedge \xi_n + \sum_{i>1} \overline{a \wedge \alpha} \wedge \overline{\xi_2} \wedge \ldots \wedge b_i \wedge \ldots \wedge \xi_n \right) \\
&= c + d(\alpha \wedge \xi_2 \wedge \ldots \wedge \xi_n),
\end{aligned}
$$

where we have used that a is of even degree. □

One computation relevant to the a-Massey products consists in checking what happens to them when one changes the ξ_i by closed forms η_i, as this gives the indeterminacy of this product.

Lemma 2.6. *Fix $j \in \{1, \ldots, n\}$ and let $\xi_j' = \xi_j + \eta_j$ for some closed element η_j. Let c be given by Eq.* (2) *and c' be given by the same equation but with ξ_j swapped by ξ_j'. Then*

$$
\begin{aligned}
c' &= c + (-1)^{(|b_j|+1)(n-j+\sum_{i>j}|b_i|)} \\
&\quad \times \left(\sum_{i=1,\, i\neq j}^{n} \overline{\xi_1} \wedge \ldots \wedge \widehat{\overline{\xi_j}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \xi_n \right) \wedge \eta_j, \qquad (5)
\end{aligned}
$$

where $\widehat{\xi_j}$ indicates that the term ξ_j is skipped in the product.

Proof. Using Lemma 2.4 and Proposition 2.2, we have

$$\begin{aligned}
c' &= (-1)^{(|b_j|+1)(n-j+\sum_{i>j}|b_i|)} \sum_{i=1}^{n} \overline{\xi_1} \wedge \ldots \wedge \widehat{\overline{\xi_j}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \xi_n \wedge (\xi_j + \eta_j) \\
&= (-1)^{(|b_j|+1)(n-j+\sum_{i>j}|b_i|)} \sum_{i=1}^{n} \overline{\xi_1} \wedge \ldots \wedge \widehat{\overline{\xi_j}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \xi_n \wedge \xi_j \\
&\quad + (-1)^{(|b_j|+1)(n-j+\sum_{i>j}|b_i|)} \sum_{i=1,\, i\neq j}^{n} \overline{\xi_1} \wedge \ldots \wedge \widehat{\overline{\xi_j}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \xi_n \wedge \eta_j \\
&= c + (-1)^{(|b_j|+1)(n-j+\sum_{i>j}|b_i|)} \\
&\quad \times \left(\sum_{i=1,\, i\neq j}^{n} \overline{\xi_1} \wedge \ldots \wedge \widehat{\overline{\xi_j}} \wedge \ldots \wedge \overline{\xi_{i-1}} \wedge b_i \wedge \xi_{i+1} \wedge \ldots \wedge \xi_n \right) \wedge \eta_j. \qquad \square
\end{aligned}$$

Observe that up to a sign, the coefficient of η_j in (5) is an element in $\langle a; b_1, \ldots, \widehat{b_j}, \ldots, b_n \rangle$, hence Lemma 2.6 proves the following inductive way to compute the indeterminacy of the a-product.

Proposition 2.7. *The indeterminacy of the a-product $\langle a; b_1, \ldots, b_n \rangle$ is a subset of*

$$\sum_{j=1}^{n} \langle a; b_1, \ldots, \widehat{b_j}, \ldots, b_n \rangle \wedge H(\mathcal{A}).$$

In particular, the indeterminacy of the triple a-product $\langle a; b_1, b_2, b_3 \rangle$ is a subset of

$$\langle b_1, a, b_2 \rangle \wedge H(\mathcal{A}) + \langle b_2, a, b_3 \rangle \wedge H(\mathcal{A}) + \langle b_3, a, b_1 \rangle \wedge H(\mathcal{A}),$$

where $\langle \bullet, \bullet, \bullet \rangle$ is the (ordinary) triple Massey product.

Remark 2.8. As a corollary to Lemma 2.6 we see that if we change ξ_i by an exact form, the cohomology class of the representative of the product does not change. Together with Lemma 2.5, this tells us that in order to compute the a-Massey product, one does not have to worry about the particular forms a and b_i chosen to represent their cohomology classes. Further, once we fix one choice of ξ_i, in order to obtain any other element in the set $\langle a; b_1, \ldots, b_n \rangle$ we only have to pick one representative for each cohomology class of degree $|\xi_i|$ and add that to ξ_i.

Next we show that the a-Massey products are well behaved under quasi-isomorphisms.

Lemma 2.9. *Let $\psi : \mathcal{B} \to \mathcal{A}$ be a quasi-isomorphism. If $\mathcal{A}$ has an a-Massey product, say $\langle a; b_1, \ldots, b_n \rangle$, and $a', b_i' \in \mathcal{B}$ are such that $[\psi(a')] = [a]$ and $[\psi(b_i')] = [b_i]$ then, the a'-Massey product $\langle a'; b_1', \ldots, b_n' \rangle$ is defined and satisfies*

$$\psi\big(\langle a'; b_1', \ldots, b_n' \rangle\big) = \langle a; b_1, \ldots, b_n \rangle. \tag{6}$$

Conversely, if $\langle a'; b'_1, \dots, b'_n\rangle$ *is an* a'*-Massey product on* $\mathcal{B}$ *and* $a = \psi(a')$ *and* $b_i = \psi(b'_i)$ *then the identity* (6) *also holds.*

Proof. We will only prove the first claim as the second is analogous. First, since ψ is a quasi-isomorphism, there are a' and $b'_i \in \mathcal{B}$ such that $[a] = [\psi(a')]$ and $[b_i] = [\psi(b'_i)]$. Further, again because ψ is a quasi-isomorphism, $a' \wedge b'_i$ is exact on $\mathcal{B}$, say $a' \wedge b'_i = d\xi'_i$, so the a'-product $\langle a'; b'_1, \dots, b'_n\rangle$ is defined in $\mathcal{B}$.

Now we prove (6). We start showing that

$$\psi\left(\langle a'; b'_1, \dots, b'_n\rangle\right) \supset \langle a; b_1, \dots, b_n\rangle.$$

Let $[c] \in \langle a; b_1, \dots, b_n\rangle$ be an element in the a-product in $\mathcal{A}$. According to Lemma 2.5,

$$[c] \in \left\langle \psi(a'); \psi\left(b'_1\right), \dots, \psi\left(b'_n\right)\right\rangle = \langle a; b_1, \dots, b_n\rangle.$$

Therefore we may write $[c] = [\sum \overline{\xi_1} \wedge \dots \wedge \psi(b'_i) \wedge \dots \wedge \xi_n]$ with $d\xi_i = \psi(a') \wedge \psi(b'_i)$, hence $d(\xi_i - \psi(\xi'_i)) = 0$ and $\xi_i - \psi(\xi'_i)$ are closed elements in $\mathcal{A}$ which represents cohomology classes in $H(\mathcal{A}) \cong H(\mathcal{B})$. Hence there are elements $\zeta_i \in \mathcal{B}$ and $z_i \in \mathcal{A}$ such that ζ_i are closed and $\psi(\xi'_i + \zeta_i) = \xi_i + dz_i$. So according to Lemma 2.6 and Remark 2.8, the class

$$[c'] = \left[\sum \left(\overline{\xi'_1 + \zeta_1}\right) \wedge \dots \wedge b'_i \wedge \dots \wedge \left(\xi'_n + \zeta_n\right)\right] \in \left\langle a'; b'_1 \dots, b'_n\right\rangle$$

satisfies

$$\begin{aligned}
\psi\left([c']\right) &= \left[\sum \overline{\psi\left(\xi'_1 + \zeta_1\right)} \wedge \dots \wedge \psi\left(b'_i\right) \wedge \dots \wedge \psi\left(\xi'_n + \zeta_n\right)\right] \\
&= \left[\sum (\overline{\xi_1 + dz_1}) \wedge \dots \wedge \psi\left(b'_i\right) \wedge \dots \wedge (\xi_n + dz_n)\right] \\
&= \left[\sum \overline{\xi_1} \wedge \dots \wedge \psi\left(b'_i\right) \wedge \dots \wedge \xi_n\right] = [c],
\end{aligned}$$

which proves the inclusion.

