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Restrained domination in graphs

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Abstract

In this paper, we initiate the study of a variation of standard domination, namely restrained domination. Let $G=(V,E)$ be a graph. A restrained dominating set is a set $S\subseteq V$ where every vertex in $V-S$ is adjacent to a vertex in S as well as another vertex in $V-S$. The restrained domination number of G, denoted by $\gamma_{\mathrm{r}}(G)$, is the smallest cardinality of a restrained dominating set of G. We determine best possible upper and lower bounds for $\gamma_{\mathrm{r}}(G)$, characterize those graphs achieving these bounds and find best possible upper and lower bounds for $\gamma_{\mathrm{r}}(G)+\gamma_{\mathrm{r}}(\overline{G})$ where G is a connected graph. Finally, we give a linear algorithm for determining $\gamma_{\mathrm{r}}(T)$ for any tree and show that the decision problem for $\gamma_{\mathrm{r}}(G)$ is NP-complete even for bipartite and chordal graphs.

1. Introduction

Graph theory terminology not presented here can be found in [1]. Let $G=(V,E)$ be a graph. For any vertex $v\in V$, the *open neighborhood of* v, denoted by $N(v)$, is defined by $\{u\in V \mid uv\in E\}$. A set S is a *dominating set* if for every vertex $u\in V-S$, there exists $v\in S$ such that $uv\in E$. The *domination number of* G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A set $S\subseteq V$ is a *restrained dominating set* if every vertex in $V-S$ is adjacent to a vertex in S and another vertex in $V-S$. The concept of restrained domination was introduced by Telle [6], albeit as a vertex partitioning problem. Note that every graph has a restrained dominating set, since $S=V$ is such a set. Let $\gamma_{\mathrm{r}}(G)$ denote the size of

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a smallest restrained dominating set. We will call a set S a $\gamma_{\rm r}$*-set* if S is a restrained dominating set of cardinality $\gamma_{\rm r}(G)$.

One possible application of the concept of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. Note that each prisoner's position is observed by a guard's position (to effect security) while each prisoner's position is seen by at least one other prisoner's position (to protect the rights of prisoners). To be cost effective, it is desirable to place as few guards as possible (in the sense above).

In Section 2, we determine this parameter for certain classes of graphs, obtain best possible upper and lower bounds for $\gamma_{\rm r}(G)$ and characterize those graphs achieving these bounds. Then, in Section 3, we find best possible upper and lower bounds for $\gamma_{\rm r}(G)+\gamma_{\rm r}(\overline{G})$ where G is a connected graph. Finally, in Section 4, we give a linear algorithm for determining $\gamma_{\rm r}(T)$ for any tree and show that the decision problem for $\gamma_{\rm r}(G)$ is NP-complete even for bipartite and chordal graphs.

2. Definitions and results

Let K_n, C_n and P_n denote, respectively, the complete graph, the cycle and the path of order n. Also, let $K_{n_1,\dots,n_t}$ denote the complete multipartite graph with vertex set $S_1\cup\cdots\cup S_t$ where $|S_i|=n_i$ for $1\leqslant i\leqslant t$. We call $K_{1,n-1}$ a *star*. A *subdivision* of an edge uv is obtained by introducing a new vertex w and replacing the edge uv with the edges uw and wv. A *spider* is a tree obtained from $K_{1,r}$, $r\geqslant 1$, by subdividing all of its edges. A *wounded spider* is a tree obtained from $K_{1,r}$, $r\geqslant 1$, by subdividing at most $r-1$ of its edges. Thus, the star, $K_{1,r}$, is also a wounded spider. In a tree, a *stem* is a vertex adjacent to a leaf (a vertex of degree one).

The following results are immediate.

Proposition 1. *If $n\neq 2$ is a positive integer, then $\gamma_{\rm r}(K_n)=1$.*

Proposition 2. *If $n\geqslant 2$ is an integer, then $\gamma_{\rm r}(K_{1,n-1})=n$.*

Proposition 3. *If n_1 and n_2 are integers such that $\min\{n_1,n_2\}\geqslant 2$, then $\gamma_{\rm r}(K_{n_1,n_2})=2$.*

Proposition 4. *If $t\geqslant 3$ is an integer, then*

$$\gamma_{\rm r}(K_{n_1,\dots,n_t})=\begin{cases}1 & \textit{if } \min\{n_1,\dots,n_t\}=1,\\ 2 & \textit{otherwise.}\end{cases}$$

It is clear from their definition that $\gamma(G)\leqslant\gamma_{\rm r}(G)$. Suppose $n\geqslant 1$ and let $k\in\{1,\dots,n-2,n\}$. Let G be the graph obtained from P_{n-k}, the path on $n-k$ vertices, by adding a set of vertices $\{v,v_1,\dots,v_{k-1}\}$ and joining the vertex v to each of the vertices in $V(P_{n-k})\cup\{v_1,\dots,v_{k-1}\}$. Then G has order n, $\gamma_{\rm r}(G)=k$ and $\gamma(G)=1$. Hence, we have

Proposition 5. *There exists a graph G for which* $\gamma_r(G)-\gamma(G)$ *can be made arbitrarily large.*

Proposition 6. *If* $n\geqslant 1$ *is an integer, then* $\gamma_r(P_n)=n-2\lfloor(n-1)/3\rfloor$.

Proof. Suppose S is a restrained dominating set of P_n, whose vertex set is $V=\{v_1,\dots,v_n\}$. Note that $v_1,v_n\in S$. Moreover, any component of $V-S$ is of size exactly two. Suppose there are m such components. Then $2m+m+1\leqslant n$ and so $m\leqslant\lfloor(n-1)/3\rfloor$. Thus $|S|=n-2m\geqslant n-2\lfloor(n-1)/3\rfloor$.

On the other hand, $V-\{v_i \mid 1\leqslant i\leqslant 3\lfloor(n-1)/3\rfloor,\ i\equiv 2 \text{ or } 3 \,(\mathrm{mod}\, 3)\}$ is a restrained dominating set of size $n-2\lfloor(n-1)/3\rfloor$. □

We omit the proof of the following result as it is similar to that of Proposition 6.

Proposition 7. *If* $n\geqslant 3$, *then* $\gamma_r(C_n)=n-2\lfloor n/3\rfloor$.

It is clear that $\gamma_r(G)\leqslant n$ for any graph G of order n. The following result shows that the star $K_{1,n-1}$ is the only connected graph G of order n for which $\gamma_r(G)=n$. We omit the (easy) proof.

Proposition 8. *Let G be a connected graph of order n. Then* $\gamma_r(G)=n$ *if and only if G is a star.*

If G is a connected graph of order n and G is not a star, then $\gamma_r(G)\leqslant n-2$. Recall that a *leaf* in a graph is a vertex of degree one, while a *stem* is a vertex adjacent to a leaf. The next two results will show for which graphs this upper bound is attained.

Theorem 9. *If T is a tree of order* $n\geqslant 3$, *then* $\gamma_r(T)=n-2$ *if and only if T is obtained from* P_4, P_5 *or* P_6 *by adding zero or more leaves to the stems of the path.*

Proof. It is easy to verify that if T is obtained from P_4, P_5 or P_6 by adding zero or more leaves to the stems, then $\gamma_r(T)=n-2$.

Conversely, let T be a tree of order n such that $\gamma_r(T)=n-2$. If $\mathrm{diam}(T)\geqslant 6$, then T contains an induced P_7, say $v_1,\dots,v_7$. But then $V(T)-\{v_2,v_3,v_5,v_6\}$ is a restrained dominating set of T of size $n-4$, which is a contradiction. Thus, $\mathrm{diam}(T)\leqslant 5$. Furthermore, since T is not a star and stars are the only trees having diameter 2, $\mathrm{diam}(T)\geqslant 3$. Consider the following three cases.