To prove the other inclusion, let $[c'] \in \langle a'; b'_1, \dots, b'_n\rangle$, and write $[c'] = [\sum \overline{\xi'_1} \wedge \dots \wedge b'_i \wedge \dots \wedge \xi'_n]$ with $d\xi'_i = a' \wedge b'_i$. Applying ψ to this expression we see that

$$\begin{aligned}
\psi\left([c']\right) &= \left[\sum \psi\left(\overline{\xi'_1}\right) \wedge \dots \wedge \psi\left(b'_i\right) \wedge \dots \wedge \psi\left(\xi'_n\right)\right] \in \left\langle \psi(a'); \psi\left(b'_1\right), \dots, \psi\left(b'_n\right)\right\rangle \\
&= \langle a; b_1, \dots, b_n\rangle,
\end{aligned}$$

as we wanted. □

The obvious implication of this lemma is that a-Massey products are obstructions to formality.

Theorem 2.10. *If a DGA has a nontrivial a-Massey product, then it is not formal.*

Proof. Indeed, if $\mathcal{A}$ has a nontrivial a-Massey product then, according to Lemma 2.9, so does its minimal model. On the other hand, $H(\mathcal{A})$ never has a nontrivial product, so the minimal models for $\mathcal{A}$ and $H(\mathcal{A})$ cannot be the same. □

Actually, the a-Massey product of degree 2 elements is the first obstruction to formality that appears as an obstruction to 3-formality [4] for a simply connected manifold, and which is different from a Massey product.

2.3. Formality of manifolds and orbifolds

The *minimal model* $\mathcal{M}$ of a connected differentiable manifold M is the minimal model for the de Rham complex $(\Omega(M), d)$ of differential forms on M. If M is simply connected, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to the space of generators of $\mathcal{M}$ in degree i, for any i. This relation also happens when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for $j > 1$ (see [3]).

A manifold M is *formal* if $(\Omega(M), d)$ is formal. Therefore, if M is formal and simply connected, then the real homotopy groups $\pi_i(M) \otimes \mathbb{R}$ are obtained from the minimal model of $H(M)$.

Many examples of formal manifolds are known: all compact symmetric spaces (e.g., spheres, projective spaces, compact Lie groups, flag manifolds), compact Kähler manifolds and simply connected manifolds of dimension six or less. The importance of formality in symplectic geometry stems from the fact that it allows to distinguish between symplectic manifolds which admit Kähler structures and some symplectic manifolds which do not [3,16].

Now we extend the definition of formality to orbifolds. Let us first introduce this concept.

Definition 2.11. An *orbifold* is a (Hausdorff, paracompact) topological space M with an atlas with charts modelled on U/G_p, where U is an open set of $\mathbb{R}^n$ and G_p is a finite group acting linearly on U with only one fixed point $p \in U$. The number n is the *dimension* of the orbifold.

Note that our definition of orbifold is more restrictive than other definitions in the literature (e.g. [15]).

An orbifold M contains a discrete set Δ of points $p \in M$ for which $G_p \neq Id$. The complement $M \setminus \Delta$ has the structure of a smooth manifold. The points of Δ are called singular points of M. For any singular point $p \in \Delta$, let B/G_p be a small neighbourhood of p, where B is a ball in $\mathbb{R}^n$. Then B/G_p is a rational homology ball (actually it is contractible), and $\partial B/G_p$ is a rational homology $(n-1)$-sphere.

Let $(U_p/G_p, \phi_p)$ with $U_p \subset \mathbb{R}^n$, G_p a finite group acting linearly on U_p (G_p is non-trivial only if $p \in \Delta$), and $\phi_p : U_p/G_p \to M$ be an atlas for the orbifold M. Then, the space $\Omega^k_{\mathrm{orb}}(M)$ of *orbifold differential k-forms* consists of the G_p-invariant k-forms on U_p which patch together to render globally defined forms. Note that there is a well-defined differential d, so that $(\Omega_{\mathrm{orb}}(M), d)$ is a DGA. Its cohomology is called the *orbifold de Rham cohomology of M*.

Definition 2.12. Let M be an orbifold. Then a *minimal model* for M is a minimal model for the DGA $(\Omega_{\mathrm{orb}}(M), d)$. The orbifold M is *formal* if its minimal model is formal.

Proposition 2.13. *Let $(\mathcal{M}, d)$ be the minimal model of an orbifold M. Then $H(\mathcal{M}) = H^*(M)$, where the latter means singular cohomology with real coefficients.*

Proof. Let $(\mathcal{A}, d)$ denote the sub-algebra of $(\Omega_{\mathrm{orb}}(M), d)$ such that $\mathcal{A}^0$ consists of functions which are constant on a neighbourhood of each singular point, and $\mathcal{A}^k$ consists of k-forms which are zero on a neighbourhood of each singular point. Let us see that

$$(\mathcal{A}, d) \hookrightarrow \big(\Omega_{\mathrm{orb}}(M), d\big)$$

is a quasi-isomorphism.

Let $\alpha \in \Omega_{\mathrm{orb}}(M)$ be closed. Let $p \in \Delta$ and consider a neighbourhood U_p of the form B/G_p, for $B \subset \mathbb{R}^n$ a ball. Then we may consider α as a closed form on B. Hence α is exact, that is, there exists a form β such that $\alpha = d\beta$. By averaging by G_p we may assume that β is G_p-equivariant, i.e., $\beta \in \Omega_{\mathrm{orb}}(U_p)$. Consider a bump function ρ which is zero off U_p and 1 in a smaller neighbourhood of p. Then $\alpha - d(\rho\beta)$ is in $\mathcal{A}$ and it is cohomologous to α. This proves surjectivity of $H(\mathcal{A}) \to H(\Omega_{\mathrm{orb}}(M))$.

Now suppose that $\alpha \in \mathcal{A}$ satisfies that $\alpha = d\beta$, with $\beta \in \Omega_{\mathrm{orb}}(M)$. Let $V_p = B/G_p$ be a neighbourhood of each $p \in \Delta$, small enough so that they are disjoint with the support of α. Consider a map $\phi : M \to M$, which is the identity off $\bigcup V_p$, sending V_p into V_p in such a way that it contracts a smaller neighbourhood of p into p. We can take ϕ orbi-smooth (that is, it has a G_p-equivariant lifting to a map $B \to B$ which is smooth). So there is a DGA morphism $\phi^* : \Omega_{\mathrm{orb}}(M) \to \Omega_{\mathrm{orb}}(M)$. Then $\alpha = \phi^*\alpha = d(\phi^*\beta)$ and $\phi^*\beta \in \mathcal{A}$. This proves injectivity of $H(\mathcal{A}) \to H(\Omega_{\mathrm{orb}}(M))$.

With the above at hand, now fix $U_p = B/G_p$ small neighbourhoods of $p \in \Delta$. Let $U = \bigcup_{p\in\Delta} U_p$. Restriction gives a map $(\mathcal{A}, d) \to (\Omega(M \setminus \bar{U}), d)$ with kernel the DGA $(\mathcal{B}, d)$ consisting of forms which vanish on $M \setminus \bar{U}$ and also vanish for positive degrees and are locally constant for degree 0, on a neighbourhood of Δ. Clearly $(\mathcal{B}, d) = \bigoplus_{p\in\Delta}(\mathcal{B}_p, d)$, where $\mathcal{B}_p$ consists of forms on U_p which vanish on the boundary (so that they can be extended by zero off U_p) and vanish for positive degrees and are constant for degree 0, on a neighbourhood of p. Hence there is an exact sequence

$$0 \to \bigoplus_{p\in\Delta}(\mathcal{B}_p, d) \to (\mathcal{A}, d) \to \big(\Omega(M \setminus \bar{U}), d\big) \to 0.$$