Case 1: $\mathrm{diam}(T)=3$. Then T has an induced P_4, say $\langle\{v_1,v_2,v_3,v_4\}\rangle$. But then neither v_2 nor v_3 can have neighbors not on the path that have other neighbors. Thus, any other vertex must be adjacent to v_2 or v_3.

Case 2: $\mathrm{diam}(T)=4$. Then T has an induced P_5, say $\langle\{v_1,v_2,v_3,v_4,v_5\}\rangle$. Any neighbor of v_2 or v_4 not on the path cannot have another neighbor not on the path. Also, if v_3 has neighbors not on the path, then $V(T)-\{v_2,v_3,v_4\}$ is a restrained dominating set of T, so that $\gamma_r(T)\leqslant n-3$, which is a contradiction. Thus, every vertex not on the path must be adjacent to v_2 or v_4.

Case 3: diam(T)=5. Let the induced path of order 6 have vertices v_1, v_2, v_3, v_4, v_5 and v_6. Certainly, v_2 and v_5 can have leaves attached to them. If v_3 (v_4, respectively) have any neighbors not on the path, then $V(T)-\{v_2, v_3, v_4\}$ ($V(T)-\{v_3, v_4, v_5\}$, respectively) is a restrained dominating set of size $n-3$, which is a contradiction. Thus, any other vertex is adjacent to v_2 or v_5. □

Theorem 10. *Let G be a connected graph of order n containing a cycle. Then $\gamma_r(G)=n-2$ if and only if G is C_4 or C_5 or G can be obtained from C_3 by attaching zero or more leaves to at most two of the vertices of the cycle.*

Proof. If G is C_4 or C_5 or can be obtained from C_3 by attaching zero or more leaves to at most two of the vertices of the cycle, then it is easy to verify that $\gamma_r(G)=n-2$.

Suppose, conversely, that $\gamma_r(G)=n-2$. Then G cannot have a cycle of length at least 6, since if $v_1, v_2, v_3, v_4, v_5, v_6$ are consecutive vertices on a cycle of length at least 6 in G, then $V(G)-\{v_1, v_2, v_4, v_5\}$ is a restrained dominating set for G, which is a contradiction.

Case 1: G contains either a 5-cycle, or a 4-cycle, containing the consecutive vertices v_1, v_2, v_3 and v_4. If one of the vertices, say v_2, has a neighbor other than v_1 and v_3, then $V(G)-\{v_1, v_2, v_3\}$ is a restrained dominating set of G, which is a contradiction. Hence, $G\cong C_4$ or $G\cong C_5$.

Case 2: G contains the triangle, v_1, v_2, v_3, v_1. If each of v_1, v_2 and v_3 has neighbors not on the cycle, then $V(G)-\{v_1, v_2, v_3\}$ is a restrained dominating set of G, which is a contradiction. Without loss of generality, assume v_1 has no neighbors except v_2 and v_3.

If v_2 (or v_3) has a neighbor not on the cycle, say u_2, which is adjacent to another vertex, say w_2, not on the cycle, then $V(G)-\{v_1, v_2, u_2\}$ is a restrained dominating set of size $n-3$, which is a contradiction. Hence, any vertex on the cycle can only be adjacent to degree one vertices not on the cycle. □

Corollary 11. *If G is a graph of order n, then $\gamma_r(G)=n$ if and only if G is a disjoint union of stars and isolated vertices. Furthermore, $\gamma_r(G)=n-2$ if and only if exactly one of the components of G is isomorphic to a graph given in Theorems* 9 *or* 10 *and every other component is a star or P_1.*

The following result is immediate.

Proposition 12. *If G is a graph, then $\gamma_r(G)=1$ if and only if $G\cong K_1+H$ where H is a graph with no isolated vertices.*

We close this section by providing a lower bound for the restrained domination number of a tree.