Working as above, $\mathcal{B}_p \hookrightarrow \Omega_{\mathrm{orb},0}(B/G_p)$ is a quasi-isomorphism, where $\Omega_{\mathrm{orb},0}(B/G_p)$ are the orbifold forms on B/G_p vanishing on the boundary. Clearly, $\Omega_{\mathrm{orb},0}(B/G_p) = \Omega_0(B)^{G_p}$ is the G_p-invariant part of the forms on B vanishing on the boundary. Thus

$$H^k(\mathcal{B}_p) = H^k\big(\Omega_0(B)\big)^{G_p} = H^k(B, \partial B)^{G_p} = \begin{cases} \mathbb{R}, & k = 2n, \\ 0, & \text{otherwise}. \end{cases}$$

This gives an exact sequence

$$H^{k-1}(M \setminus \bar{U}) \to \bigoplus_{p\in\Delta} H^k(B, \partial B) \to H^k(\mathcal{A}) \to H^k(M \setminus \bar{U}).$$

Together with the exact sequence for singular cohomology

$$H^{k-1}(M \setminus \bar{U}) \to \bigoplus_{p\in\Delta} H^k(B, \partial B) = H^k(M, M \setminus \bar{U}) \to H^k(M) \to H^k(M \setminus \bar{U}),$$

this shows the desired result. □

Remark 2.14. The concept of formality is already defined for nilpotent CW-complexes [6]. In the case that M is an orbifold whose underlying space is nilpotent, Definition 2.12 of formality for M agrees with that in [6]. For this it is enough to see that if $(\Omega_{PL}(M), d)$ denotes the complex of piecewise polynomial differential forms (for a suitable triangulation of M), then the inclusion $(\Omega_{\mathrm{orb}}(M), d) \hookrightarrow (\Omega_{PL}(M), d)$ gives a quasi-isomorphism.

3. Symplectic resolutions

3.1. Symplectic orbifolds and their resolutions

Now we introduce the concepts of symplectic orbifold and symplectic resolution and show that any symplectic orbifold can be resolved into a smooth symplectic manifold.

Definition 3.1. A *symplectic orbifold* (M, ω) is a $2n$-dimensional orbifold M together with a 2-form $\omega \in \Omega^2_{\mathrm{orb}}(M)$ such that $d\omega = 0$ and $\omega^n \neq 0$ at every point.

Definition 3.2. A *symplectic resolution* of a symplectic orbifold (M, ω) is a smooth symplectic manifold $(\tilde{M}, \tilde{\omega})$ and a map $\pi : \tilde{M} \to M$ such that:

(a) π is a diffeomorphism $\tilde{M} \setminus E \to M \setminus \Delta$, where $\Delta \subset M$ is the singular set and $E = \pi^{-1}(\Delta)$ is the *exceptional set*.
(b) The exceptional set E is a union of possibly intersecting smooth symplectic submanifolds of $\tilde{M}$ of codimension at least 2.
(c) $\tilde{\omega}$ and $\pi^*\omega$ agree in the complement of a small neighbourhood of E.

In [13], it is given a method to obtain resolutions of symplectic orbifolds arising as quotients pre-symplectic semi-free S^1-actions. The following result gives an alternative method which is valid for any symplectic orbifold, and which is inspired in the resolution of isolated quotient singularities of complex manifolds.

Theorem 3.3. *Any symplectic orbifold has a symplectic resolution.*

Proof. Let p be a singular point of M. Take an orbifold chart $U_p = B_p/G_p$ around p, where $B_p \subset \mathbb{R}^{2n}$ is a ball. The symplectic form ω is a closed non-degenerate 2-form $\omega \in \Omega^2(U)$ invariant by G_p. By the equivariant Darboux theorem, there is a symplectomorphism $\varphi : (B_p, \omega) \to (B, \omega_0) \subset \mathbb{R}^{2n}$, where B is the standard ball and ω_0 the standard symplectic form in $\mathbb{R}^{2n}$, and a linear free G_p action on $\mathbb{R}^{2n} \setminus \{0\}$ for which the map above is G_p-equivariant (the proof of the existence of usual Darboux coordinates in [10, pp. 91–93] carries over to this case, only being careful that all the objects constructed should be G_p-equivariant). Therefore, the orbifold admits charts of the form B/G_p, where B a symplectic ball of $(\mathbb{R}^{2n}, \omega_0)$, such that G_p acts linearly by symplectomorphisms, that is $G_p \in Sp(2n, \mathbb{R})$.

Moreover, since the group $G_p \subset Sp(2n, \mathbb{R})$ is finite, we may take a metric on $\mathbb{R}^{2n}$ compatible with ω_0 and average it with respect to G_p. This gives a metric compatible with ω_0 and invariant by G_p therefore producing a G_p-invariant complex structure on B, so that we may interpret $B \subset \mathbb{C}^n$ and we have that $G_p \subset U(n)$. This induces a complex structure I on B/G_p, and

$$B/G_p \subset \mathbb{C}^n/G_p$$

has the complex and symplectic structure induced from the canonical complex and symplectic structures of $\mathbb{C}^n$.

For $n = 1$, the only finite subgroups of $U(1)$ are cyclic groups $\mathbb{Z}_m \subset U(1)$, and hence $B/G_p = \mathbb{C}/\mathbb{Z}_m$ is already non-singular. So in this case M has already the structure of smooth symplectic manifold.

For $n > 1$, we work as follows. For each $p \in \Delta$ consider a Kähler structure in a ball $U_p = B/G_p$ around p as above, where $B \subset \mathbb{C}^n$. The singular complex variety $X = \mathbb{C}^n/G_p$ is an affine algebraic variety with a single singularity at the origin. We can take an algebraic resolution of the singularity (it always can be done [8] by successive blow-up along smooth centers, starting with a single blowing-up at p), which is a quasi-projective variety $\pi_X : \tilde{X} \to X$. The exceptional set is a complex submanifold $E = \pi_X^{-1}(0)$. Consider some embedding $\tilde{X} \subset \mathbb{P}^N$ and let Ω be the induced Kähler form on $\tilde{X}$. Now let $\tilde{U} = \pi_X^{-1}(U)$. We glue $\tilde{U}$ to $M \setminus \{p\}$ by identifying $\tilde{U} \setminus E$ with $U_p \setminus \{p\}$ via $\varphi^{-1} \circ \pi_X$, to get a smooth manifold

$$\tilde{M} = \big(M \setminus \{p\}\big) \cup \tilde{U}.$$

There is an obvious projection $\pi : \tilde{M} \to M$.

We want to define a symplectic structure $\tilde{\omega}$ on $\tilde{M}$ which equals ω on $M \setminus U_p$. Consider the form $\varphi_*\omega$ on $U = B/G_p$ and its pull-back to $\tilde{U}$ via π_X, $\omega' = \pi_X^*\varphi_*\omega$. The annulus

$$A = \big(\tfrac{2}{3}\bar{B} \setminus \tfrac{1}{3}B\big)/G_p \subset B/G_p$$

is homotopy equivalent to S^{2n-1}/G_p, so we have that $\Omega - \omega' = d\alpha$ on A, for some $\alpha \in \Omega^1(A)$. Let ρ be a bump function which equals zero in $(B \setminus \frac{2}{3}B)/G_p$, and which equals one in $\frac{1}{3}B/G_p$. Define

$$\tilde{\omega} = \omega' + \epsilon d(\rho\alpha). \tag{7}$$

Then $\tilde{\omega} = \omega' = \pi^*\omega$ on $(B \setminus \frac{2}{3}B)/G_p$, so it can be glued with ω to define a smooth closed 2-form on $\tilde{M}$. On $\frac{1}{3}B/G_p$, we have

$$\tilde{\omega} = \omega' + \epsilon(\Omega - \omega') = (1 - \epsilon)\pi^*\omega + \epsilon\Omega. \tag{8}$$

Clearly, such $\tilde{\omega}$ is symplectic on $\frac{1}{3}B/G_p$ (actually it is a Kähler form there). Finally, as A is compact, the norm of $d(\rho\alpha)$ on A is bounded. As ω' is symplectic on A, choosing $\epsilon > 0$ small enough, we get that $\tilde{\omega}$, defined in (7), is also symplectic. □

Observe that for the symplectic resolution constructed in this theorem, we can say more about the exceptional set since it is modelled in the resolution of a singularity on an algebraic variety. Indeed, besides the conditions (a)–(c) from Definition 3.2, $\tilde{M}$ also satisfies:

(d) There exists a complex structure $\tilde{I}$ on a neighbourhood of E so that $(\tilde{\omega}, \tilde{I})$ is a Kähler structure.
(e) For the complex structure $\tilde{I}$ from (d) and $p \in \Delta$ one can find a complex structure, in a neighbourhood U of p making it Kähler and such that the resolution map $\pi : \tilde{U} \to U$ is holomorphic.