Theorem 13. *If T is a tree of order $n \geqslant 3$, then $\gamma_r(T) \geqslant \Delta(T)$. Furthermore, $\gamma_r(T) = \Delta(T)$ if and only if T is a wounded spider which is not a star.*

Proof. Let T be a tree of order $n \geqslant 3$. Since T has at least $\Delta(T)$ leaves and any restrained dominating set must contain all the leaves, $\gamma_r(T) \geqslant \Delta(T)$. Clearly, for any wounded spider T which is not a star, we have $\gamma_r(T) = \Delta(T)$. So suppose T is a tree for which $\gamma_r(T) = \Delta(T)$. Let v be a vertex in T of degree $\Delta(T)$ and let S be γ_r-set of T. Clearly, S must contain all the leaves of T and hence is precisely the set of all leaves. Suppose V is the vertex set of T. Then $|V - S| \geqslant 2$ and each vertex $x \in V - S$ is adjacent to at least one $x' \in S$ such that $x \neq y$ implies $x' \neq y'$. So, each vertex in $V - S - \{v\}$ is adjacent to v and exactly one vertex in S. Hence, T is a wounded spider which is not a star. $\square$

3. A Nordhaus–Gaddum-type result

Nordhaus and Gaddum provided best possible bounds on the sum of the chromatic numbers of a graph and its complement in [5]. A corresponding result for the domination number was presented by Jaeger and Payan [3]: if G is a graph of order $n \geqslant 2$, then $3 \leqslant \gamma(G) + \gamma(\overline{G}) \leqslant n + 1$. An improved upper bound is due to Joseph and Arumugam [4]: if G is a graph of order n such that neither G nor $\overline{G}$ has isolated vertices, then $\gamma(G) + \gamma(\overline{G}) \leqslant (n+4)/2$.

We now prove best possible bounds on the sum of the restrained domination numbers of a graph and its complement.

Theorem 14. *If G is a graph of order $n \geqslant 2$, then $4 \leqslant \gamma_r(G) + \gamma_r(\overline{G})$. If G is a graph of order $n \geqslant 2$ such that $G \ncong P_3$ and $\overline{G} \ncong P_3$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 2$. Furthermore, these bounds are best possible.*

Proof. For the lower bound, we only need to show that if $\gamma_r(G) = 1$, then $\gamma_r(\overline{G}) \geqslant 3$. Suppose $\gamma_r(G) = 1$. Then, by Proposition 12, $G \cong K_1 + H$, where H is graph without any isolated vertices. If $\gamma_r(\overline{H}) = 1$, then, by Proposition 12, $\overline{H} \cong K_1 + H'$, where H' has no isolated vertices. This, however, implies that H has an isolated vertex, which is a contradiction. Thus, $\gamma_r(\overline{H}) \geqslant 2$ and, therefore, $\gamma_r(\overline{G}) \geqslant 3$.

Next we prove the upper bound. The cases when $n = 2$ or 3 are easy. Assume, therefore, $n \geqslant 4$. Since the complement of a disconnected graph is connected, we may assume that $G = (V, E)$ is connected. Choose $uv \in E$ and let $N = N_G(u) \cap N_G(v)$, $A = V - N - \{u, v\}$ and $I = \{w \mid w \text{ is an isolated vertex in } \langle A \rangle_{\overline{G}}\}$.

Case 1: $|I| \leqslant 1$ *and* $N = \emptyset$. If $I = \emptyset$, then $\{u, v\}$ is a restrained dominating set of $\overline{G}$ and the upper bound holds. Assume, therefore, that $|I| = 1$. Then there is a vertex, say w, in $(N(u) - \{v\}) \cup (N(v) - \{u\})$ such that $\deg(w) \geqslant 2$. Suppose, without loss of generality, that w is adjacent to u. It follows that $V - \{u, w\}$ is a restrained dominating set of G, while $\{u, v\} \cup I$ is a restrained dominating set of $\overline{G}$. Thus, $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant (n-2) + 2 + |I| \leqslant n + 1 < n + 2$.