3.2. Cohomology of resolutions

We study the singular cohomology of symplectic resolutions with real coefficients. For the symplectic resolution $\tilde{M}$, this is isomorphic to the de Rham cohomology. For the orbifold M, this is isomorphic to the orbifold de Rham cohomology.

Proposition 3.4. *Let $\pi : \tilde{M} \to M$ be a symplectic resolution of a symplectic orbifold. For each $p \in \Delta$, let $E_p = \pi^{-1}(p)$ be the exceptional set. Then there is a split short exact sequence*

$$H^k(M) \xrightarrow{\pi^*} H^k(\tilde{M}) \xrightarrow{i^*} \prod_{p\in\Delta} H^k(E_p),$$

for $k > 0$.

Proof. We may assume that Δ consists only of one point p, since the general case follows from that by doing the resolution of the singularities one by one and taking the inverse limit in the noncompact case.

Let us see now that i^* is surjective. For the singular point p, let $U_p = B_p/G_p$ be a small ball around p and $\tilde{U}_p = \pi^{-1}(U_p)$ the corresponding neighbourhood of the exceptional set E_p. Then G_p acts freely and linearly on $B_p \setminus \{p\}$. By choosing an invariant metric, we see that B_p is foliated by spheres invariant under the G_p action, so not only is $B_p \setminus \{p\}$ a deformation retract of the rational homology sphere S^{2n-1}/G_p but also $U_p = (B_p \setminus \{p\})/G_p$ is a deformation retract of S^{2n-1}/G_p. In particular $\tilde{U}_p \setminus E_p \cong U_p \setminus \{p\}$ has the same real cohomology as S^{2n-1}.

Using the long exact sequence for relative cohomology for the pair $(\tilde{U}_p, \partial\tilde{U}_p)$, one easily sees that $H^k(\tilde{U}_p, \partial\tilde{U}_p) \cong H^k(\tilde{U}_p) \cong H^k(E_p)$, for $0 < k < 2n-1$. For $k = 2n-1$, we get

$$\begin{aligned} 0 \to H^{2n-1}(\tilde{U}_p, \partial\tilde{U}_p) \to H^{2n-1}(\tilde{U}_p) = 0 \to H^{2n-1}(\partial\tilde{U}_p) \\ = \mathbb{R} \to H^{2n}(\tilde{U}_p, \partial\tilde{U}_p) \to H^{2n}(\tilde{U}_p) = 0, \end{aligned}$$

so that $H^{2n-1}(\tilde{U}_p, \partial\tilde{U}_p) = 0$ and $H^{2n}(\tilde{U}_p, \partial\tilde{U}_p) = \mathbb{R}$. Actually, as $\tilde{U}_p$ is a compact oriented connected manifold with boundary, $H^{2n}(\tilde{U}_p, \partial\tilde{U}_p)$ is generated by the fundamental class $[\tilde{U}_p, \partial\tilde{U}_p]$. So we have a map given as the composition

$$f_p : H^k(E_p) \cong H^k(\tilde{U}_p) \cong H^k(\tilde{U}_p, \partial\tilde{U}_p) = H^k(\tilde{M}, \tilde{M} \setminus \tilde{U}_p) \hookrightarrow H^k(\tilde{M}),$$

for $0 < k < 2n$. It is easy to see that $i^* \circ f_p$ is the identity, thus proving the surjectivity of i^*.

Now we prove that π^* is injective. We define a map $\psi : H^k(\tilde{M}) \to H^k(M)$ for $0 < k < 2n-1$, as follows. The Mayer–Vietoris exact sequence of $M = (M \setminus \{p\}) \cup U_p$ gives

$$\cdots \to H^{k-1}\big(S^{2n-1}/G_p\big) \to H^k(M) \to H^k\big(M \setminus \{p\}\big) \oplus H^k(U_p) \to H^k\big(S^{2n-1}/G_p\big) \to \cdots,$$

so that $H^k(M) \cong H^k(M \setminus \{p\})$, for $0 < k < 2n-1$. We define ψ as the composition

$$\psi\colon\ H^k(\tilde{M}) \to H^k(\tilde{M} \setminus E_p) \xrightarrow{\cong} H^k\big(M \setminus \{p\}\big) \xrightarrow{\cong} H^k(M), \quad k < 2n-1, \tag{9}$$

using that $\pi:\tilde{M}\setminus E_p\to M\setminus\{p\}$ is a diffeomorphism. Clearly, $\psi\circ\pi^*$ is the identity, so that π^* is injective for $0<k<2n-1$. For $k=2n-1$ and $k=2n$, we have that $H^k(E_p)=0$, and there is a diagram whose rows are exact sequences:

$$\begin{array}{ccccccccccccc} 0 & \to & H^{2n-1}(M) & \to & H^{2n-1}(M\setminus\{p\}) & \to & H^{2n-1}(S^{2n-1}/G_p) & \to & H^{2n}(M) & \to & H^{2n}(M\setminus\{p\}) & \to & 0 \\ & & \downarrow & & \| & & \| & & \downarrow & & \| & & \\ 0 & \to & H^{2n-1}(\tilde{M}) & \to & H^{2n-1}(\tilde{M}\setminus E_p) & \to & H^{2n-1}(S^{2n-1}/G_p) & \to & H^{2n}(\tilde{M}) & \to & H^{2n}(\tilde{M}\setminus E_p) & \to & 0, \end{array}$$

which proves the assertion.

It remains to see that the sequence is exact in the middle. Clearly $\pi^*\circ i^*=0$. Also, for $k=2n-1,2n$ the statement is clear, since the previous paragraph proves that in this case $H^k(M)=H^k(\tilde{M})$. For $0<k<2n-1$ we work as follows. We write $\tilde{M}$ as a union of open sets, $\tilde{M}=(M\setminus\{p\})\cup\tilde{U}_p$, whose intersection $(M\setminus\{p\})\cap\tilde{U}_p=U_p\setminus\{p\}$ is homotopic to S^{2n-1}/G_p. Since $\tilde{U}_p$ is a deformation retract of E_p, the Mayer–Vietoris exact sequence gives

$$\cdots\to H^{k-1}\big(S^{2n-1}/G_p\big)\to H^k(\tilde{M})\to H^k(\tilde{M}\setminus E_p)\oplus H^k(E_p)\to H^k\big(S^{2n-1}/G_p\big)\to\cdots.$$

For $0<k<2n-1$ we have isomorphisms $H^k(\tilde{M})\cong H^k(M\setminus\{p\})\oplus H^k(E_p)\cong H^k(M)\oplus H^k(E_p)$. Actually, this map equals the map (ψ,i^*), with ψ defined in (9). Hence the sequence is exact in the middle. □

The last piece of data we need to describe the product in $H(\tilde{M})$ is the pairing

$$H^k(\tilde{U}_p)\otimes H^{2n-k}(\tilde{U}_p,\partial\tilde{U}_p)\to H^{2n}(\tilde{U}_p,\partial\tilde{U}_p)\cong\mathbb{R},$$

where the last isomorphism is given by integration on the fundamental class $[\tilde{U}_p,\partial\tilde{U}_p]$. Combining this pairing with the isomorphisms

$$H^k(\tilde{U}_p,\partial\tilde{U}_p)\cong H^k(\tilde{U}_p)\cong H^k(E_p),\quad 1\leqslant k\leqslant 2n-1,$$

we have a map (the local intersection product),

$$F_p:H^k(E_p)\otimes H^{2n-k}(E_p)\to\mathbb{R},\tag{10}$$

for $k=1,\dots,2n-1$.