Case 2: $|I|\leqslant 1$ *and* $N\neq\emptyset$. Since $A\cup\{u\}$ is a restrained dominating set of G and $\{u,v\}\cup N\cup I$ is a restrained dominating set of $\overline{G}$, we have $\gamma_r(G)+\gamma_r(\overline{G})\leqslant(|A|+1)+(2+|N|+|I|)=(|A|+|N|+2)+1+|I|\leqslant n+2$.

Case 3: $|I|\geqslant 2$, *say* $u',v'\in I$. Note that since u' (v', respectively) is in I, then in G, the vertex u' (v', respectively) is adjacent to every vertex in $A-\{u'\}$ ($A-\{v'\}$, respectively). Choose an edge $e=xy$ in G with $x\in N\cup\{u,v\}$ and $y\in A$. Choose one vertex w in $\{u,v\}-\{x\}$ and one vertex w' in $\{u',v'\}-\{y\}$. Then $\{w,w'\}$ is a restrained dominating set of G. Again, $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 2+n$.

All that remains is to show that the lower and upper bounds are best possible.

That the lower bound is best possible, may be seen as follows. For $n=2$ or 3, consider the complete graph K_n. For $n=4$, consider the path P_4. For $n\geqslant 5$, let n_1 and n_2 be integers such that $\min\{n_1,n_2\}\geqslant 2$ and $n_1+n_2=n-1$. Let $\overline{H}\cong K_{n_1,n_2}$ and $G\cong K_1+H$. Then, by Propositions 3 and 12, $\gamma_r(G)=1$, $\gamma_r(\overline{G})=3$ and $\gamma_r(G)+\gamma_r(\overline{G})=4$.

Since $\gamma_r(K_{1,n-1})+\gamma_r(\overline{K}_{1,n-1})=n+2$, our upper bound is best possible. □

4. Complexity issues for γ_r

In this section several complexity results are given. A linear time algorithm is given which computes the value for $\gamma_r(G)$ for any tree T. In the general case, however, the decision problem for $\gamma_r(G)$ is NP-complete, even when restricted to bipartite and chordal graphs.

For the first result, a dynamic programming style algorithm is constructed using the methodology of Wimer [7].

We make use of the well-known fact that the class of (rooted) trees can be constructed recursively from copies of the single vertex K_1, using only one rule of composition, which combines two trees (T_1,r_1) and (T_2,r_2) by adding an edge between r_1 and r_2 and calling r_1 the root of the resulting larger tree. We denote this as follows: $(T,r_1)=(T_1,r_1)\circ(T_2,r_2)$.

In particular, if S is a restrained dominating set of T, then S splits into two subsets S_1 and S_2 according to this decomposition. We express this as follows: $(T,S)=(T_1,S_1)\circ(T_2,S_2)$.

In order to construct an algorithm to compute $\gamma_r(T)$ for any tree T, we characterize the classes of possible tree-subset pairs (T,S) which can occur. For this problem there are four classes:

$[1]=\{(T,S)|r\in S, S$ is a restrained dominating set of $T\}$.

$[2]=\{(T,S)|r\notin S, S$ is a dominating set of T, S is not a restrained dominating set of T, but is a restrained dominating set of $T-r\}$.

$[3]=\{(T,S)|r\notin S, S$ is a restrained dominating set of $T\}$.

$[4]=\{(T,S)|r\notin S, S$ is not a dominating set of T but is a dominating set of $T-r$ and for all $x\notin S$, there exists $y\notin S$ such that $xy\in E(T)\}$.

	[1]	[2]	[3]	[4]
[1]	[1]	X	[1]	[1]
[2]	[2]	[3]	[3]	X
[3]	[3]	[3]	[3]	X
[4]	[3]	[4]	[4]	X

Fig. 1.