Proposition 3.5. *Let $\pi:\tilde{M}\to M$ be a symplectic resolution of a compact connected symplectic orbifold, and let $\Psi:H^k(\tilde{M})\to H^k(M)\oplus(\bigoplus H^k(E_p))$ be the isomorphism given by the split exact sequence in Proposition* 3.4*, for $k>0$. Consider $a\in H^k(\tilde{M})$ and $b\in H^l(\tilde{M})$, with $k,l>0$, and denote $\Psi(a)=(a_1,(a_p)_{p\in\Delta})$ and $\Psi(b)=(b_1,(b_p)_{p\in\Delta})$. Then*

$$\Psi(a\cup b)=\begin{cases}(a_1\cup b_1,(a_p\cup b_p)_{p\in\Delta}), & k+l<2n,\\ \big(a_1\cup b_1+\sum_{p\in\Delta}F_p(a_p,b_p),0\big), & k+l=2n.\end{cases}$$

Proof. As in the proof of Proposition 3.4, it is enough to do the case where there is only one singular point p.

Consider $a_1, b_1 \in H^*(M)$, and let $a = \Psi^{-1}(a_1, 0) = \pi^*(a_1)$, $b = \Psi^{-1}(b_1, 0) = \pi^*(b_1)$. Then $\Psi^{-1}(a_1 \cup b_1, 0) = \pi^*(a_1 \cup b_1) = a \cup b$, i.e. $\Psi(a \cup b) = (a_1 \cup b_1, 0)$.

Now consider $a_1 \in H^k(M)$, $b_p \in H^l(E_p)$, and let $a = \Psi^{-1}(a_1, 0) = \pi^*(a_1)$, $b = \Psi^{-1}(0, b_p)$. Then b lies in the image of $H^l(\tilde{U}_p, \partial\tilde{U}_p) = H^l(\tilde{M}, \tilde{M} \setminus \tilde{U}_p) \hookrightarrow H^l(\tilde{M})$. So $a \cup b$ restricted to $\tilde{M} \setminus \tilde{U}_p$ is zero, i.e., $\psi(a \cup b) = 0$. On the other hand, the restriction of a to E_p is zero, so $(a \cup b)|_{E_p} = 0$. Therefore $\Psi(a \cup b) = 0$.

Finally, consider $a_p \in H^k(E_p)$, $b_p \in H^l(E_p)$, and let $a = \Psi^{-1}(0, a_p)$, $b = \Psi^{-1}(0, b_p)$. Clearly, $(a \cup b)|_{E_p} = a_p \cup b_p$. On the other hand, if $k + l < 2n$, $\psi(a \cup b) = 0$, since $(a \cup b)|_{M \setminus U_p} = 0$. If $k + l = 2n$, then $a \cup b$ is the image of $a_p \cup b_p$ under the map

$$H^{2n}(\tilde{U}_p, \partial\tilde{U}_p) \to H^{2n}(\tilde{M}, \tilde{M} \setminus \tilde{U}_p) \to H^{2n}(\tilde{M}) = \mathbb{R}.$$

Then $\Psi(a \cup b) = (F_p(a_p, b_p), 0)$. □

Remark 3.6. The pairing F_p is non-degenerate. We can prove this as follows: take a compact orbifold M with just one singular point p of the required type (see the proof of Theorem 3.9 where a construction of such an orbifold is done). Then $\tilde{M}$ is a compact oriented manifold, hence the intersection product $H^k(\tilde{M}) \otimes H^{n-k}(\tilde{M}) \to \mathbb{R}$ satisfies Poincaré duality. By Proposition 3.5, under the isomorphism $\Psi : H^k(\tilde{M}) = H^k(M) \oplus H^k(E_p)$, the intersection product of $H^k(\tilde{M})$ decomposes as the intersection product on $H^k(M)$ and the pairing F_p on $H^k(E_p)$. Hence both should be non-degenerate.

The non-degeneracy of F_p implies that $\dim H^k(E_p) = \dim H^{2n-k}(E_p)$. In particular, $H^1(E_p) = 0$.

3.3. The Lefschetz property and resolutions

Now we study how the Lefschetz property behaves under symplectic resolutions and prove that the resolution $(\tilde{M}, \tilde{\omega})$, constructed in Theorem 3.3, satisfies the Lefschetz property if and only if (M, ω) does.

Let $\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ be a symplectic resolution. Let p be a singular point of M, then we have a local intersection

$$F_p : H^k(E_p) \otimes H^{2n-k}(E_p) \to \mathbb{R}.$$

Definition 3.7. We say that the resolution satisfies the local Lefschetz property at E_p if the map

$$[\tilde{\omega}]^{n-k} : H^k(E_p) \to H^{2n-k}(E_p) \tag{11}$$

is an isomorphism for $k = 1, \dots, 2n - 1$.

Note that the above definition only depends on the restriction of $[\tilde{\omega}]$ to E_p.

Proposition 3.8. *Let $\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ be a symplectic resolution of a symplectic orbifold of dimension $2n$. Suppose that π satisfies the local Lefschetz property at every divisor E_p, $p \in \Delta$. Then, for any $k = 1, \dots, n$, the kernel of*

$$[\tilde{\omega}]^{n-k} : H^k(\tilde{M}) \to H^{2n-k}(\tilde{M})$$

is isomorphic to the kernel of

$$[\omega]^{n-k} : H^k(M) \to H^{2n-k}(M).$$

In particular, if (M, ω) satisfies the Lefschetz property so does $(\tilde{M}, \tilde{\omega})$.

Proof. We may suppose that we only do the resolution at one point. The general case follows from this one. Also we may assume that $\dim M = 2n \geqslant 4$. By property (c) in Definition 3.2, $\tilde{\omega}$ and $\pi^*\omega$ agree on a neighbourhood of the complement of the exceptional divisor, so denoting by $\Psi : H^k(\tilde{M}) \to H^k(M) \oplus H^k(E_p)$ the isomorphism coming from Proposition 3.4, we have

$$\Psi\big([\tilde{\omega}]\big) = \big([\omega], [\tilde{\omega}|_{E_p}]\big).$$

By Proposition 3.5, the map

$$[\tilde{\omega}]^{n-k} : H^k(\tilde{M}) \to H^{2n-k}(\tilde{M})$$

decomposes under the isomorphism Ψ as the direct sum of the two maps,

$$[\omega]^{n-k} : H^k(M) \to H^{2n-k}(M),$$

and

$$[\tilde{\omega}]^{n-k} : H^k(E_p) \to H^{2n-k}(E_p).$$

If the local Lefschetz property is satisfied, the second map is an isomorphism. The result follows. □

Theorem 3.9. *The symplectic resolution $\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ constructed in Theorem 3.3 satisfies the local Lefschetz property at E_p, for each singular point $p \in M$. So (M, ω) satisfies the Lefschetz property if and only if $(\tilde{M}, \tilde{\omega})$ does.*

Proof. Take the complex projective variety $\bar{X} = \mathbb{P}^n / G_p$ with the linear action of G_p which extends that on $\mathbb{C}^n \subset \mathbb{P}^n$. Resolve its singularities [8] at the infinity to obtain a projective variety Z with a single isolated singularity at p. Let $\pi : \tilde{Z} \to Z$ be the resolution of the singularity at p. Then $\tilde{Z}$ is a smooth projective variety, hence it satisfies the hard-Lefschetz property, that is, if Ω denotes the Kähler form of $\tilde{Z}$, then

$$[\Omega]^{n-k} : H^k(\tilde{Z}) \to H^{2n-k}(\tilde{Z}) \tag{12}$$

is an isomorphism.

Let $U = B/G_p$ be a small neighborhood of $p \in Z$, and let $\tilde{U} = \pi^{-1}(U)$. Then Proposition 3.5 implies that the map (12) decomposes as a direct sum of the maps $[\omega_Z]^{n-k} : H^k(Z) \to H^{2n-k}(Z)$ and $[\Omega]^{n-k} : H^k(E_p) \to H^{2n-k}(E_p)$, where $E_p = \pi^{-1}(p)$ and ω_Z is the Kähler form of Z. So the map

$$[\Omega]^{n-k} : H^k(E_p) \to H^{2n-k}(E_p) \tag{13}$$

is an isomorphism.