Next, we must consider the effect of composing a tree T_1 having a set S_1 which is of class $[i]$ with a tree T_2 having a set which is of class $[j]$ for every possible combination of classes $1 \leqslant i, j \leqslant 4$. That is, we must describe the appropriate class of the combined set $S_1 \cup S_2$ in the composed tree $T = T_1 \circ T_2$. This is described in Fig. 1. An 'X' in the table signifies that this composition cannot happen, that is, no set S can ever decompose into two subsets S_1 and S_2 of the classes indicated.

From Fig. 1, we can now write out a system of recurrence relations, as follows:

$$\begin{aligned}
&[1] = [1] \circ [1] \cup [1] \circ [3] \cup [1] \circ [4],\\
&[2] = [2] \circ [1],\\
&[3] = [2] \circ [2] \cup [2] \circ [3] \cup [3] \circ [1] \cup [3] \circ [2] \cup [3] \circ [3] \cup [4] \circ [1],\\
&[4] = [4] \circ [2] \cup [4] \circ [3].
\end{aligned}$$

To illustrate this, a tree-subset pair of class [1] can be read as follows: a tree-subset pair (T, S) which is of class [1] can be obtained only by composing a pair (T_1, S_1) of class [1] with a pair (T_2, S_2) of class [1] or by composing a pair (T_1, S_1) of class [1] with a pair (T_2, S_2) of class [3] or by composing a pair (T_1, S_1) of class [1] with a pair (T_2, S_2) of class [4].

To prove the correctness of this dynamic programming algorithm for computing $\gamma_{\mathrm{r}}(T)$ for any tree T, we would have to prove a theorem asserting that each of these recurrences are correct. Space limitations prevent us from doing this here, but it is easy to do. It is even easier to verify the correctness of Fig. 1, which can be done by inspection. The final step in specifying a γ_{r}-algorithm is to define the initial vector. In this case, for trees, the only basis graph is the tree with single vertex K_1. We need to know the minimum cardinality of a set S in a class of type [1]–[4] in the graph K_1, if any exists. It is easy to see that the initial vector is $[1, -, -, 0]$ where '$-$' means undefined.

We now have all the ingredients for a γ_{r}-algorithm, where the input is the parent array *parent*$[1 \ldots p]$ for the input tree and where the output is the 4-tuple corresponding to the root (i.e. vertex 1) of T which is computed repeatedly by applying the recurrence system to each vertex in the parent array, with the initial vector $[1, -, -, 0]$ being associated with every vertex in the parent array as the computation begins.

The basic structure for the algorithm is a simple iteration.

```
procedure γr;
for i:=1 to p do
   initialize vector [i,1... 4] to [1,-,-,0];
for j:=p downto 2 do
begin
  k:=  parent[j];
  vector[k,1]:= min{vector[k,1]+vector[j,1], vector[k,1]+vector[j,3],
                    vector[k,1]+vector[j,4]};
  vector[k,2]:= vector[k,2]+vector[j,1];
  vector[k,3]:= min{vector[k,2]+vector[j,2], vector[k,2]+vector[j,3],
                    vector[k,3]+vector[j,1], vector[k,3]+vector[j,2],
                    vector[k,3]+vector[j,3], vector[k,4]+vector[j,1]};
  vector[k,4]:= min{vector[k,4]+vector[j,2], vector[k,4]+vector[j,3]};
end;
γr(T):= min {vector[1,1], vector[1,3]};
end;{γr}.
```

It is clear that procedure γ_r has linear execution time.

To show that the decision problem for arbitrary graphs is NP-complete, we need to use a well-known NP-completeness result, called Exact Three Cover (X3C), which is defined as follows.

EXACT COVER BY 3-SETS (X3C)

Instance. A finite set X with $|X| = 3q$ and a collection $\mathscr{C}$ of 3-element subsets of X.

Question. Does $\mathscr{C}$ contain an exact cover for X, that is, a subcollection $\mathscr{C}' \subseteq \mathscr{C}$ such that every element of X occurs in exactly one member of $\mathscr{C}'$? Note that if $\mathscr{C}'$ exists, then its cardinality is precisely q.