Now let $\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ be a symplectic resolution at a point p with local model B/G_p, as carried out in Theorem 3.3. Then

$$\tilde{\omega}|_{\tilde{V}} = (1-\epsilon)\pi^*\omega + \epsilon\Omega,$$

in a neighborhood $\tilde{V} = \pi^{-1}(V)$ of E_p. But in V, $\omega = d\gamma$ for a 1-form γ, which we can suppose G_p-invariant, so $\pi^*\omega$ is exact in V. So, restricting to $\tilde{V}$, $[\tilde{\omega}] = [\epsilon\Omega]$. As the map (13) is an isomorphism, so is the map

$$[\tilde{\omega}]^{n-k} : H^k(E_p) \to H^{2n-k}(E_p),$$

completing the theorem. □

3.4. Resolutions and a-Massey products

We show that a-Massey products are also well behaved with respect to symplectic resolutions.

Theorem 3.10. *Let $\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ be a symplectic resolution of a symplectic orbifold (M, ω). If M has a non-trivial a-product, then so does $\tilde{M}$.*

Proof. As before, we can assume, without loss of generality, that there is only one singular point p.

Let $\mathcal{A} \subset \Omega_{\text{orb}}(M)$ be the algebra of smooth forms which are constant (for degree 0) and zero (for degree > 0) in a neighborhood of the critical point p. Then the map

$$\mathcal{A} \hookrightarrow \Omega_{\text{orb}}(M)$$

is a quasi-isomorphism. According to Lemma 2.9, there is a non-zero a-product $\langle a; b_1, \dots, b_m \rangle$ on $\mathcal{A}$. The inclusion

$$\pi^* : \mathcal{A} \to \Omega(\tilde{M})$$

is a map of DGAs which induces the injection $\pi^* : H^k(M) \to H^k(\tilde{M})$ for $k > 0$ (it also induces an injection for $k = 0$). To prove our result we will show that

$$\pi^*\langle a; b_1, \dots, b_m \rangle = \big\langle \pi^*(a); \pi^*(b_1), \dots, \pi^*(b_m) \big\rangle.$$

The inclusion

$$\pi^*\langle a; b_1, \dots, b_m \rangle \subset \big\langle \pi^*(a); \pi^*(b_1), \dots, \pi^*(b_m) \big\rangle$$

is obvious. So we only have to prove the converse.

Let $[c'] = [\sum \overline{\xi_1'} \wedge \ldots \wedge \pi^*(b_i) \wedge \ldots \wedge \xi_m'] \in \langle \pi^*(a); \pi^*(b_1), \ldots, \pi^*(b_m)\rangle$, where $\pi^*(a) \wedge \pi^*(b_i) = d\xi_i'$. And let $\xi_i \in \mathcal{A}$ be such that $d\xi_i = a \wedge b_i$. Then $d(\xi_i' - \pi^*(\xi_i)) = 0$, so $\xi_i' - \pi^*(\xi_i)$ represents a cohomology class.

Using Proposition 3.4, we may decompose $[\xi_i' - \pi^*(\xi_i)] = \pi^* s_{i,1} + s_{i,p}$, where $s_{i,1} \in H^k(M)$ and $s_{i,p} \in H(\tilde{U}_p, \partial \tilde{U}_p) \subset H(\tilde{M})$. Here we choose U_p to be disjoint of the support of b_i for all i. We represent $s_{i,1}$ by a form $\zeta_i \in \mathcal{A} \subset \Omega_{\mathrm{orb}}(M)$ and $s_{i,p}$ by a form $\eta_i \in \Omega(\tilde{U}_p, \partial\tilde{U}_p)$ (which can be thought of as a form on $\tilde{M}$ supported inside $\tilde{U}_p$). So we can write

$$\xi_i' - \pi^*(\xi_i) = \pi^*(\zeta_i) + \eta_i + dz_i, \tag{14}$$

where $z_i \in \Omega(\tilde{M})$. Therefore, $d(\xi_i + \zeta_i) = a \wedge b_i$ and

$$[c] = \Big[\sum \overline{(\xi_1 + \zeta_1)} \wedge \ldots \wedge b_i \wedge \ldots \wedge (\xi_m + \zeta_m)\Big] \in \langle a; b_1, \ldots, b_m\rangle$$

is such that

$$\begin{aligned}
\pi^*[c] &= \Big[\sum \overline{\pi^*(\xi_1 + \zeta_1)} \wedge \ldots \wedge \pi^* b_i \wedge \ldots \wedge \pi^*(\xi_m + \zeta_m)\Big] \\
&= \Big[\sum \big(\overline{\xi_1' - \eta_1}\big) \wedge \ldots \wedge \pi^* b_i \wedge \ldots \wedge \big(\xi_m' - \eta_m\big)\Big] \\
&= \Big[\sum \overline{\xi_1'} \wedge \ldots \wedge \pi^* b_i \wedge \ldots \wedge \xi_m'\Big] \\
&= [c'],
\end{aligned}$$

where in the second equality we have used (14), Lemma 2.6 and Remark 2.8, and in the third equality we used that $\eta_i \wedge \pi^* b_j = 0$ since these forms have disjoint supports. This shows the reverse inclusion and finishes the theorem. □

4. Symplectic blow-up

In this section we recall results about the behaviour of the Lefschetz property under ordinary symplectic blow-up, as introduced by McDuff [9], and we study the behaviour of a-products under this construction.

In what follows, we let M^{2n} be a symplectic manifold/orbifold and $N^{2(n-k)} \subset M$ be a symplectic submanifold which does not intersect the orbifold singularities. We let $\pi : \tilde{M} \to M$ be the symplectic blow-up of M along N. Then the cohomology of $\tilde{M}$ is given by

$$H^i(\tilde{M}) = H^i(M) \oplus H^{i-2}(N)[\sigma] + H^{i-4}(N)[\sigma]^2 + \cdots + H^{i-2k+2}(N)[\sigma]^{k-1},$$

where σ is a closed 2-form such that σ^{k-1} has non-zero integral over the $\mathbb{C}P^{k-1}$ fibers of the exceptional divisor. The multiplication rules are the obvious ones using the restriction of elements on $H^i(M)$ to $H^i(N)$ together with the extra relation

$$[\sigma]^k = -PD(N) - c_{k-1}[\sigma] - \cdots - c_1[\sigma]^{k-1},$$

where c_i are the Chern classes of the normal bundle of N, and $PD(N)$ is the Poincaré dual of N.

Similarly to the case of symplectic resolutions the summand $H(M) \hookrightarrow H(\tilde{M})$ is given by the image of the pull-back $\pi^*: H(M) \to H(\tilde{M})$. However, unlike the case of resolutions, in general one cannot choose representatives for the cohomology classes in $H(M) \hookrightarrow H(\tilde{M})$ with support away from the exceptional set.

4.1. The Lefschetz property

There is a contrast between the behaviour of the maps $[\omega]^{n-k}: H^k(M) \to H^{2n-k}(M)$ under resolution of singularities and under ordinary symplectic blow-up. While we have proved that in the former case these maps have the same kernel, the same is not true for the latter. Indeed, in [2], the first author proved that one can reduce the dimension of the kernel of the map $[\omega]^{n-k}$ by blowing-up along specific submanifolds. The result from [2] adapted to the case we study is the following:

Theorem 4.1. *Given a symplectic orbifold (M^{2n}, ω) and a symplectic surface $\Sigma^2 \subset M$ disjoint from the singular set, let $\pi: \tilde{M} \to M$ be the symplectic blow-up of M along Σ. Then there is a symplectic form $\tilde{\omega}$ on $\tilde{M}$ such that in $H^2(\tilde{M})$*

$$\ker\bigl([\tilde{\omega}]^{n-2}\cup\bigr) = \pi^*\bigl(\ker\bigl(\bigl[\omega^{n-2}\bigr]\cup\bigr) \cap \ker\bigl(PD(\Sigma)\cup\bigr)\bigr),$$

where $PD(\Sigma)$ denotes the Poincaré dual of Σ. Furthermore, in $H^k(\tilde{M})$, for $k > 2$, we have

$$\ker\bigl([\tilde{\omega}]^{n-k}\cup\bigr) = \pi^*\bigl(\ker\bigl(\bigl[\omega^{n-k}\bigr]\cup\bigr)\bigr).$$

4.2. Symplectic blow-up and a-Massey products

Similarly, a-Massey products also behave differently under symplectic blow-up. We focus our attention on the triple a-product.

Theorem 4.2. *Let (M^{2n}, ω) be a symplectic orbifold, $N^{2(n-k)} \hookrightarrow M$ be a symplectic submanifold disjoint from the orbifold singularities and $\tilde{M}$ the symplectic blow-up of M along N. Then:*

1. *If M has a nontrivial triple a-product, say, $\langle a; b_1, b_2, b_3\rangle$, and $|a| + \lfloor(|b_i| - 1)/2\rfloor + \lfloor(|b_j| - 1)/2\rfloor \leqslant k - 1$, for all i, j, where $\lfloor\alpha\rfloor$ denotes integer part of α, then so does $\tilde{M}$.*
2. *If $H^{\mathrm{odd}}(N) = \{0\}$, $k > 5$ and N has a nontrivial triple a-product, then so does $\tilde{M}$.*

Proof. We start with the proof of the first claim. Let $\langle a; b_1, b_2, b_3\rangle$ be a nontrivial a-product in M. This means that $a \wedge b_i$ is exact and

$$0 \notin \bigl\{[b_1 \wedge \xi_2 \wedge \xi_3 + \overline{\xi_1} \wedge b_2 \wedge \xi_3 + \overline{\xi_1} \wedge \overline{\xi_2} \wedge b_3] \mid d\xi_i = a \wedge b_i\bigr\}.$$

Since the form $a \wedge b_i$ is exact in M, $\pi^*a \wedge \pi^*b_i$ is exact in $\tilde{M}$, hence the a-product is defined on $\tilde{M}$. According to Lemma 2.6 and Remark 2.8, once we fix the $\pi^*\xi_i$, the a-product is obtained by adding closed forms to $\pi^*\xi_i$ and only depends on the cohomology class of the closed forms added. In particular, we can assume that these closed forms are of the standard form $\eta_i = \sum \eta_{ij}\sigma^j$, so that the generic element of the product is given by

$$\begin{aligned}&\big[\pi^*b_1\wedge\big(\pi^*\xi_2+\eta_2\big)\wedge\big(\pi^*\xi_3+\eta_3\big)+\big(\overline{\pi^*\xi_1+\eta_1}\big)\wedge\pi^*b_2\wedge\big(\pi^*\xi_3+\eta_3\big)\\&\quad+\big(\overline{\pi^*\xi_1+\eta_1}\big)\wedge\big(\overline{\pi^*\xi_2+\eta_2}\big)\wedge\pi^*b_3\big].\end{aligned}$$

Due to the hypothesis about k and $|a|$, $|b_i|$, we see that the component of the above class lying in $H(M)\subset H(\tilde{M})$ is precisely the original a-product as there are no powers of σ higher than $k-1$ appearing when the product is computed. Since the original product was nontrivial, so is the product induced on $H(\tilde{M})$.

To prove the second claim, we let $\langle a; b_1, b_2, b_3\rangle$ be a nontrivial a-product on N. This can only be the case if a and b_i are even degree forms, due to Lemma 2.5, as $H^{\mathrm{odd}}(N)=\{0\}$. Further, $H^{\mathrm{odd}}(N)=\{0\}$ together with Proposition 2.7 implies that $\langle a; b_1, b_2, b_3\rangle$ has no indeterminacy and hence is a single cohomology class. Now consider the closed forms $a\wedge\sigma, b_i\wedge\sigma\in H^{\mathrm{even}}(\tilde{M})$. The relations $a\wedge b_i=d\xi_i$ imply that

$$a\wedge\sigma\wedge b_i\wedge\sigma=d\xi_i\wedge\sigma^2,$$

and hence

$$0\neq\langle a; b_1, b_2, b_3\rangle[\sigma]^5\in\langle a\wedge\sigma; b_1\wedge\sigma, b_2\wedge\sigma, b_3\wedge\sigma\rangle\cap H(N)[\sigma]^5. \tag{15}$$

According to Proposition 2.7, the indeterminacy of this product is a subset of

$$\begin{aligned}&\langle b_1\wedge\sigma, a\wedge\sigma, b_2\wedge\sigma\rangle H^{|b_3|-1}(\tilde{M})+\langle b_2\wedge\sigma, a\wedge\sigma, b_3\wedge\sigma\rangle H^{|b_1|-1}(\tilde{M})\\&\quad+\langle b_3\wedge\sigma, a\wedge\sigma, b_1\wedge\sigma\rangle H^{|b_2|-1}(\tilde{M}).\end{aligned}$$

Since $H^{\mathrm{odd}}(N)=\{0\}$, all the triple Massey products above lie in $H(M)\subset H(\tilde{M})$ and also $H^{|b_i|-1}(\tilde{M})=H^{|b_i|-1}(M)$, so the indeterminacy of the $(a\wedge\sigma)$-product is a subset of $H(M)$, but the representative (15) does not belong to this set, hence the product does not vanish. □

Remark 4.3. We must notice that for the case that we want to consider, that is, when M is simply connected and 8-dimensional, the hypothesis of the first part of the Theorem 4.2 only holds if we are blowing-up along a symplectic submanifold of dimension 2 and a and b_i are forms of degree 2. The second item was included for sake of completeness and can only happen in higher dimensions. Indeed, the first even dimension where a-Massey products can appear is 8, hence in order for item 2 of the theorem above to be used one should need the ambient manifold to be at least 20-dimensional.

5. Examples

In this section we give an example of a simply connected symplectic 8-manifold which satisfies the Lefschetz property but is not formal. In order to explain our example, we recall the example given by the last two authors in [5].

Example 5.1. Consider G, the product of the complex Heisenberg group H, with $\mathbb{C}$. As a manifold G is diffeomorphic to $\mathbb{C}^4$ but with a group structure induced by the following embedding in $\mathrm{GL}(5,\mathbb{C})$

$$(z_1,z_2,z_3,z_4)\mapsto\begin{pmatrix}1 & z_1 & z_3 & 0 & 0\\ 0 & 1 & z_2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & z_4\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

Letting ξ be a cubic root of 1, we have that 1 and ξ generate a lattice $\Lambda\subset\mathbb{C}$ and then we obtain a cocompact lattice $\Gamma\subset G$ given by the matrices whose entries lie in Λ. Further, the map

$$\rho:G\to G,\quad \rho(z_1,z_2,z_3,z_4)=\bigl(\xi z_1,\xi z_2,\xi^2 z_3,\xi z_4\bigr),$$

generates a $\mathbb{Z}_3$ action on G which preserves the lattice Γ and the group structure. Therefore it induces a $\mathbb{Z}_3$ action on the compact nilmanifold $\Gamma\setminus G$. This action is free away from 81 fixed points corresponding to $z_i=n/(1-\xi)$, for $n=0$, 1 and 2.