Theorem 15 (Garey and Johnson [2]). X3C *is NP-complete.*

RESTRAINED DOMINATING SET (RDS)

Instance. A graph $G = (V, E)$ and a positive integer $k \leqslant |V|$.

Question. Is there a restrained dominating set of cardinality at most k?

Theorem 16. RDS *is NP-complete, even for bipartite graphs.*

Proof. It is clear that RDS is in NP.

To show that RDS is an NP-complete problem, we will establish a polynomial transformation from X3C. Let $X = \{x_1, \ldots, x_{3q}\}$ and $\mathscr{C} = \{C_1, \ldots, C_m\}$ be an arbitrary instance of X3C.

We will construct a bipartite graph G and a positive integer k such that this instance of X3C will have an exact three cover if and only if G has a restrained dominating set of cardinality at most k.

We now describe the construction of G. Corresponding to each $x_i \in X$ associate the path x_i, y_i, z_i. Corresponding to each C_j associate a K_2 with vertices c_j and d_j. The construction of the bipartite graph G is completed by joining x_i and c_j if and only if $x_i \in C_j$. Finally, set $k = m + 4q$.

Suppose $\mathscr{C}$ has an exact 3-cover, say $\mathscr{C}'$. Then $\bigcup_{i=1}^{3q}\{z_i\} \cup \bigcup_{j=1}^{m}\{d_j\} \cup \{c_j \mid C_j \in \mathscr{C}'\}$ is a restrained dominating set of cardinality $m + 4q$. This construction can clearly be accomplished in polynomial time.

Suppose, conversely, that D is a restrained dominating set of cardinality at most $m + 4q$. Then the vertices in the set L, defined by $\bigcup_{i=1}^{3q}\{z_i\} \cup \bigcup_{j=1}^{m}\{d_j\}$, are all end vertices of G and have to be in D. Hence, $|D| - |L| \leqslant (m + 4q) - (m + 3q) = q$. Let $I = \{i \in \{1,\dots,3q\} \mid x_i \in D \text{ or } y_i \in D\}$ and let $J = \{j \in \{1,\dots,m\} \mid c_j \in D\}$. Then, since D is a dominating set of G, $(\bigcup_{i\in I}\{x_i, y_i\} \cup \bigcup_{j\in J} N[c_j]) \cap \{x_1,\dots,x_{3q}\} \supseteq \{x_1,\dots,x_{3q}\}$. We conclude that $|I| + 3|J| \geqslant 3q$. Also, $|I| + |J| \leqslant |D| - |L| \leqslant q$. Hence, $3|I| + 3|J| \leqslant |I| + 3|J|$, so that $I = \emptyset$. We conclude that $x_i, y_i \notin D$ for $i = 1,\dots,3q$. Since $x_i, i = 1,\dots,3q$, is dominated by D, we must have that $|J| = q$ and that $\mathscr{C}' = \{C_j \mid j \in J\}$ is an exact cover for X. □

Theorem 17. RDS *is NP-complete, even for chordal graphs.*

Proof. The proof is similar to the proof of Theorem 16, except that edges are added so that $\{c_1,\dots,c_m\}$ forms a clique. □

References

[1] G. Chartrand, L. Lesniak, Graphs and Digraphs, Second ed., Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[2] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.

[3] F. Jaeger, C. Payan, Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple, C. R. Acad. Sci. Ser. A 274 (1972) 728–730.

[4] J.P. Joseph, S. Arumugam, A note on domination in graphs, Preprint.

[5] E.A. Nordhaus, J.W. Gaddum, On complementary graphs, Amer. Math. Monthly 63 (1956) 175–177.

[6] J.A. Telle, Vertex partitioning problems: characterization, complexity and algorithms on partial k-trees, Ph.D. Thesis, University of Oregon, CIS-TR-94-18.

[7] T.V. Wimer, Linear algorithms on k-terminal graphs, Ph.D. Thesis, Clemson University, 1987.