The orbifold $M=\Gamma\setminus G/\mathbb{Z}_3$ has a symplectic structure. Indeed, if we consider the left-invariant complex 1-forms $u^1=dz_1$, $u^2=dz_2$, $u^3=dz_3-z_1\,dz_2$, $u^4=dz_4$ defined on G, we see that $du^1=du^2=du^4=0$, $du^3=u^{12}$ and that

$$\omega=iu^{1\bar 1}+u^{23}+u^{\bar 2\bar 3}+iu^{4\bar 4}$$

is a $(\Gamma\times\mathbb{Z}_3)$-invariant symplectic 2-form, hence induces a symplectic structure on the orbifold M, where we are using the sort hand notation $u^{ij}=u^i\wedge u^j$, $u^{\bar i}=\overline{u^i}$, $u^{i\bar j}=u^i\wedge\overline{u^j}$, etc. The orbifold M is simply connected and has vanishing odd Betti numbers [5]. Furthermore, it has a nonvanishing a-Massey product. Indeed, if we let

$$a=u^{1\bar 1},\qquad b_1=u^{2\bar 2},\qquad b_2=u^{2\bar 4},\qquad b_3=u^{\bar 2 4},$$

then a and b_i are closed and invariant under the $\mathbb{Z}_3$ action, so define closed forms on M. Further

$$a\wedge b_1=-du^{12\bar 3},\qquad a\wedge b_2=du^{\bar 1 3\bar 4},\qquad a\wedge b_3=-du^{1\bar 3 4}.$$

Hence we can compute the a-Massey product

$$\begin{aligned}\langle a;b_1,b_2,b_3\rangle&=-u^{12\bar 3}\wedge u^{\bar 1 3\bar 4}\wedge u^{\bar 2 4}-u^{\bar 1 3\bar 4}\wedge u^{1\bar 3 4}\wedge u^{2\bar 2}+u^{1\bar 3 4}\wedge u^{12\bar 3}\wedge u^{2\bar 4}\\&=2u^{1\bar 1 2\bar 2 3\bar 3 4\bar 4}.\end{aligned}$$

Since $H^5(M)=\{0\}$, the Massey products $\langle b_i,a,b_j\rangle\in H^5(M)$ vanish and, according to Proposition 2.7, the product above has indeterminacy zero, thus it is a non-trivial a-Massey product. Finally, according to Theorems 3.3 and 3.10, the symplectic resolution of M is a simply connected non-formal 8-dimensional symplectic manifold.

Example 5.2. As shown in [5], the orbifold M obtained in the previous example has vanishing odd Betti numbers, so in order to check whether it satisfies the Lefschetz property, one only needs to consider $[\omega]^2 : H^2(M) \to H^6(M)$. We show that while for M this map is not an isomorphism, one can blow up M along three symplectic tori to obtain an orbifold which does satisfy the Lefschetz property, but which still has non-trivial a-Massey products.

We start determining the second cohomology of M. This is given by the $\mathbb{Z}_3$-invariant part of the Lie algebra cohomology of G and has an ordered basis given by

$$\{u^{1\bar{1}}, u^{4\bar{4}}, u^{23}, u^{\bar{2}\bar{3}}, u^{\bar{1}2}, u^{13}, u^{1\bar{2}}, u^{\bar{1}\bar{3}}, u^{1\bar{4}}, u^{\bar{1}4}, u^{2\bar{2}}, u^{2\bar{4}}, u^{\bar{2}4}\},$$

where u^i are the invariant 1-forms introduced in the previous example. In this basis the pairing $[\omega]^2 : H^2(M;\mathbb{C}) \times H^2(M;\mathbb{C}) \to \mathbb{C}$ is given by Table 1.

Table 1

	$u^{1\bar{1}}$	$u^{4\bar{4}}$	u^{23}	$u^{\bar{2}\bar{3}}$	$u^{\bar{1}2}$	u^{13}	$u^{1\bar{2}}$	$u^{\bar{1}\bar{3}}$	$u^{1\bar{4}}$	$u^{\bar{1}4}$	$u^{2\bar{2}}$	$u^{2\bar{4}}$	$u^{\bar{2}4}$
$u^{1\bar{1}}$	**0**	**−1**	**−i**	**−i**	0	0	0	0	0	0	0	0	0
$u^{4\bar{4}}$	**1**	**0**	**−i**	**−i**	0	0	0	0	0	0	0	0	0
u^{23}	**−i**	**−i**	**0**	**1**	0	0	0	0	0	0	0	0	0
$u^{\bar{2}\bar{3}}$	**−i**	**−i**	**1**	**0**	0	0	0	0	0	0	0	0	0
$u^{\bar{1}2}$	0	0	0	0	**0**	**−i**	0	0	0	0	0	0	0
u^{13}	0	0	0	0	**−i**	**0**	0	0	0	0	0	0	0
$u^{1\bar{2}}$	0	0	0	0	0	0	**0**	**i**	0	0	0	0	0
$u^{\bar{1}\bar{3}}$	0	0	0	0	0	0	**i**	**0**	0	0	0	0	0
$u^{1\bar{4}}$	0	0	0	0	0	0	0	0	**0**	**−1**	0	0	0
$u^{\bar{1}4}$	0	0	0	0	0	0	0	0	**−1**	**0**	0	0	0
$u^{2\bar{2}}$	0	0	0	0	0	0	0	0	0	0	**0**	0	0
$u^{2\bar{4}}$	0	0	0	0	0	0	0	0	0	0	0	**0**	0
$u^{\bar{2}4}$	0	0	0	0	0	0	0	0	0	0	0	0	**0**

Hence, the kernel of ω^2 has real basis $\{iu^{2\bar{2}}, u^{2\bar{4}} + u^{\bar{2}4}, i(u^{2\bar{4}} - u^{\bar{2}4})\}$. Now we split each u^i into real and imaginary parts $u^j = e^{2j-1} + ie^{2j}$, so that the kernel of ω^2 is generated by e^{34}, $e^{37} - e^{48}$ and $e^{47} + e^{38}$.

In terms of the real basis $\{e_i\}$, where e_i is the invariant vector field dual to e^i, the Lie algebra $\mathfrak{g}$ of G has the following structure:

$$-[e_1, e_3] = -[e_2, e_4] = e_5, \qquad -[e_1, e_4] = -[e_2, e_3] = e_6,$$

and the symplectic form is

$$\omega = e^{12} + e^{35} - e^{46} + e^{78}.$$

Observe that since the lattice Γ is given by matrices whose entries are in the lattice Λ generated by 1 and ξ, the vector fields e_{2i-1} have period 1 ($1 \in \Lambda$), while the vector fields e_{2i} have period $\sqrt{3} = \frac{1+2\xi}{i}$ (note that $1 + 2\xi \in \Lambda$).

For the example at hand, we consider the abelian Lie sub-algebras of $\mathfrak{g}$ generated by

$$\{e_3 + e_7, e_4 + e_8\}, \qquad \{e_3 + \sqrt{3}e_8, e_7\} \quad \text{and} \quad \{e_3 + e_7, e_8\}.$$

Each of these Lie algebras integrates to a Lie subgroup of G and the lattice Γ restricts to a cocompact lattice on each of the subgroups. Therefore, each of the abelian algebras gives rise to a fibration of $\Gamma \setminus G$ by embedded tori. One can clearly see that these tori are symplectic and by a general position argument, we can choose three tori, T_i, one torus on each family, so that they do not intersect each other and also they do not pass through the fixed points of the $\mathbb{Z}_3$ action. Thus, their images via the quotient map $\Gamma \setminus G \to M$ are three disjoint embedded tori which do not meet the orbifold singularities.

By Theorem 4.1, there is a symplectic form $\tilde{\omega}$ on $\tilde{M}$, the blow-up of M along the three tori, such that the kernel of $[\tilde{\omega}]^2 : H^2(\tilde{M}) \to H^6(\tilde{M})$ is given by

$$\pi^*\big(\ker\big(\big[\omega^2\big]\cup\big) \cap \ker\big(PD(T_1)\cup\big) \cap \ker\big(PD(T_2)\cup\big) \cap \ker\big(PD(T_3)\cup\big)\big),$$

but by choice, each of the $PD(T_i)$ pairs nontrivially with one of the elements in the basis $\{e^{34}, e^{37} - e^{48}, e^{47} + e^{38}\}$ for $\ker([\omega^2]\cup)$. Hence $\tilde{\omega}^2 : H^2(\tilde{M}) \to H^6(\tilde{M})$ is an isomorphism and the orbifold $\tilde{M}$ satisfies the Lefschetz property. Further, due to Theorem 4.2, $\tilde{M}$ has a nontrivial a-Massey product, hence it is not formal.

According to Theorems 3.3, 3.9 and 3.10, the symplectic resolution of $\tilde{M}$ satisfies the Lefschetz property and has a non-trivial a-Massey product. This example shows that the Lefschetz property is not related to formality in dimension 8.

Acknowledgments

This work has been partially supported through grants MCyT (Spain) MTM2004-07090-C03-01 and MTM2005-08757-C04-02 as well as EPSRC (UK).

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